Forward Starting Option in the B&S framework

Analytical and Numerical solutions

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Outline

Black and Scholes market

Black and Scholes market

General Framework

- 1 risky asset: S(t)
- 1 riskless asset: B(t)
- 1 T-derivative with payoff: $\chi = \phi(S(T))$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$
$$dB(t) = rB(t)dt$$

where W(t) is a Wiener process.

Defining $f(S(t)) = ln(S(t))$, and applying Ito's formula to $f(.)$, we are able to find the closed form solution:

$$S(T) = S(t)e^{\,(\mu - \frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))} \tag{1}$$

Pricing a European Call option

Kolmogorov equation and Feynman-Kac formula

- European call option: χ = (S(T)-K)$^+$;
- $\Pi(t)$ is the price of the call option over time;

Assuming there exists a generic smooth function F(t, S(t)) = $\Pi(t)$ that is $\mathscr{F}_t - \textit{measurable}$, then the price of the derivative over time is:

- **PDE approach:**

$$\frac{\partial F(t,S(t))}{\partial t} + r\frac{\partial F(t,S(t))}{\partial S(t)}S(t) + \frac{1}{2}\frac{\partial^2 F(t,S(t))}{\partial S(t)^2}\sigma^2 S(t)^2 - rF(t,S(t)) = 0$$
$$F(T,S(T)) = (S(T)-K)^+$$

- **Martingale approach:**

$$F(t,S(t)) = e^{-r(T-t)}E^Q[(S(T)-K)^+|\mathscr{F}_t] \tag{2}$$

Analytical solution

Using the Girsanov theorem, we are able to retrieve the dynamic of S(t) under Q:

$$dS(t) = rS(t)dt + \sigma S(t)d\bar{W}(t) \tag{3}$$

Where $\bar{W}$ is a Q-Wiener process.

Price of a European call option in the B&S market

We define $K = \kappa S(t)$ where $\kappa \in (0,1)$, then the analytical solution is:

$$C_E(t,s) = S(t)N(d_1(t)) - e^{-r(T-t)}\kappa S(t)N(d_2(t))$$
$$d_{1,2} = \frac{ln\frac{1}{\kappa} + (r \pm \frac{1}{2}\sigma)(T-t)}{\sigma\sqrt{T-t}}$$

European call option

Pricing a Forward Starting Option

Analytical solution - Part 1

Let $t^* > t$ and therefore $\mathscr{F}_t \subset \mathscr{F}_{t^*}$. The risk neutral price of a forward starting option is:

$$C_{FS}(t, S(t)) = e^{-r(T-t)} E^Q \left[(S(T) - \kappa S(t^*))^+ | \mathscr{F}_t \right] \tag{4}$$

Exploiting the **tower property of conditional expectation** we can rewrite the expectation in (4) as:

$$E^Q \left[(S(T) - \kappa S(t^*))^+ | \mathscr{F}_t \right] = E^Q \left[E^Q \left[(S(T) - \kappa S(t^*))^+ | \mathscr{F}_{t^*} \right] | \mathscr{F}_t \right] \tag{5}$$

Then, the price of the forward starting option can be computed as follow:

$$\begin{aligned}
C_{FS}(t, S(t)) &= e^{-r(T-t^*)} e^{-r(t^*-t)} E^Q \left[E^Q \left[(S(T) - \kappa S(t^*))^+ | \mathscr{F}_{t^*} \right] | \mathscr{F}_t \right] \\
&= e^{-r(t^*-t)} E^Q \left[e^{-r(T-t^*)} E^Q \left[(S(T) - \kappa S(t^*))^+ | \mathscr{F}_{t^*} \right] | \mathscr{F}_t \right] \\
&= e^{-r(t^*-t)} E^Q \left[C_E(t^*, S(t^*)) | \mathscr{F}_t \right] \\
&= e^{-r(t^*-t)} \left[E^Q \left[S(t^*) | \mathscr{F}_t \right] N(d_1(t^*)) - e^{-r(T-t^*)} \kappa E^Q \left[S(t^*) | \mathscr{F}_t \right] N(d_2(t^*)) \right]
\end{aligned}$$

Analytical solution - Part 2

Since we know the Q-dynamic of S(t), then:

$$E^Q\left[S(t^*)|\mathscr{F}_t\right] = S(t)e^{r(t^*-t)} \tag{6}$$

Price of a Forward Starting Option in the B&S market

$$\begin{aligned} C_{FS}(t,S(t)) &= e^{-r(t^*-t)}\left[S(t)e^{r(t^*-t)}N(d_1(t^*)) - e^{-r(T-t^*)}\kappa S(t)e^{r(t^*-t)}N(d_2(t^*))\right] \\ &= S(t)N(d_1(t^*)) - e^{-r(T-t^*)}\kappa S(t)N(d_2(t^*)) \end{aligned}$$

Where:

$$d_{1,2} = \frac{ln\frac{1}{\kappa} + (r \pm \frac{1}{2}\sigma)(T-t^*)}{\sigma\sqrt{T-t^*}} \tag{7}$$

Comparing numerical and analytical solutions

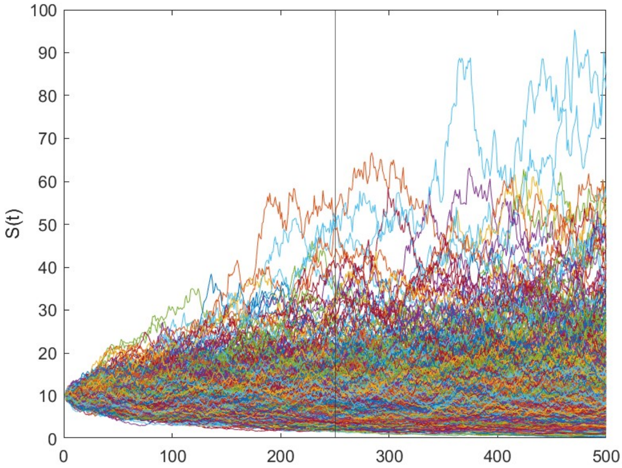
Parametrization

Parameters

- $\kappa = 0.2$;
- $\mathrm{S}(0) = \mathrm{s} = 10$;
- $\sigma = 0.8$;
- $\mathrm{t} = 0$;
- $t^* = 0.5$;
- $\mathrm{T} = 1$;
- $\mathrm{r} = 0.01$;
- $\Delta\mathrm{t} = 0.002$ (discretization steps);

The exact solution of the GBM (under Q) is:

$$\hat{S}(t + \Delta t) = \hat{S}(t) e^{\,(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} Z} \quad Z \sim N(0, 1) \tag{8}$$


S(t)
100
90
80
70
60
50
40
30
20
10
0
0
100
200
300
400
500

Montecarlo approximation of European Call Option

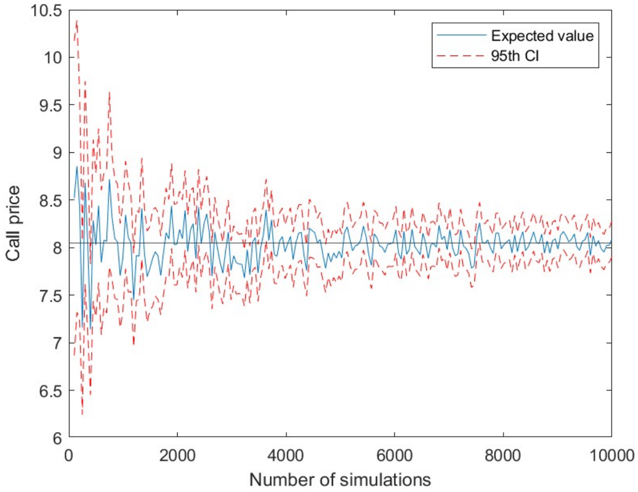
- **Closed form**:

$$C_E(0,s) = sN(d_1(0)) - e^{-rT}\kappa sN(d_2(0))$$
$$d_{1,2} = \frac{ln\frac{1}{\kappa} + (r \pm \frac{1}{2}\sigma)T}{\sigma\sqrt{T}}$$

- **Numerical approximation** (pseudo-code):

$$\begin{aligned} C_E(0,s) &= e^{-rT}E^Q[(S(T)-K)^+|\mathscr{F}_t] \\ &= e^{-rT}[mean[max[S(\hat{T}) - \kappa s, 0]]] \end{aligned}$$

Montecarlo Properties


Expected value
95th CI
Call price
Number of simulations
10.5
10
9.5
9
8.5
8
7.5
7
6.5
6
0
2000
4000
6000
8000
10000

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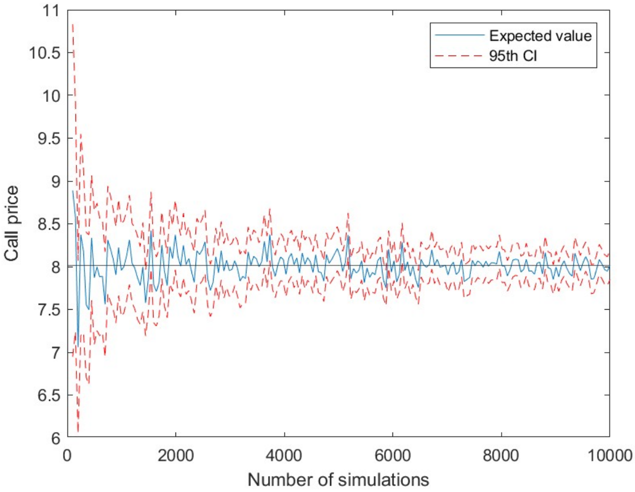
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Appendix

Price of a European call option in the B&S market

Price of a European call option in the B&S market

$$C_E(t, S(t)) = S(t)N(d_1(t, S(t))) - e^{-r(T-t)}KN(d_2(t, S(t)))$$

$$d_{1,2} = \frac{ln\frac{S(t)}{K} + (r \pm \frac{1}{2}\sigma)(T-t)}{\sigma\sqrt{T-t}}$$

▸▸ Analytical solution

Montecarlo simulation

Theorem 1: Montecarlo integration (LLN)

Let $S(T)^{(s)}$ for s = 1, ..., S be a random sample from any distribution with finite mean and variance, and define:

$$\hat{g}_s = \frac{1}{S}\sum_{s=1}^{S} max[S(T)^{(s)} - K, 0] \tag{9}$$

Then $\hat{g}_s \xrightarrow{p} \mathrm{E}[g_s]$ as $S \to \infty$.

Theorem 2: Numerical standard error (CLT)

Using the same setup as in theorem 1:

$$\sqrt{S}\{\hat{g}_s - E[g_s]\} \to N(0, \sigma_g^2) \tag{10}$$

as S $\to \infty$, where $\sigma_g^2 = Var[g_s]$.

MC simulation 1 MC simulation 2