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Part V: Camera Geometry and Multi-View

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From the beginning, in 2D

- Typically we represent a point (pixel) as:

$$P = [x, y]^t = \begin{bmatrix} x \\ y \end{bmatrix}$$

- Often convenient to use homogeneous coordinates:

$$P = [x, y]^t = [sx, sy, s]$$

- s is a scaling factor, commonly 1.0 $\quad P = [x, y]^t = [x, y, 1]$

- For simplicity we can omit "t"

Affine transformations

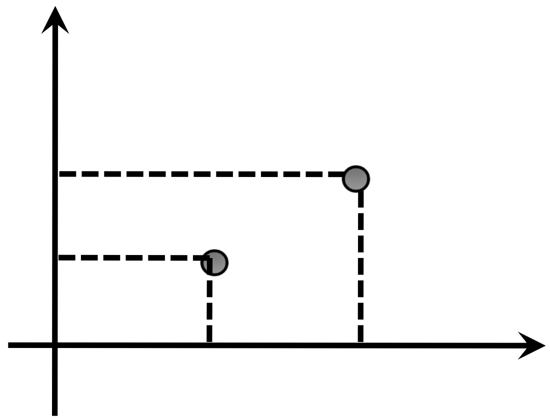
- Spatial transformations represented as:
  - Multiplication of a matrix and a homogeneous point
- Different types
  - Scaling
  - Rotation
  - Translation
  - Combination of them

Scaling

- Linear transformation applied to all points. It can be uniform or non uniform:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

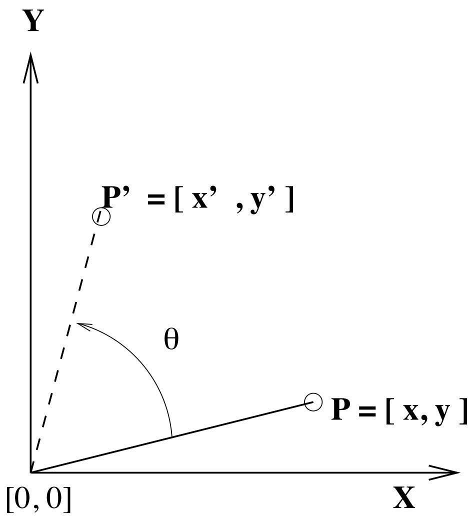
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} c_x & 0 \\ 0 & c_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

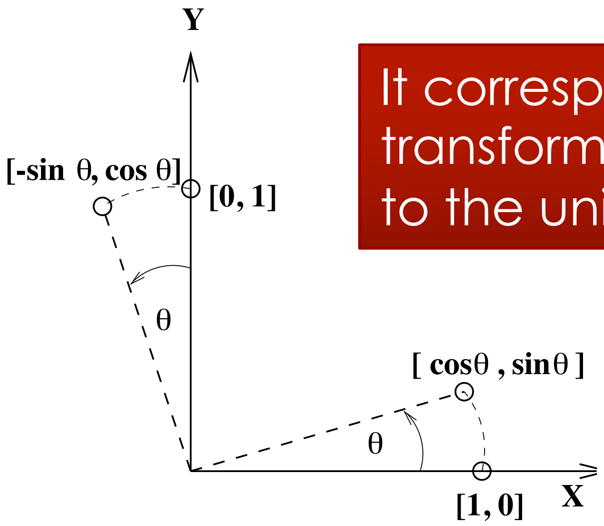


Rotation

- $P=[x,y]$ rotated by ϑ

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\vartheta - y\sin\vartheta \\ x\sin\vartheta + y\cos\vartheta \end{bmatrix}$$





It corresponds to the transformation applied to the unit vectors

Translation

- Uniform shift of the object points
- Equivalent to changing the origin
- The displacement $D([x,y]) = [x+x_0, y+y_0]$
- Using the homogeneous coordinates:

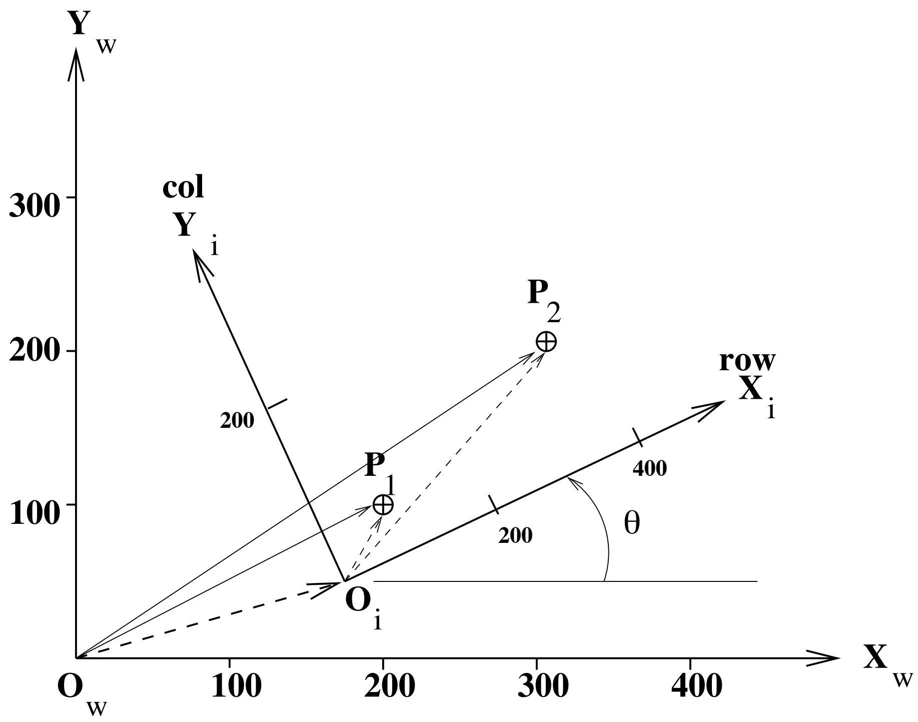
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + x_0 \\ y + y_0 \\ 1 \end{bmatrix}$$

Rotation, scaling, and translation

- Image $I(r,c)$
- Planar surface $W(x,y)$
- Need to map: real world ←→ image plane
- Combination of rotation R, scaling S, translation D
- 4 parameters:
  - Rotation angle
  - Scale factor
  - Translation vector

Rotation, scaling, and translation

- ${}^{w}P_j = D_{x0,y0} S_s R_\theta \, {}^{i}P_j$
- $w \rightarrow$ *world*, $i \rightarrow$ *image*, $j \rightarrow$ *generic point*



Rotation, scaling, and translation

- To obtain the transformation matrix
  - Take two points, called control points
  - Control points must be clearly visible
  - Determine the position vector in the world and in the image plane
- Now solve the system…

$$\begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

For P_1
Then the same for P_2

General affine transformation

- It handles parameters in a single matrix

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- 6 params determined using 3 matching pairs
- Errors may occur
- The higher the number of points, the smaller the error
- This is defined as the **CAMERA MATRIX**

General affine transformation

- Error (deviation) using least-squares method:

$$\varepsilon(a_{11},a_{12},a_{13},a_{21},a_{22},a_{23}) = \sum_{j=1}^{n}((a_{11}x_j + a_{12}y_j + a_{13} - u_j)^2 + (a_{21}x_j + a_{22}y_j + a_{23} - v_j)^2$$

- To determine the matrix coefficients, take partial derivatives of the error function and set them to zero.

- The resulting equation system is:

$$\begin{bmatrix} \sum x_j^2 & \sum x_j y_j & \sum x_j & 0 & 0 & 0 \\ \sum x_j y_j & \sum y_j^2 & \sum y_j & 0 & 0 & 0 \\ \sum x_j & \sum y_j & \sum 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum x_j^2 & \sum x_j y_j & \sum x_j \\ 0 & 0 & 0 & \sum x_j y_j & \sum y_j^2 & \sum y_j \\ 0 & 0 & 0 & \sum x_j & \sum y_j & \sum 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} \sum u_j x_j \\ \sum u_j y_j \\ \sum u_j \\ \sum v_j x_j \\ \sum v_j y_j \\ \sum v_j \end{bmatrix}$$

Least square method

- Developed to solve overdetermined equation systems
- Over-determined = more equations than unknowns
- Goal: minimize the sum of the squares of the errors
- **Error** is the distance between the observed value (*u* and *v* in our case) and the value obtained by applying the transformation (*x* and *y* after transformation)
- A simple case: $y = f(x) = Ax + B \;\rightarrow\; \varepsilon = \sum_{j=i}^{N} (Ax_j + B - y_j)^2$
- The solution is found by computing the minimum of the error = compute the partial derivative with respect to each unknown and set them to zero

$$\frac{\partial \varepsilon}{\partial A}, \frac{\partial \varepsilon}{\partial B}$$

Going 3D

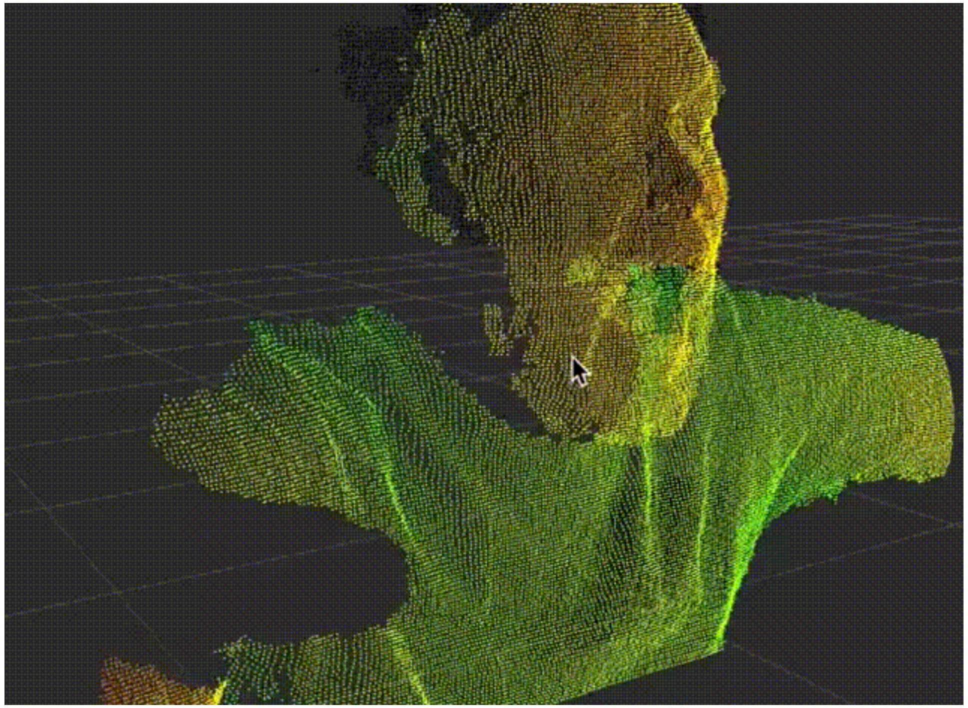
- One view in some cases is not sufficient
  - Depth of a point cannot be perceived
- Need to acquire images from different perspectives for:
  - 3D mesh reconstruction
  - Point Cloud acquisition
  - Position estimation
  - Structure from motion
  - Mosaicking
  - Etc…

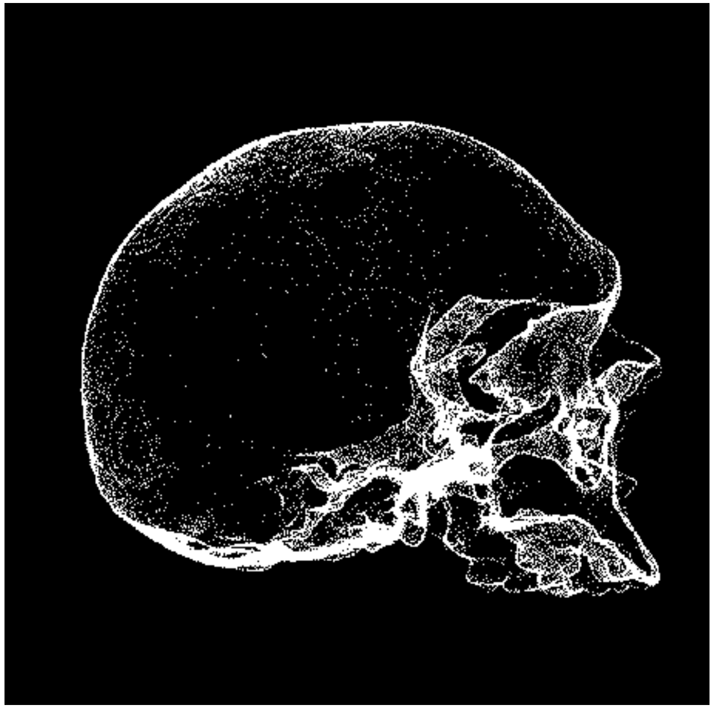
Going 3D

- 3D analysis handles the structure of objects and their real motion in space
  - More robust than 2D estimation
- 3D computation requires **at least two 2D projections**
- **Calibration** is needed
  - **Intrinsic** parameters
  - **Extrinsic** parameters

Examples

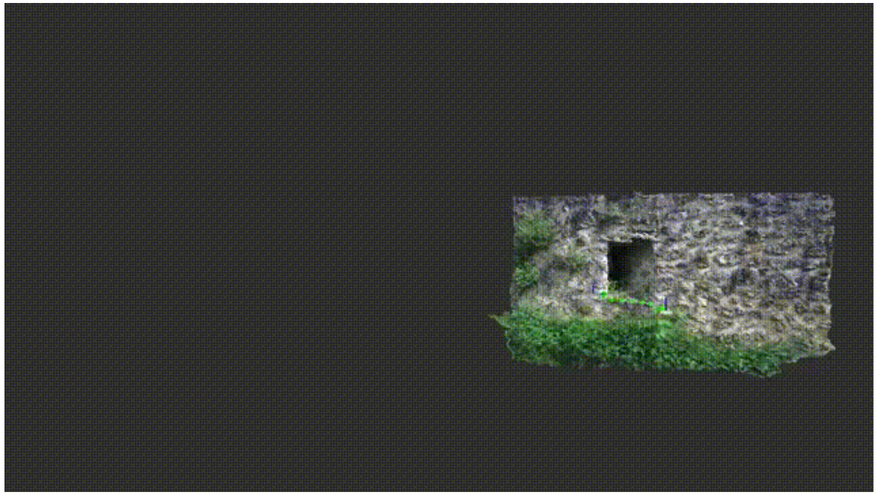
Point Clouds

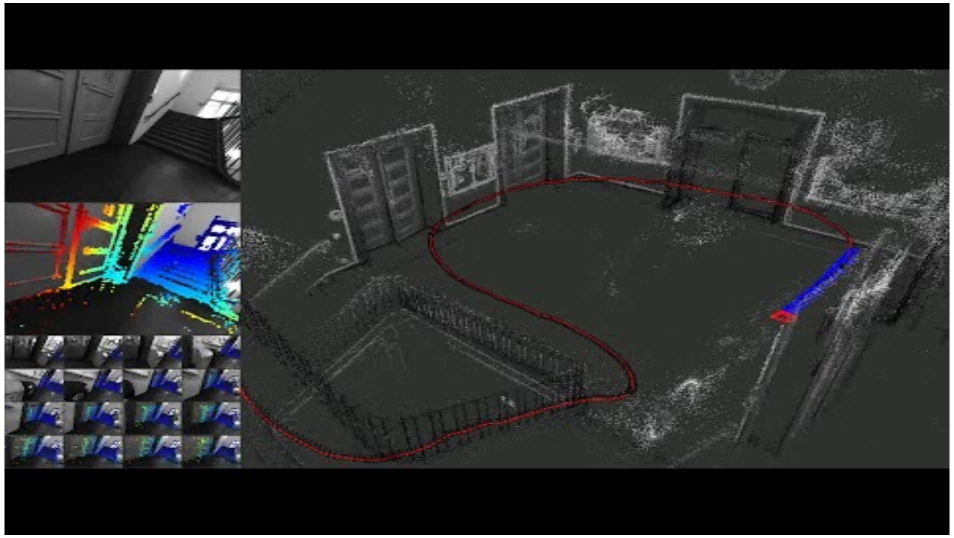




Examples

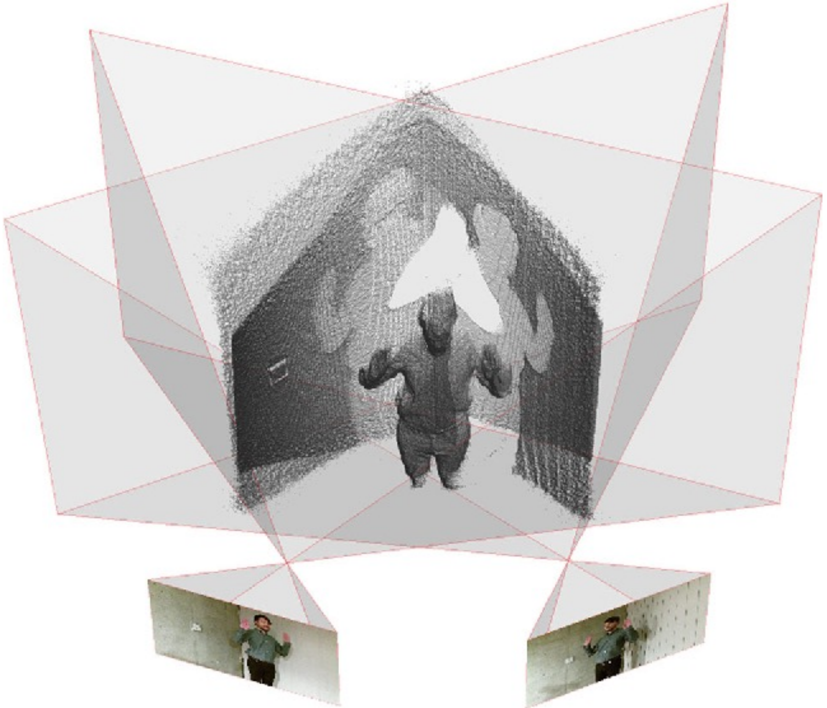
Structure From Motion

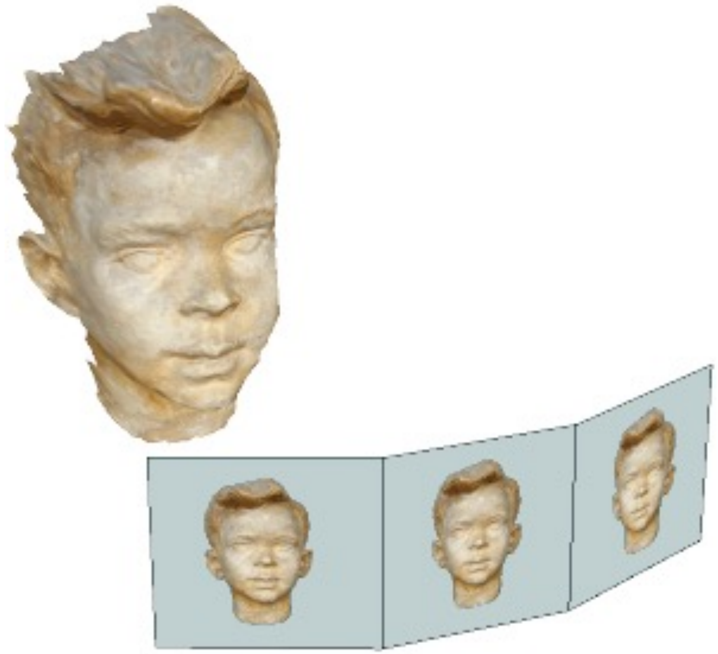




Examples

Mesh Reconstruction





Intrinsic and extrinsic parameters

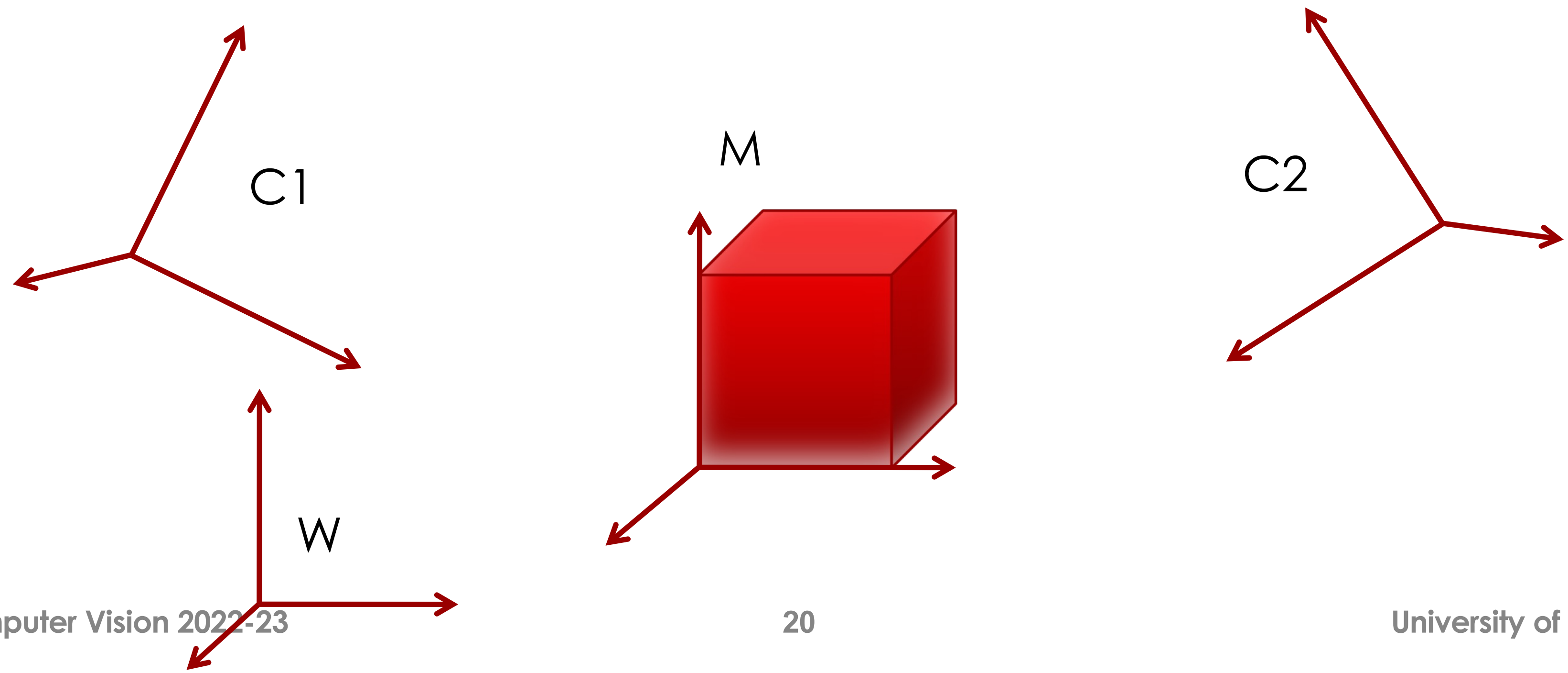
- Intrinsic parameters
  - Specific for the sensor
  - Link pixel coordinates with the corresponding coordinates in the camera reference system
  - Optical, geometric, digital features
    - Focal length
    - Image distortion
    - Size of the pixel
- Extrinsic parameters:
  - Position of the camera in the world coordinates
    - Translation vector
    - Rotation matrix

3D affine transformations

- Extension to the 3D of what we have seen for 2D
- Rotation, translation, scaling
- In homogeneous coordinates, go from
  - $[P_x, P_y, P_z]$ to $[sP_x, sP_y, sP_z, s]$
- Each point has in general 4 coordinate systems
  - Model
  - World
  - C1
  - C2

3D affine transformations

- ${}^{W}P = \mathbf{TR}\,{}^{M}P$, same for ${}^{1}P$ and ${}^{2}P$
- ${}^{1}P \,!{=}\, {}^{2}P$
- C1 and C2 can have opposite views, thus there is in general no one-to-one mapping of all points



Translation

$$^{2}P = T(x_0, y_0, z_0)\,^{1}P$$

$$\begin{bmatrix} ^{2}P_x \\ ^{2}P_y \\ ^{2}P_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{1}P_x \\ ^{1}P_y \\ ^{1}P_z \\ 1 \end{bmatrix}$$

- Add a translation vector to map point ^{1}P in coordinate of C1 to ^{2}P in coordinate C2

Scaling

$$^{2}P = S(s_x, s_y, s_z)\,^{1}P$$

$$\begin{bmatrix} ^{2}P_x \\ ^{2}P_y \\ ^{2}P_z \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{1}P_x \\ ^{1}P_y \\ ^{1}P_z \\ 1 \end{bmatrix}$$

- Scaling can be applied differently on the image coordinates

Rotation

- Rotation *x, y,* or *z* axis
- In case of rotation about *z*, it is the same as in 2D

$$
{}^{2}P = R(X,\vartheta)\,{}^{1}P
$$

$$
\begin{bmatrix} {}^{2}P_x \\ {}^{2}P_y \\ {}^{2}P_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta & 0 \\ 0 & \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{1}P_x \\ {}^{1}P_y \\ {}^{1}P_z \\ 1 \end{bmatrix}
$$

$$
{}^{2}P = R(Y,\vartheta)\,{}^{1}P
$$

$$
\begin{bmatrix} {}^{2}P_x \\ {}^{2}P_y \\ {}^{2}P_z \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{1}P_x \\ {}^{1}P_y \\ {}^{1}P_z \\ 1 \end{bmatrix}
$$

$$
{}^{2}P = R(Z,\vartheta)\,{}^{1}P
$$

$$
\begin{bmatrix} {}^{2}P_x \\ {}^{2}P_y \\ {}^{2}P_z \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 & 0 \\ -\sin\vartheta & \cos\vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{1}P_x \\ {}^{1}P_y \\ {}^{1}P_z \\ 1 \end{bmatrix}
$$

General configuration

- The generic 3D affine transformation can be expressed through

$$\begin{bmatrix} {}^2P_x \\ {}^2P_y \\ {}^2P_z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^1P_x \\ {}^1P_y \\ {}^1P_z \\ 1 \end{bmatrix}$$

Camera model

- It consists of the transformation that maps the 3D point into the image plane, so:

$$^{I}P = {}^{I}_{W}C\,{}^{W}P$$

- According to our knowledge:

$$\begin{bmatrix} s\,^{I}P_r \\ s\,^{I}P_c \\ s \end{bmatrix} = {}^{I}_{W}C \begin{bmatrix} ^{W}P_x \\ ^{W}P_y \\ ^{W}P_z \\ 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & 1 \end{bmatrix} \begin{bmatrix} ^{W}P_x \\ ^{W}P_y \\ ^{W}P_z \\ 1 \end{bmatrix}$$

- Using the 3x4 camera matrix we can handle rotation, translation and scaling

Nice, but

- In general camera coordinates differ from world coordinates

- Need to apply a roto-translation transformation to go from ${}^{W}P$ to ${}^{C}P$

$$\begin{bmatrix} {}^{C}P_x \\ {}^{C}P_y \\ {}^{C}P_z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{W}P_x \\ {}^{W}P_y \\ {}^{W}P_z \\ 1 \end{bmatrix}$$

$${}^{C}P = {}^{C}_{W}TR(\alpha,\beta,\gamma,t_x,t_y,t_z)\,{}^{W}P$$

  - Furthermore, ${}^{C}P$ is about the camera coordinates, and not the image coordinates ${}^{F}P$

From world to image coordinates

- From world to camera, from camera to image

$$^{F}P = {}^{F}_{C}\Pi(f)\,{}^{C}P$$

$$^{F}P = {}^{F}_{C}\Pi(f)\,{}^{C}_{W}TR(\alpha,\beta,\gamma,t_x,t_y,t_z)\,{}^{W}P$$

$$\begin{bmatrix} s\,^{F}P_r \\ s\,^{F}P_c \\ s \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & 1 \end{bmatrix} \begin{bmatrix} ^{W}P_x \\ ^{W}P_y \\ ^{W}P_z \\ 1 \end{bmatrix}$$

From mm to pixels

- It consists of a scaling factor that is related to the real size of the pixel
  - d_x is the horizontal size
  - d_y is the vertical size
- Not only that, origin is usually bottom-left
- In the image, origin is top-left

$$^{I}P = {}^{I}_{F}S \, {}^{F}P$$

$${}^{I}_{F}S = \begin{bmatrix} 0 & -1/d_y & 0 \\ 1/d_x & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the end

- The final expression becomes then

$$[p_r, p_c]^T = {}^I P = {}_F^I S\, {}_C^F \Pi(f)\, {}_W^C TR(\alpha, \beta, \gamma, t_x, t_y, t_z)\, {}^W P$$

$$\begin{bmatrix} s\,{}^I P_r \\ s\,{}^I P_c \\ s \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & 1 \end{bmatrix} \begin{bmatrix} {}^W P_x \\ {}^W P_y \\ {}^W P_z \\ 1 \end{bmatrix}$$

Calibration

- To make a real measurement in 3D
- The camera matrix we derive is appropriate but the coefficients of the matrix have to be determined
- 11 parameters using least squares

1. Typically an object of known size is used
2. A set of points in the image/world are taken
3. 6 pairs are ok, but a higher number is preferable

Shapiro-Stockman: Page 431

Calibration

- From the camera matrix
  - Given a 3D point $[{}^{W}P_x, {}^{W}P_y, {}^{W}P_z] = [x, y, z]$
  - Given the 2D projection $[{}^{I}P_r, {}^{I}P_c] = [u, v]$
  - For each point in the calibration process we have:

$$\begin{bmatrix} x_j & y_j & z_j & 1 & 0 & 0 & 0 & 0 & -x_j u_j & -y_j u_j & -z_j u_j \\ 0 & 0 & 0 & 0 & x_j & y_j & z_j & 1 & -x_j v_j & -y_j v_j & -z_j v_j \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{24} \\ c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$$

Calibration

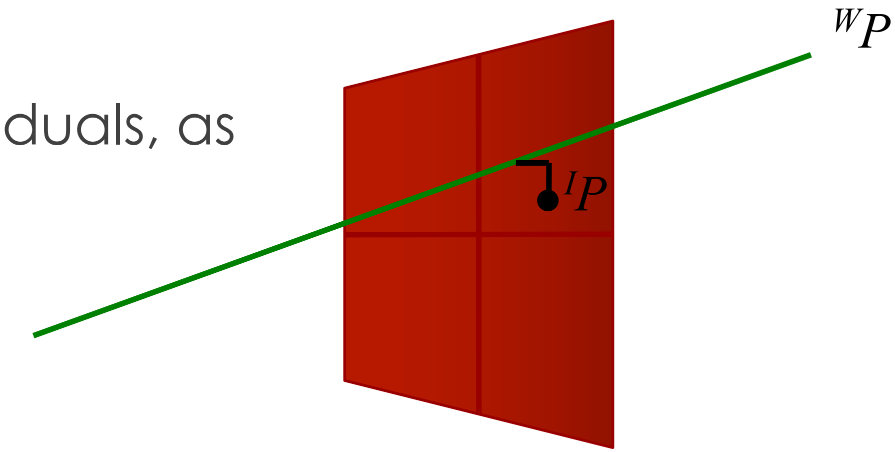
- Overdetermined system: 11 unknowns and 12 or more equations
- No vector can satisfy all the equations
- Need to use a least squares approach
- Objective: minimize the sum of all the square differences

Calibration

- The error between the actual image point measurements and the world points comes from

$$^{I}P = {}^{I}_{W}C\,{}^{W}P$$

- Need to minimize the residuals, as in the example:



Compute 3D position of a point

- If the cameras are calibrated we can determine the 3D position of a generic point $[x, y, z]$ given two projections $[r_1, c_1]$ & $[r_2, c_2]$

$$\begin{bmatrix} sr_1 \\ sc_1 \\ s \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} tr_2 \\ tc_2 \\ t \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Compute 3D position of a point

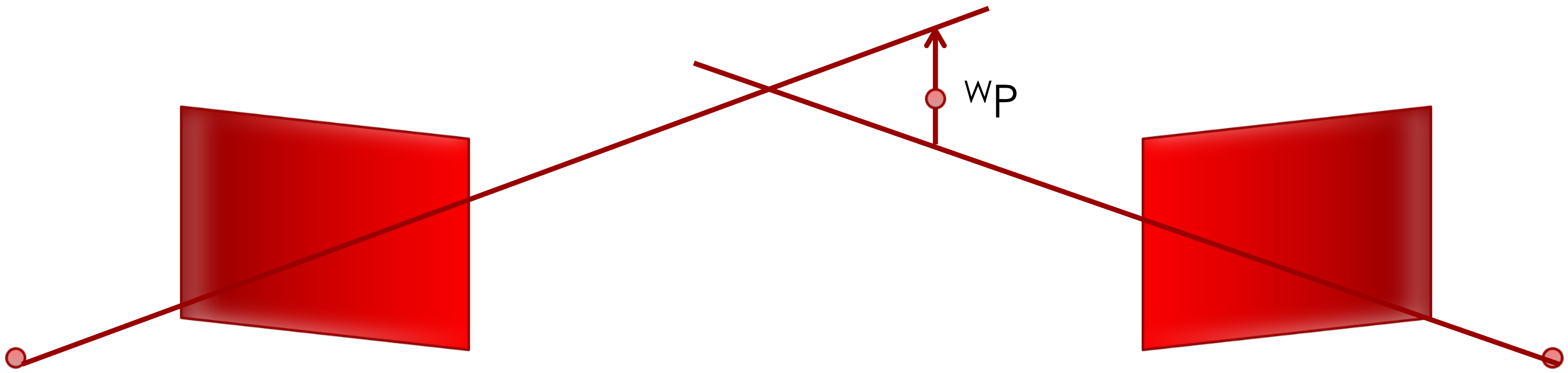
- If we drop the scaling factors we obtain four equations to compute r_1, r_2, c_1, c_2

$$r_1 = (b_{11} - b_{31} r_1)x + (b_{12} - b_{32} r_1)y + (b_{13} - b_{33} r_1)z + b_{14}$$
$$c_1 = (b_{21} - b_{31} c_1)x + (b_{22} - b_{32} c_1)y + (b_{23} - b_{33} c_1)z + b_{24}$$
$$r_2 = (c_{11} - c_{31} r_2)x + (c_{12} - c_{32} r_2)y + (c_{13} - c_{33} r_2)z + c_{14}$$
$$c_2 = (c_{21} - c_{31} c_2)x + (c_{22} - c_{32} c_2)y + (c_{23} - c_{33} r_2)z + c_{24}$$

- We have three unknowns
- From any three of these four eq. we can find a solution
- The solution is subject to errors though

Approximation errors

- If we took the four equations all together they would be inconsistent
  - Approximation in image points
  - Approximation in camera model
  - In practice the rays do not intersect where they should in the real world

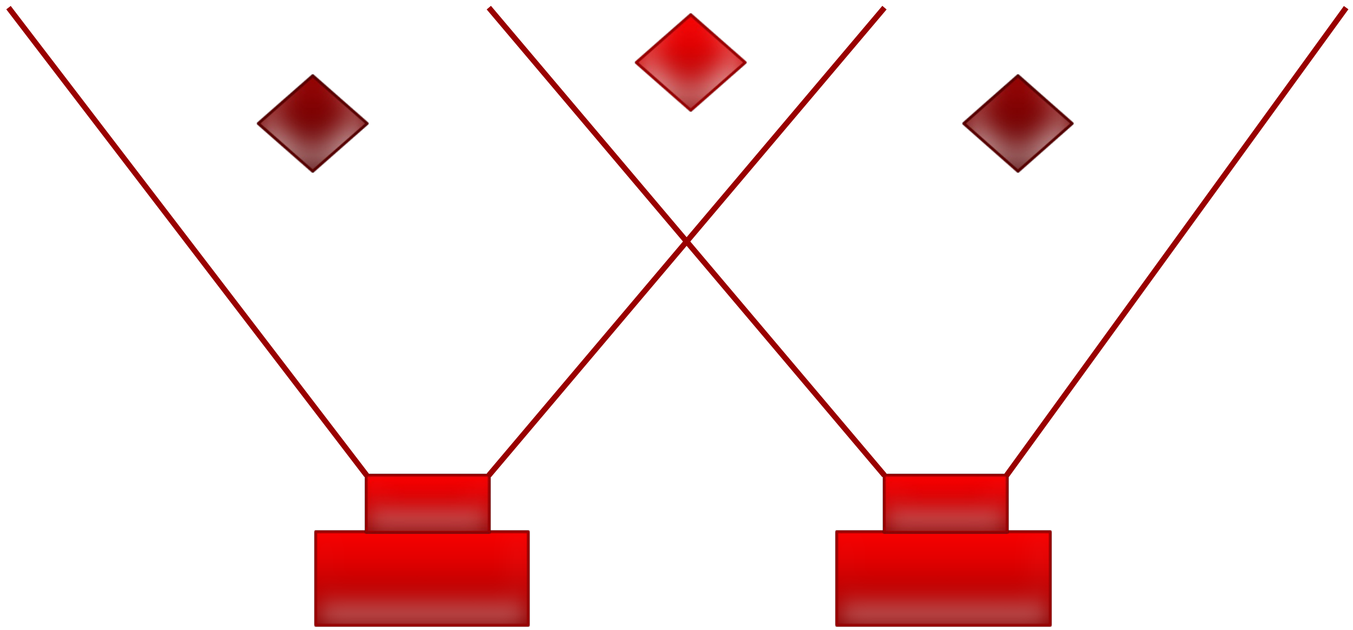


Binocular stereo

- Two images grab a scene from different positions
- Human Visual System is stereo
- Identification of two main problems
  - Compute correspondences
  - Reconstruction

Two co-planar cameras

- Only for the points "seen" by both cameras, the depth information can be extracted



Two co-planar cameras

- Offset in the view, due to the camera shift
- Same point is projected in a different place
- Depending on the depth of the point, shift is different
- The closer the object, the higher the shift
- Parallax: Apparent motion → closer objects move faster

Compute correspondences

- Find points in the images that correspond to the projection of the same point in the real scene

- Image coupling is possible because the two images do not differ much between each other.

- False correspondences may be present → introduction of other constraints

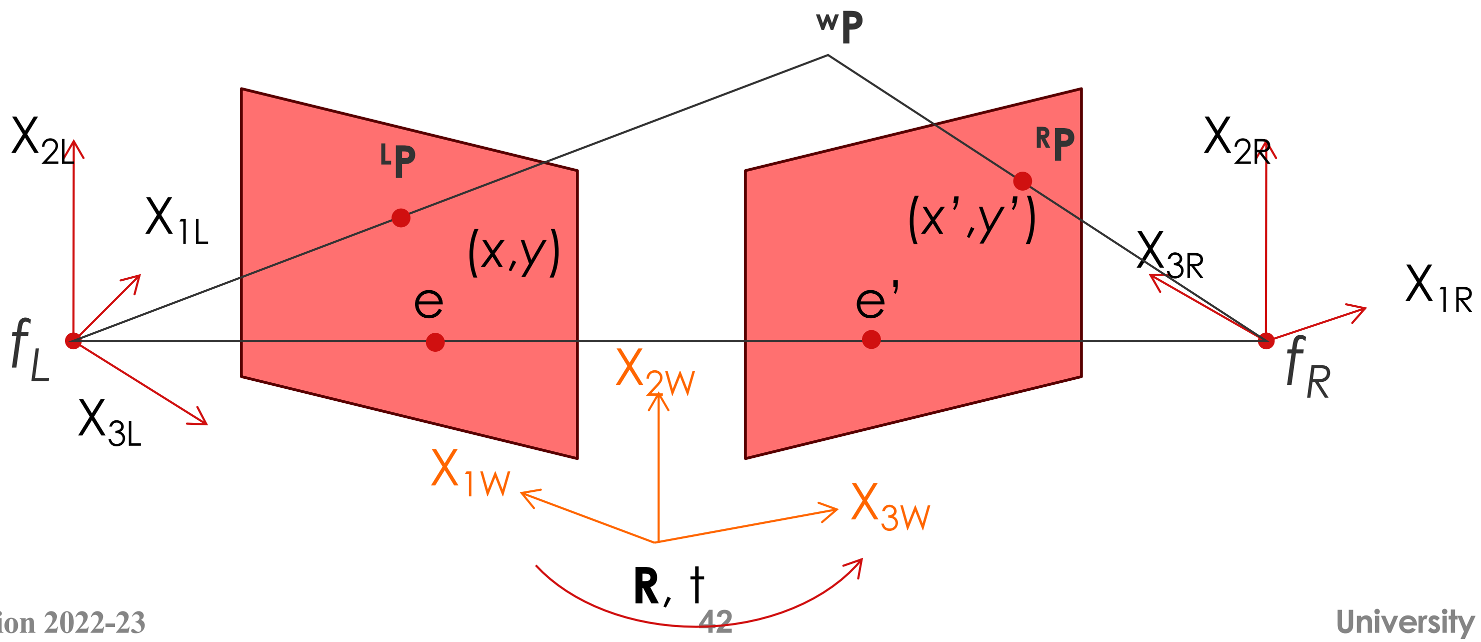
- **Epipolar constraint:**
  - Correspondences can be met along a horizontal line called "epipolar line"

Reconstruction

- Once the points in the two images are matched, reconstruction is possible.
- Before computing correspondences it is necessary to **calibrate** the cameras to determine:
  - Extrinsic parameters
  - Intrinsic parameters

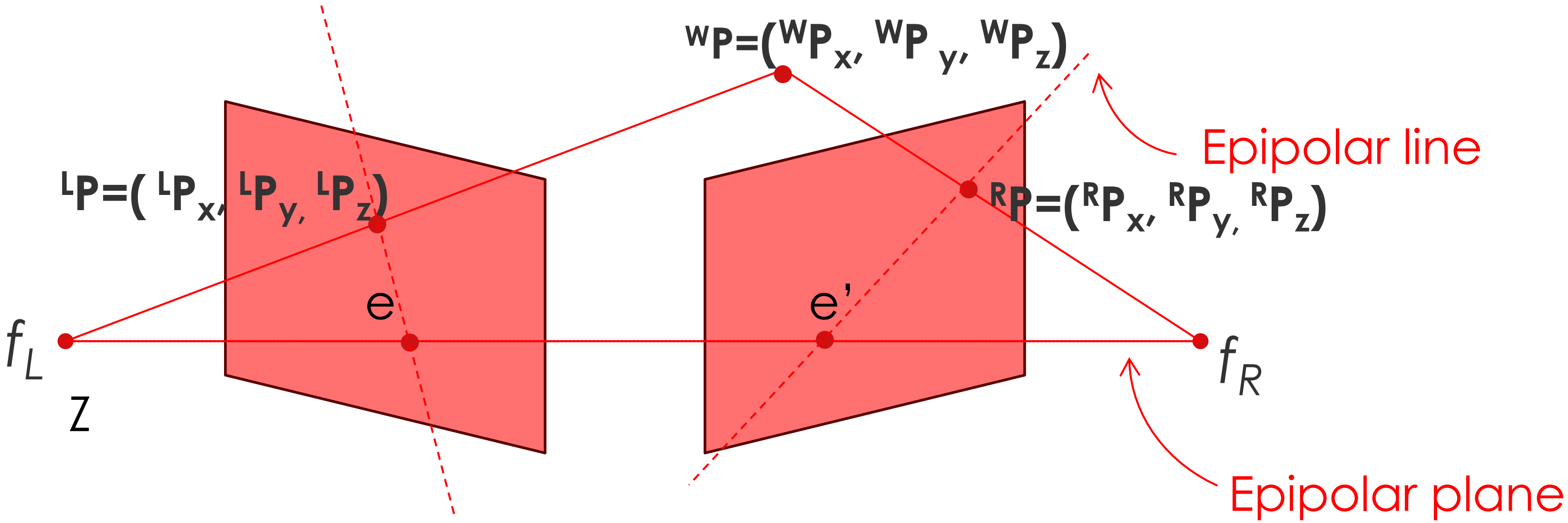
Stereo vision

- Based on the so-called epipolar geometry
- Estimation of the depth of a point using the horizontal disparity of the projections
- The problem consists in computing:
  - Correspondences of intensity points (correlation)
  - Correspondences of features (edges, contours)

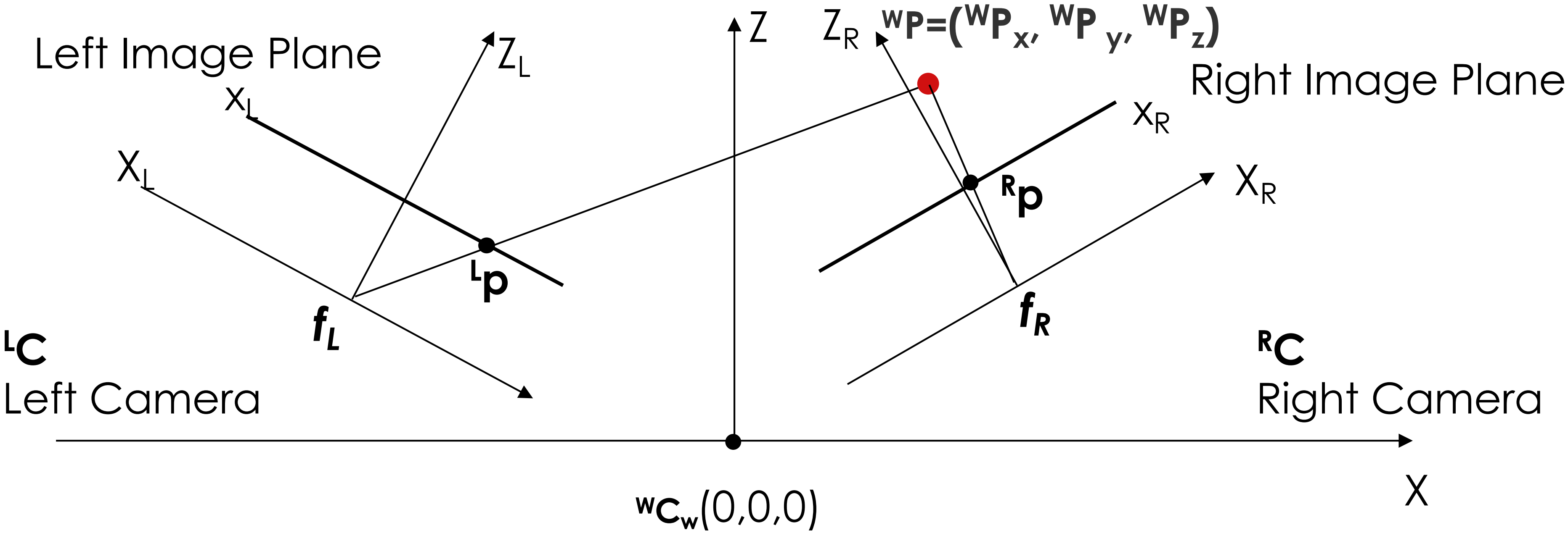


Epipolar geometry

- Epipolar plane given by: P, f_L and f_R
- The intersection between the epipolar plane with the image planes defines the **epipolar lines**
- All epipolar lines pass through a point called **epipole**.
- The projection of each point in the epipolar plane falls onto the epipolar line.
- Epipolar planes are a set of planes that share the line (baseline) connecting f_L and f_R.



Reconstruction from two static images



- Hypothesis: $f_L = f_R = f$
- We obtain:

$$^{R}\mathbf{P} = (^{R}P_x, {}^{R}P_y, {}^{R}P_z), \quad ^{L}\mathbf{P} = (^{L}P_x, {}^{L}P_y, {}^{L}P_z), \quad ^{W}\mathbf{P} = (^{W}P_x, {}^{W}P_y, {}^{W}P_z)$$

$$^{R}\mathbf{P} = {}^{R}\mathbf{R}\,{}^{W}\mathbf{P} + {}^{R}\mathbf{T}, \quad ^{L}\mathbf{P} = {}^{L}\mathbf{R}\,{}^{W}\mathbf{P} + {}^{L}\mathbf{T}$$

- where $^{R}\boldsymbol{R}$, $^{R}\boldsymbol{T}$, $^{L}\boldsymbol{R}$, and $^{L}\boldsymbol{T}$ are the extrinsic parameters of the cameras that indicate the positioning of $^{R}\mathbf{C}$ and $^{L}\mathbf{C}$ with respect to $^{W}\mathbf{C}$

Reconstruction from two static images

- From the previous eq:

$$
\begin{aligned}
{}^{\mathrm{L}}\mathbf{P} &= {}^{\mathrm{L}}\mathbf{R}\,{}^{\mathrm{R}}\mathbf{R}^{-1}\,{}^{R}\mathbf{P} - {}^{\mathrm{L}}\mathbf{R}\,{}^{\mathrm{R}}\mathbf{R}^{-1}\,{}^{R}\mathbf{T} + {}^{\mathrm{L}}\mathbf{T} \\
&= \mathbf{M}\,{}^{R}\mathbf{P} + \mathbf{B}
\end{aligned}
$$

- Using the simplified perspective projections (Z>>f) we obtain the coordinates on the two image planes

$$
\begin{aligned}
{}^{L}p_x &= \frac{f\,{}^{L}P_x}{{}^{L}P_z}, & {}^{L}p_y &= \frac{f\,{}^{L}P_y}{{}^{L}P_z} \\
{}^{R}p_x &= \frac{f\,{}^{R}P_x}{{}^{R}P_z}, & {}^{R}p_y &= \frac{f\,{}^{R}P_y}{{}^{R}P_z}
\end{aligned}
\qquad (1)
$$

- From which we obtain:

$$
\frac{{}^{L}P_z}{f}\begin{bmatrix} {}^{L}p_x \\ {}^{L}p_y \\ f \end{bmatrix} = \frac{{}^{R}P_z}{f}\mathbf{M}\begin{bmatrix} {}^{R}p_x \\ {}^{R}p_y \\ f \end{bmatrix} + \mathbf{B} \qquad (2)
$$

Estimation of the 3D position

- **Stereo Matching Problem** :
  Estimate the coordinates of ${}^{W}P$, given $({}^{L}p_x, {}^{L}p_y)$ and $({}^{R}p_x, {}^{R}p_y)$, and the extrinsic parameters of the cameras.

- **Algorithm**:
  1. From (2) determine the depth of ${}^{W}\mathbf{P}$ with respect to ${}^{R}P_z$ e ${}^{L}P_z$
  2. From (1) calculate the other coordinates for ${}^{L}\mathbf{P}$ e ${}^{R}\mathbf{P}$
  3. Ordinary Least Squares to estimate ${}^{W}\mathbf{P}$ in ${}^{W}\mathbf{C}$

- If ${}^{L}\mathbf{C}$ and ${}^{R}\mathbf{C}$ are parallel and aligned:

$$
{}^{W}P_z = \frac{fb}{{}^{L}p_x - {}^{R}p_x}
$$

Where *b* is the distance between the two cameras

From points to images

- ${}^{L}p_x$-${}^{R}p_x$ is called disparity
- The disparity image is the difference between the left and right image
- To compute the disparity image, need to find a match pixel by pixel
- Issues:
  - Regions can be occluded
  - Regions can be disoccluded
  - Uniform areas do not allow finding correspondences

From points to images

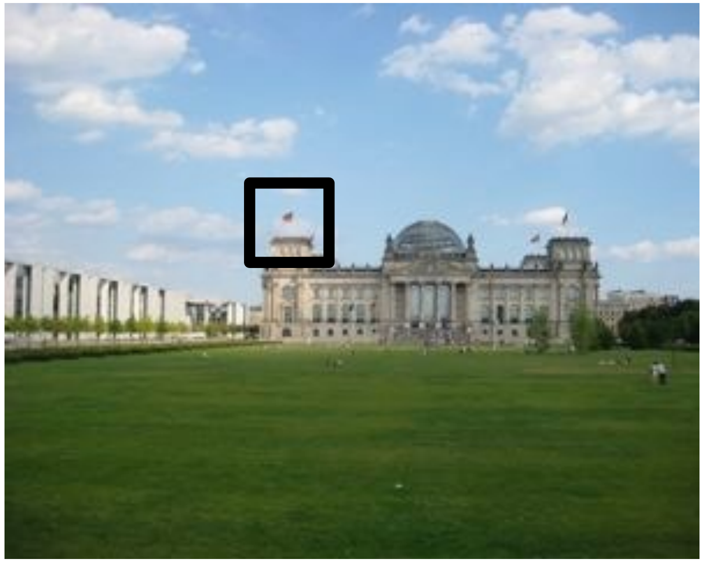
- To find a match, an error function must be evaluated
  - Intensity difference
  - Evaluate edges, windows, segmented known areas

Window-based analysis

- Take a window in the left (right) image
- Along the epipolar line find the windows that best match the right and left image
- Compute an error function (MSE, SAD, SSD)
- Find the minimum
- Winner-take-all → that's the disparity!

Winner-take-all with SAD

- Moving the window of analysis, compute:



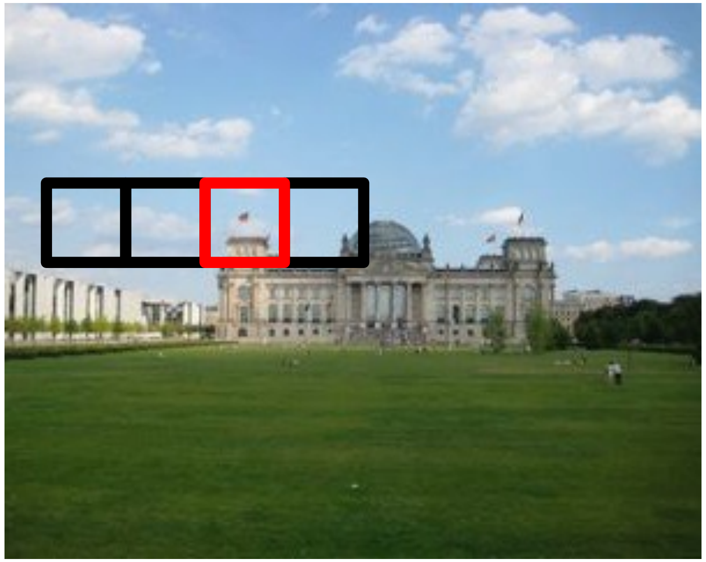


Image Normalization

- Snapshots may be taken under different conditions
- Reflectance of surfaces could be not ideal (not Lambertian)
- It is preferable to work with normalized values
- A window-based analysis helps preventing acquisition issues

$$\bar{I} = \frac{1}{|W(x,y)|} \sum_{(u,v) \in W(x,y)} I(u,v)$$

Compute the average pixel in the window

$$\|I\|_{W(x,y)} = \sqrt{\sum_{(u,v) \in W(x,y)} [I(u,v)]^2}$$

Compute the window magnitude

$$\hat{I}(x,y) = \frac{I(x,y) - \bar{I}}{\|I - \bar{I}\|_{W(x,y)}}$$

Calculate the normalized value

Normalization and SSD

- Normalization makes windows comparable
- Now comparison can be computed, using for example, SAD or the Sum of Squared Differences (SSD)

$$SSD(x,y,d) = \sum_{(u,v)\in W(x,y)} [\hat{I}_L(u,v) - \hat{I}_R(u-d,v)]^2$$

Correlation

- Compare the intensity in a neighborhood
- Take a window around point (u,v)
  - Arrange the values in a vector $w \in \Re^p$ by scanning the window in the first image row by row
  - Same for w' in the second image
  - Evaluate the correlation using

$$C(d) = \frac{1}{|w - \overline{w}|} \frac{1}{|w' - \overline{w}'|} (w - \overline{w})(w' - \overline{w}')$$

- Maximizing the correlation means minimizing the SSD

Page 240 in Forsyth - Ponce

Correlation

- If the window is 3x3, p=9
- It's a comparison between vectors, and the correlation measure is the angle between

$$w - \overline{w} \qquad \text{and} \qquad w' - \overline{w}'$$

$$C(d) = \sum \hat{I}(u,v)\hat{I}'(u-d,v) = w \cdot w' = \cos\theta$$

- In the normalized case, the correlation is maximum if the original brightness of the two windows is shifted by an offset and a scale factor

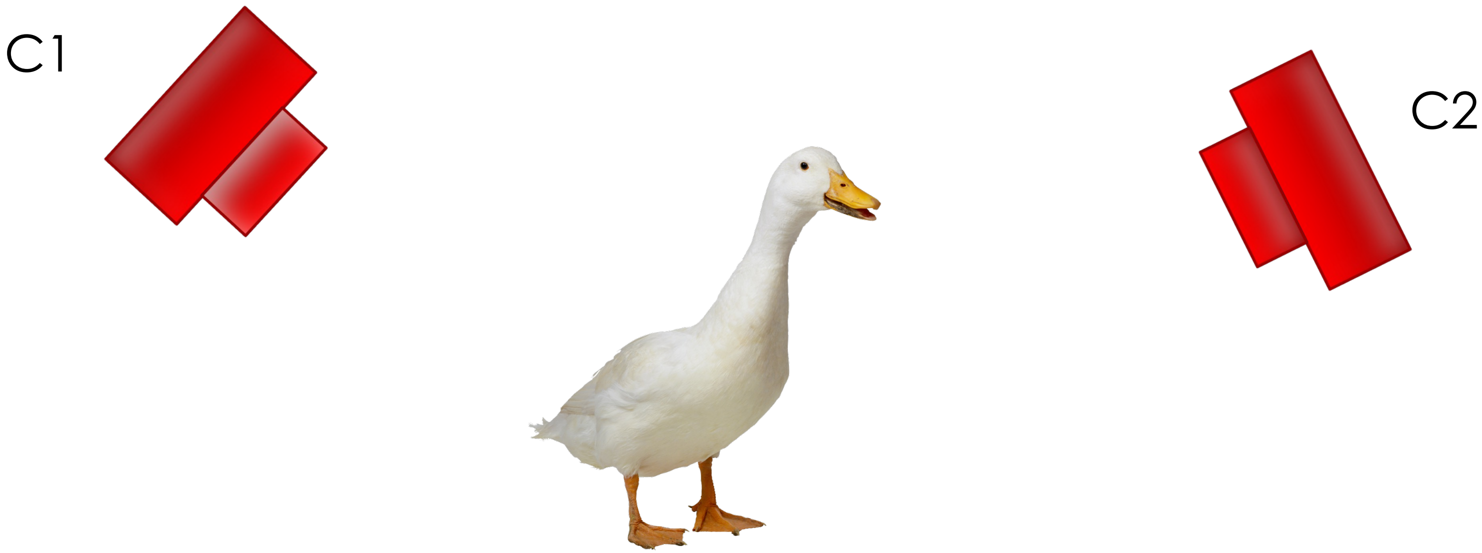
$$I' = \lambda I + \mu$$

Comments

- Computing the correlation at each frame for the whole image can be expensive
- Luckily window shifts overlap
- Keep the information about windows in memory
- Usually the computation is carried out considering a range of the disparity
- In computing the correlation, it is assumed that the scene is parallel to the image plane

General stereo configuration

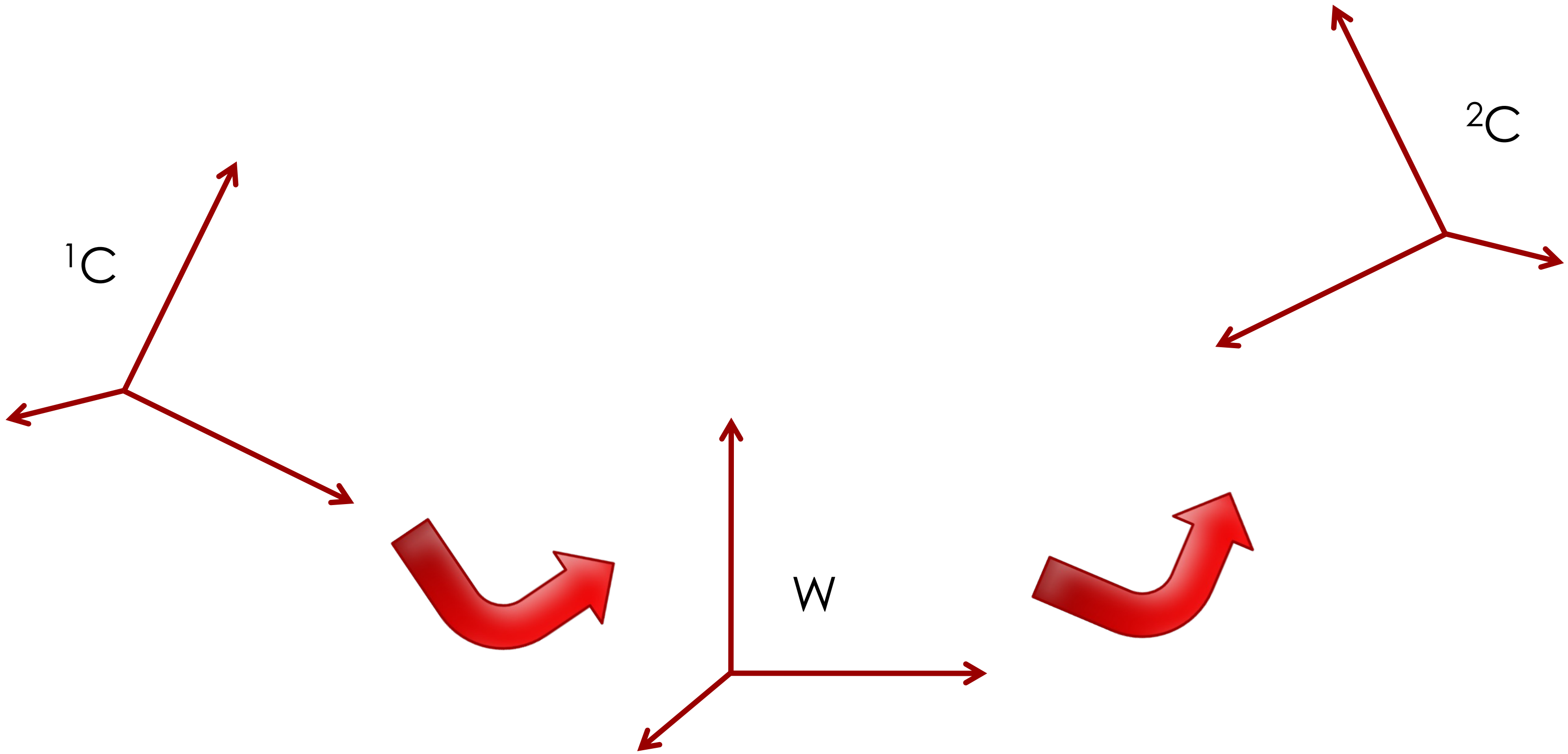
- Stereo rig we have seen:
  - Two cameras
  - Aligned
  - Parallel views
- In general
  - Two cameras with arbitrary positioning in 3D space



What do we need?

- Position of C1 and some internal parameters such as the focal length
  - This information is embedded in the *camera matrix*
  - The camera matrix defines a ray for each point ${}^{w}P$ mapped on ${}^{1}P$
- Same thing for C2
- Correspondences: ${}^{1}P \leftarrow\rightarrow {}^{2}P$ of ${}^{w}P$
- Compute ${}^{w}P$ from ${}^{1}P$ and ${}^{2}P$

The world is the shared information



Fundamental matrix

- Represents the epipolar geometry in case two generic views
- F is a 3x3 matrix that maps p into p'

$$p'^T F p = 0$$

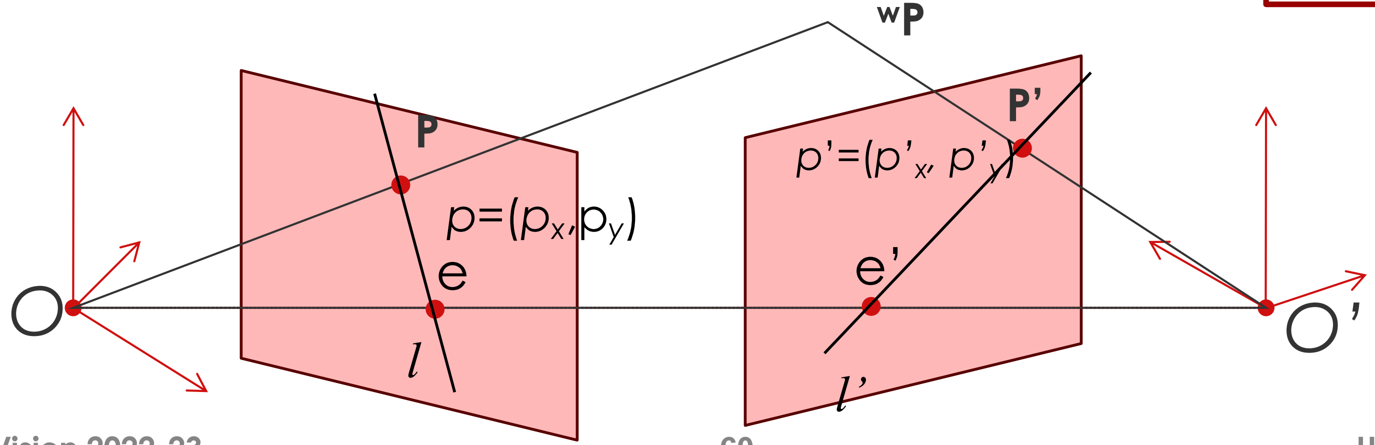
- Independent from the scene structure
- Can be computed using correspondences
- No need to know the intrinsic parameters

Fundamental matrix

- What does the fundamental matrix represent?
- Say we know the intrinsic parameters of the cameras
- The two cameras are shifted by a certain translation vector t and rotated by a matrix R

$Op \cdot [OO' \times O'p'] = 0$ $\quad p \cdot [t \times Rp'] = 0$ $\quad p = (u, v, 1)^T$ $\quad p' = (u', v', 1)^T$ $\quad p^T [t_{\times} R] p' = 0$ $\quad p^T \mathrm{E} p' = 0$

Essential Matrix



Fundamental matrix

- The Essential matrix assumes intrinsic parameters are known
- Intrinsic parameters can be modeled through a matrix as:

$$p = \mathrm{K}\hat{p} \quad p' = \mathrm{K}'\hat{p}' \quad \rightarrow F = K^{-T} E K'^{-1}$$

- We say p and p' <u>correspond</u>, being different projections of the same point
- So, for each point p in one view, there's a corresponding epipolar line l' in the other image
- p' lies on the epipolar line l'

The camera matrix

- Starting from the fundamental matrix it is possible to derive the camera matrices (^1M and ^2M) of the two views
- Conversely to F, camera matrices relate 3D space coordinates with the image
  - Depend on image coordinates
  - Depend on world coordinates
- A change of the world coordinates affects ^1M and ^2M, not F
- This tells us:
  - ^1M and ^2M determine a unique F
  - F determines ^1M and ^2M up to a multiplication by a certain matrix H

Ambiguity

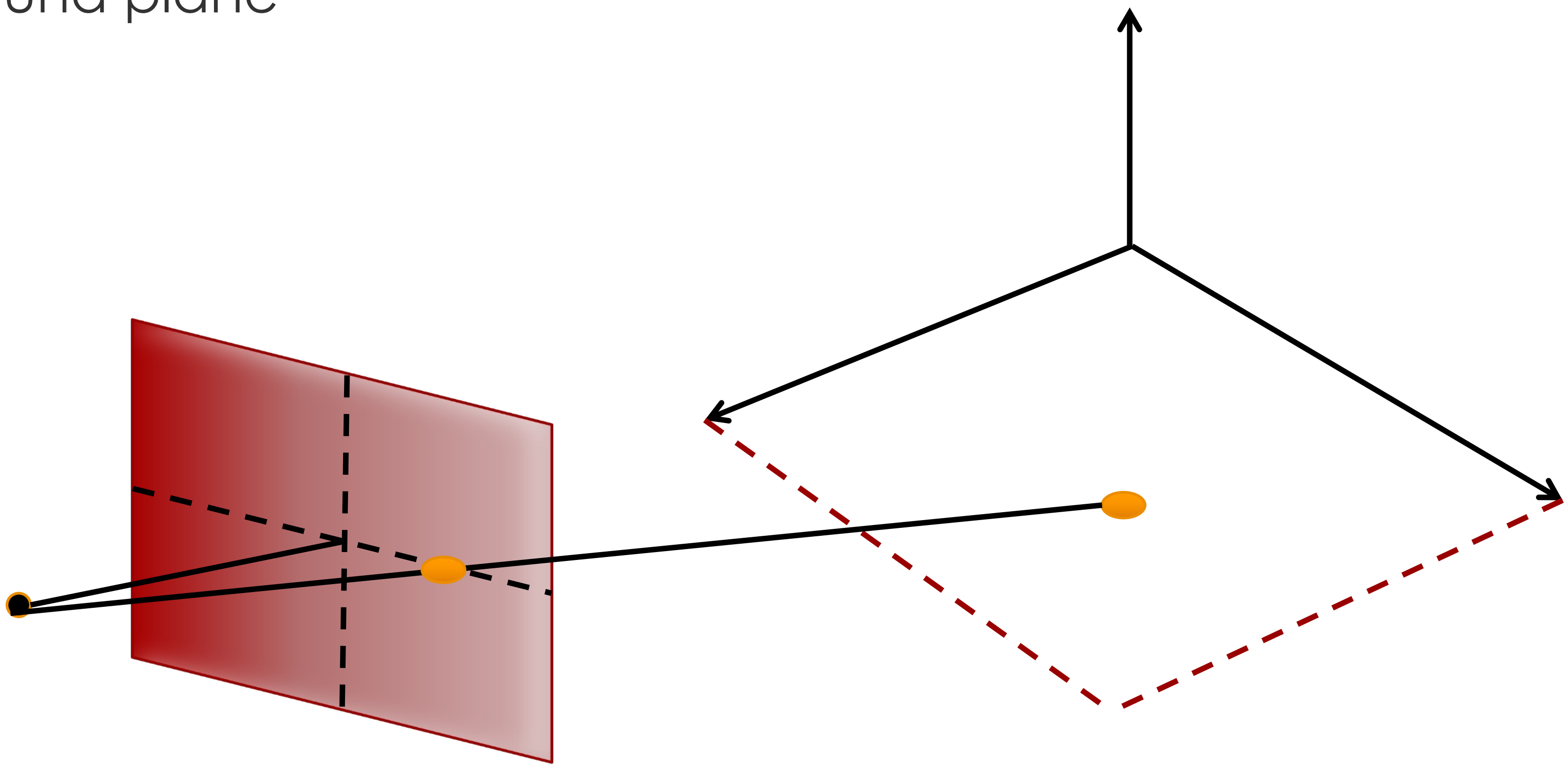
- Given F, for an object, impossible to determine
  - The absolute position (in the world)
  - Orientation (NSWE)
  - Scale
- However, up to a projective transformation, the ambiguity in reconstruction can be solved
- In fact the projected points don't change if:

$$MP = (MH^{-1})(HP)$$

- H is a projective transformation, that does not affect the projection of ${}^{w}P$ onto the image plane

Localization on a Plane: 2D Homography

- In many applications, the mapping between the cameras can be limited to the computation of correspondences using a common 2D ground plane



2D Homography

- It's an invertible transformation between two planes
  - The image plane
  - The ground plane
- The homography matrix is defined as H that satisfies $p' = Hp$

- Where p is the point in the world ground plane and p' is the point in the image plane

- Since any vector crossed with itself gives 0, we can rewrite it as:

$$p' \times (Hp) = 0$$

See Hartley – Zissermann, p.90

Computation of the Matrix

- Given $p'=(x',y',w')^T$ in homogeneous coordinates the cross product becomes

$$p_i' \times Hp_i = \begin{pmatrix} y_i'\overline{h}^{3T}p_i - w_i'\overline{h}^{2T}p_i \\ w_i'\overline{h}^{1T}p_i - x_i'\overline{h}^{3T}p_i \\ x_i'\overline{h}^{2T}p_i - y_i'\overline{h}^{1T}p_i \end{pmatrix}$$

  - This can be rewritten (since $h_j^T p_i = p_i^T h_j$) *as*:

$$\begin{bmatrix} 0^T & -w_i'p_i^T & y_i'p_i^T \\ w_i'p_i^T & 0^T & -x_i'p_i^T \\ -y_i'p_i^T & x_i'p_i^T & 0^T \end{bmatrix}\begin{bmatrix} \overline{h}^1 \\ \overline{h}^2 \\ \overline{h}^3 \end{bmatrix} = 0$$

Computation of the matrix

- Only two of the equations are independent, so the third equation can be discarded since it results as the multiplication of (x' times the first row) + (y' times the second row) up to scale.

$A_i h=0$

$$\begin{bmatrix} 0^T & -w'_i p_i^T & -y'_i p_i^T \\ w'_i p_i^T & 0^T & -x'_i p_i^T \end{bmatrix} \begin{bmatrix} \bar{h}^1 \\ \bar{h}^2 \\ \bar{h}^3 \end{bmatrix} = 0$$

- A_i is a 2x9 matrix
- h^i are the lines of matrix $H=\{h_{11}, \ldots, h_{33}\}$
- For each point we have two equations. For simplicity we can assume w'=1

Solving for H

- Each correspondence gives two independent equations
- We then need 4 corresponding points, in order to get Ah=0, where A is made up by four A_i contributed from each correspondence
- We want to obtain the coefficients of H, where A is 8x9
- H is determined up to scale
- Scale can be arbitrarily chosen inserting a requirement on the norm, such as norm(h)=1

Overdetermined system

- As in calibration, using more points can be useful.
- If > 4, the system is overdetermined
- Due to measurement errors, the solution Ah = 0 is only satisfied by the trivial case of $h_{ij}=0$
- The non-trivial solution is to minimize the norm $\left\| A\bar{h} \right\|$
- Subject to the constraint $\left\| \bar{h} \right\| = 1$
- This corresponds to finding the minimum of $\left\| A\bar{h} \right\| / \left\| h \right\|$
- The solution is found applying the DLT (Direct Linear Transformation)

The inhomogeneous solution

- The matrix can be solved also by imposing the condition $h_j=1$ for some entry of the matrix H, since the matrix is defined up to scale.
- By setting for example $h_{33}=1$ the resulting system becomes:

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w_i' & -y_i w_i' & -w_i w_i' & -x_i y_i' & y_i y_i' \\ x_i w_i' & y_i w_i' & w_i w_i' & 0 & 0 & 0 & -x_i x_i' & -y_i x_i' \end{bmatrix} h = \begin{bmatrix} -w_i y_i' \\ w_i x_i' \end{bmatrix}$$

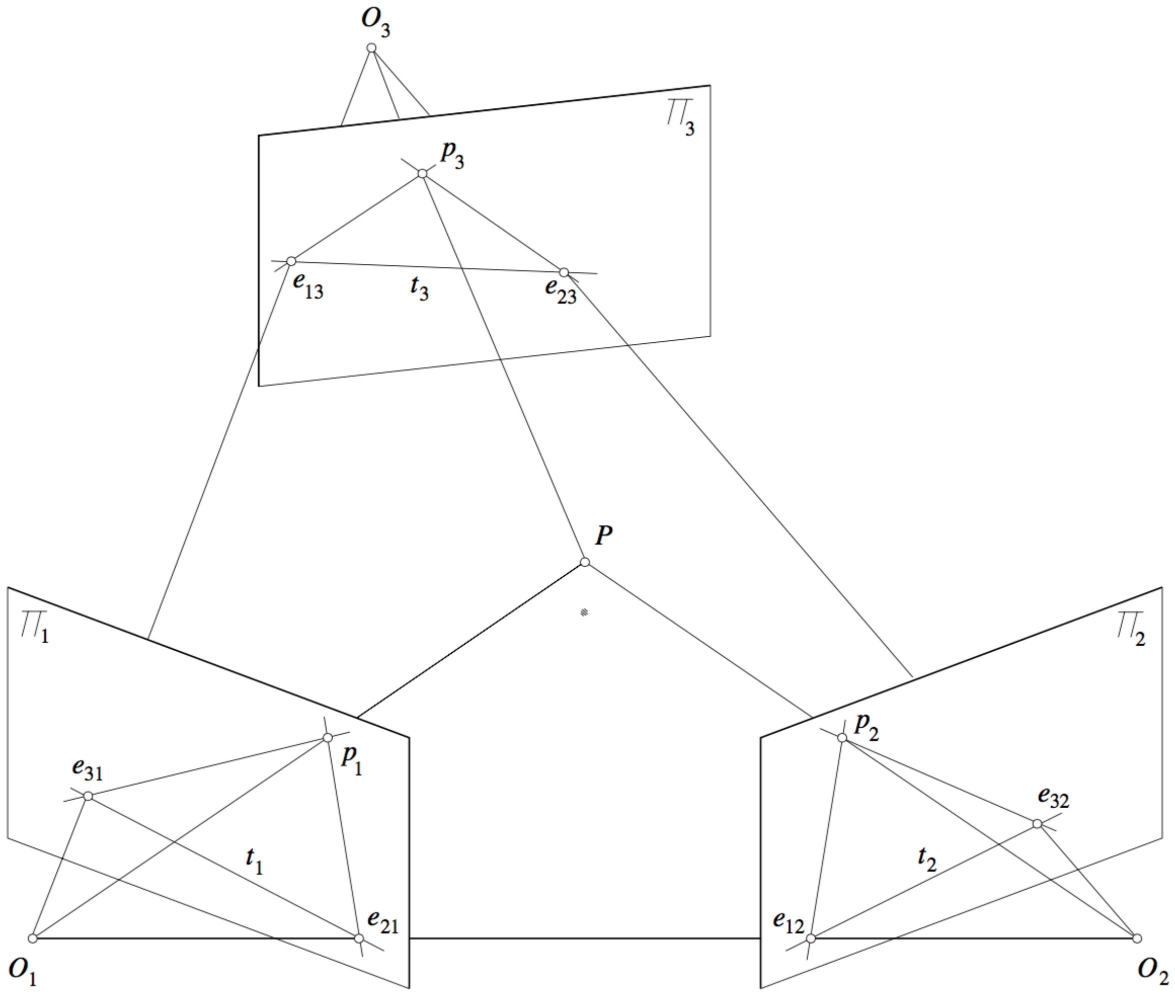
- In this case we obtain 2 equations per point in the form of Mh=b
- h has 8 unknowns, b is a 8-vector
- Standard solution for 4 points or least squares for more points
- Not recommended if h_j are close to zero

Multiple view geometry

- So far two examples:
  - Stereo
  - Generic two-view
- Extension to multiple views is possible (not trivial) for applications like 3D reconstruction
  - Need to acquire the object from multiple perspectives
  - Different views may not overlap significantly
  - Need to register the pictures
  - Instead of pixels, we talk about *voxels*

See Forsyth- Ponce, ch.11

What about three views?



What about three views?!

- Let us assume we know the internal parameters
- Having 3 cameras means having ${}^1p$, ${}^2p$, and ${}^3p$ observing the same point P
- The 3 origins, ${}^1O$, ${}^2O$, and ${}^3O$ intersect forming a trifocal plane that creates three trifocal lines when intersecting the image planes
- Each line passes through the epipoles determined by the two other cameras
- Each pair of cameras defines an epipolar constraint, as:

$$\begin{cases} p_1^T E_{12} p_2 = 0 \\ p_2^T E_{23} p_3 = 0 \\ p_3^T E_{31} p_1 = 0 \end{cases}$$

What about three views?

- The three equations are not independent since

$$e_{31}^T E_{12} e_{32} = e_{12}^T E_{23} e_{13} = e_{23}^T E_{31} e_{21} = 0$$

- In fact, e_{31} and e_{32} are the first and second images of the optical center ^3O of the third camera, and are therefore in epipolar correspondence

- Any of the two eq are independent, which means that it is possible to compute the position of ^1p given the position of ^2p and ^3p, if the essential matrices are known.

- Using eq 1 and 3 from the previous slide, we have a system in the unknown coordinates of ^1p

Problem of Transfer

- To understand the position of a point in W we only need two projections
- Knowing the essential/fundamental matrices it is possible to predict the position of a point in a picture given the other two (or more) projections