Mathematical calculations for the Double Pendulum Simulation

Here I will discuss how the analytical system works.

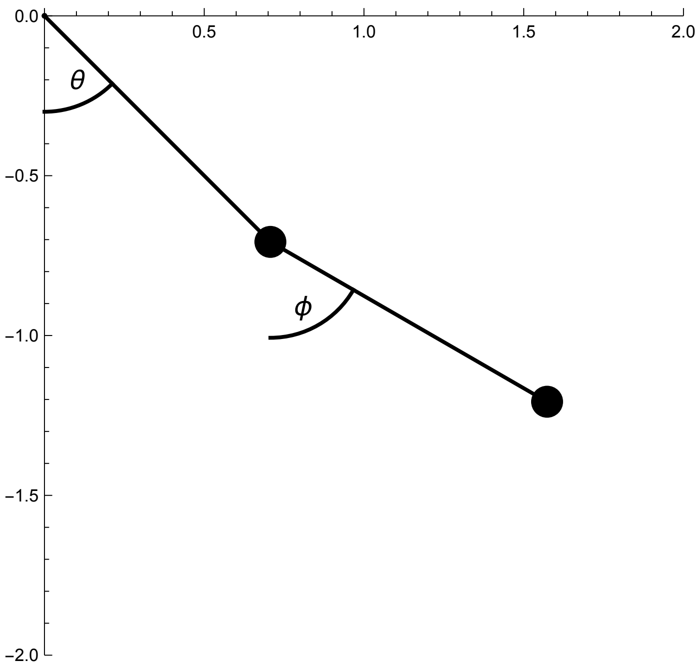
Let’s start by considering an example system:
We define the angle θ and the angle ϕ respect to the vertical, as shown in the figure below.
Each rod as a length of l and mass m.

In[1]:=
```
(*Example*)
a=π/4;
b=π/3;

x1=Sin[a];
y1=-Cos[a];
x2=x1+ Sin[b];
y2=y1- Cos[b];

Graphics[{Thick,Line[{{0,0},{x1,y1}}],Line[{{x1,y1},{x2,y2}}],
  Disk[{x1,y1},0.05],Disk[{x2,y2},0.05],Black,Point[{0,0}],
  Circle[{0,0},0.3,{3/2 π,3/2 π+a}],Circle[{x1,y1},0.3,{3/2 π,3/2 π+b}],
  Text[Style["θ",14],{0.1,-0.2}],Text[Style["ϕ",14],{x1+0.1,y1-0.2}]
},PlotRange→{{0, 2}, {0,- 2}},Axes→True,AxesOrigin→{0,0}]
```

Out[7]=



So we have the generalized coordinates $q = (\theta, \phi)$, which uniquely describe the two points

$x_1 = (l \sin\theta, -l\cos\theta)$, $x_2 = x_1 + (l\sin\phi, -l\cos\phi)$.

For simplicity and to reduce computations we will set $m = 1$ and $l = 1$.

From $\dot{x}_1$ and $\dot{x}_2$ we can calculate the kinetic energy:

In[8]:=
```
T [θ_, ϕ_, θ_', ϕ_'] := 1/2 (2 θ'^2 + ϕ'^2 + 2 Cos[θ - ϕ] θ' ϕ')
```

And the potential energy:

In[9]:=
```
V[θ_, ϕ_] := g (2 Cos[θ] + Cos[ϕ])
```

So we can get the Lagrangian:

In[10]:=
```
L[θ_, ϕ_, θ_', ϕ_'] := T[θ, ϕ, θ', ϕ'] - V[θ, ϕ]
Collect[FullSimplify[L [θ, ϕ, θ', ϕ']], {θ', ϕ', g}]
```

Out[11]=

$$g\,(-2\,\mathrm{Cos}[\theta] - \mathrm{Cos}[\phi]) + (\theta')^2 + \mathrm{Cos}[\theta-\phi]\,\theta'\,\phi' + \frac{(\phi')^2}{2}$$

By calculating the partial derivatives of the Lagrangian L respect to $\dot{\theta}$ and $\dot{\phi}$ we get the associated moments p_θ and p_ϕ.

In[12]:=
```
M = {Coefficient[D[L[θ, ϕ, θ', ϕ'] , θ'], {θ′, ϕ′}],
   Coefficient[D[L[θ, ϕ, θ', ϕ'], ϕ'], {θ′, ϕ′}]};
MatrixForm[M]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} 2 & \mathrm{Cos}[\theta-\phi] \\ \mathrm{Cos}[\theta-\phi] & 1 \end{pmatrix}$$

We now have that $(p_\theta, p_\phi) = M\,(\dot{\theta}, \dot{\phi})$.

We can invert the matrix to find $\dot{\theta}$ and $\dot{\phi}$ respect to p_θ and p_ϕ.

In[14]:=
```
Inv = Inverse[M];
MatrixForm[Inv]
```

Out[15]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{-2+\mathrm{Cos}[\theta-\phi]^2} & \frac{\mathrm{Cos}[\theta-\phi]}{-2+\mathrm{Cos}[\theta-\phi]^2} \\ \frac{\mathrm{Cos}[\theta-\phi]}{-2+\mathrm{Cos}[\theta-\phi]^2} & -\frac{2}{-2+\mathrm{Cos}[\theta-\phi]^2} \end{pmatrix}$$

We can calculate the Jacobian of the system:

In[16]:=
```
J [θ_, ϕ_, θ_', ϕ_', Pθ_, Pϕ_ ] := Pθ θ' + Pϕ ϕ' - L[θ, ϕ, θ', ϕ']
FullSimplify[J[θ, ϕ, θ', ϕ', Pθ, Pϕ]]
```

Out[17]=

$$g\,(2\,\mathrm{Cos}[\theta] + \mathrm{Cos}[\phi]) - (\theta')^2 + \mathrm{P}\phi\,\phi' - \frac{(\phi')^2}{2} + \theta'\,(\mathrm{P}\theta - \mathrm{Cos}[\theta-\phi]\,\phi')$$

By substituting $\dot{\theta}$ and $\dot{\phi}$ in the Jacobian we get the Hamiltonian of the system.

In[18]:=
```
H [θ_, ϕ_, Pθ_, Pϕ_] := {Pθ θ' + Pϕ ϕ' - L[θ, ϕ, θ', ϕ']} /.
  {θ' → Inv⟦1, 1⟧ Pθ + Inv⟦1, 2⟧ Pϕ , ϕ' → Inv⟦2, 1⟧ Pθ + Inv⟦2, 2⟧ Pϕ }
Collect[FullSimplify[H [θ, ϕ, Pθ, Pϕ]], {Pθ, Pϕ, g}]
```

Out[19]=

$$\left\{-\frac{P\theta^2}{-3+\text{Cos}[2\ (\theta-\phi)]}-\frac{2\ P\phi^2}{-3+\text{Cos}[2\ (\theta-\phi)]}+\frac{2\ P\theta\ P\phi\ \text{Cos}[\theta-\phi]}{-3+\text{Cos}[2\ (\theta-\phi)]}+g\ (2\ \text{Cos}[\theta]+\text{Cos}[\phi])\right\}$$

From the Hamiltonian we can simply get the 4 equation of motion by calculating the partial derivatives of the Hamiltonian:

In[20]:=
```
Collect[FullSimplify[-D[H[θ, ϕ, Pθ, Pϕ], Pθ]], {Pθ, Pϕ, g}]
Collect[FullSimplify[-D[H[θ, ϕ, Pθ, Pϕ], Pϕ]], {Pθ, Pϕ, g}]
Collect[FullSimplify[D[H[θ, ϕ, Pθ, Pϕ], θ]], {Pθ, Pϕ, g}]
Collect[FullSimplify[D[H[θ, ϕ, Pθ, Pϕ], ϕ]], {Pθ, Pϕ, g}]
```

Out[20]=

$$\left\{\frac{P\theta}{-2+\text{Cos}[\theta-\phi]^2}-\frac{P\phi\ \text{Cos}[\theta-\phi]}{-2+\text{Cos}[\theta-\phi]^2}\right\}$$

Out[21]=

$$\left\{\frac{2\ P\phi}{-2+\text{Cos}[\theta-\phi]^2}-\frac{P\theta\ \text{Cos}[\theta-\phi]}{-2+\text{Cos}[\theta-\phi]^2}\right\}$$

Out[22]=

$$\left\{-\frac{g\ (-3+\text{Cos}[2\ (\theta-\phi)])^2\ \text{Sin}[\theta]}{2\ \left(-2+\text{Cos}[\theta-\phi]^2\right)^2}-\frac{P\theta^2\ \text{Cos}[\theta-\phi]\ \text{Sin}[\theta-\phi]}{\left(-2+\text{Cos}[\theta-\phi]^2\right)^2}-\frac{2\ P\phi^2\ \text{Cos}[\theta-\phi]\ \text{Sin}[\theta-\phi]}{\left(-2+\text{Cos}[\theta-\phi]^2\right)^2}+\frac{P\theta\ P\phi\ (5+\text{Cos}[2\ (\theta-\phi)])\ \text{Sin}[\theta-\phi]}{2\ \left(-2+\text{Cos}[\theta-\phi]^2\right)^2}\right\}$$

Out[23]=

$$\left\{-\frac{2\ P\theta\ P\phi\ (5+\text{Cos}[2\ (\theta-\phi)])\ \text{Sin}[\theta-\phi]}{(-3+\text{Cos}[2\ (\theta-\phi)])^2}+\frac{2\ P\theta^2\ \text{Sin}[2\ (\theta-\phi)]}{(-3+\text{Cos}[2\ (\theta-\phi)])^2}+\frac{4\ P\phi^2\ \text{Sin}[2\ (\theta-\phi)]}{(-3+\text{Cos}[2\ (\theta-\phi)])^2}-g\ \text{Sin}[\phi]\right\}$$

So those are our equations to calculate the movements of the pendulums. We can actually write them a bit better by defining:

$c = \cos(\theta - \phi)$
$s = \cos(\theta - \phi)$
$d = 2 - c^2$
$d_2 = d^2$
$p = p_\theta\, p_\phi$
$n = \left(\left(p_\theta^2 + 2\, p_\phi^2\right) c - p\left(2 + c^2\right)\right) s$

In this way we have the four equations:

$$\dot{\theta} = \frac{-p_\theta + p_\phi\, c}{d}$$

$$\dot{\phi} = \frac{p_\theta - 2\, p_\phi\, c}{d}$$

$$\dot{p}_\theta = -\frac{n}{d_2} + 2\,g\,s$$
$$\dot{p}_\phi = \frac{n}{d_2} + g\,s$$

These equations are ready to be put in our system integrator.