Replicating Cooley & Hansen's model: "The inflation tax in a real business cycle model"

Alfero Mattia

July 9, 2023

1 Introduction

The model presented in these report is Cooley and Hansen (1989). It's a simple one-sector stochastic optimal growth model with a real economy identical to that studied by Gary Hansen (1985), but money is introduced into the model using a cash-in-advance constraint.
In this model households consume a consumption good, work in firms, invest in physical capital and hold cash. Firms produce the consumption good using capital and labour.
The main objective of the paper is to show how an economy evolves in environments with different level and volatility of prices.

2 The model

Firms' production function is Cobb-Douglass:

$$Y_t = z_t K_t^{\theta} H_t^{1-\theta} \tag{1}$$

where Y_t is the output, K_t and H_t are the aggregate capital and labour, respectively. The technology (z_t) follows a law of motion given by $ln(z_{t+1}) = \gamma ln(z_t) + \epsilon_{t+1}^z$ with mean $ln(\bar{z})(1-\gamma)$ and variance σ_ϵ^2.

The representative household uses previously acquired money balances to purchase the consumption good. That is, a household's consumption choice must satisfy the constraint:

$$p_t c_t = m_{t-1} + (g_t - 1)M_{t-1} \tag{2}$$

where p_t is the price level, c_t the consumption, m_t the household's money balance and M_t is the per capita money supply (lump-sum transfer), evolving according to $M_t = g_t M_{t-1}$, everything at time t. Money growth (g_t) follows a law of motion given by $ln(g_{t+1}) = \alpha ln(z_t) + \epsilon_{t+1}^g$ with mean $ln(\bar{g})(1-\alpha)$ and variance σ_ϵ^2.

As in Hansen (1985), labor is assumed to be indivisible. Thus, a fraction π_t of the households will work h_0 hours and the remaining $(1-\pi_t)$ households will be unemployed during period t. A lottery determines which of the households work and which do not. Thus, hours worked in period t are given by:

$$h_t = \pi_t h_0 \tag{3}$$

The representative household maximizes $u(c_t, l_t) = ln(c_t) + Aln(l_t)$ under the cash in advance constraint on consumption purchase (2) and the following budget constraint:

$$c_t + i_t + \frac{m_t}{p_t} = w_t h_t + r_t k_t + \frac{m_{t-1} + (g_t - 1)M_{t-1}}{p_t} \tag{4}$$

The representative household must choose among consuming c_t, investing i_t and holding cash $\frac{m_t}{p_t}$, using the income from labor $w_t h_t$ and capital $r_t k_t$, the real value of money from the previous period $\frac{m_{t-1}}{p_t}$, and the lump-sum money transfer $\frac{(g_t-1)M_{t-1}}{p_t}$

A change in variables is introduced so that the problem solved by the households will be stationary: $\tilde{m}_t = \frac{m_t}{M_t}$ and $\tilde{p}_t = \frac{p_t}{M_t}$. Moreover, the capital k_t evolves according to $k_{t+1} = (1-\delta)k_t + i_t$.
Thus, we can rewrite the (2) and (4) as:

$$\tilde{p}_t c_t = \frac{\tilde{m_{t-1}} + (g_t - 1)}{g_t} \tag{5}$$

$$c_t + k_{t+1} + \frac{\tilde{m}_t}{p_t} = w_t h_t + r_t k_t + (1-\delta)k_t + \frac{\tilde{m_{t-1}} + (g_t - 1)}{\tilde{p}_t g_t} \tag{6}$$

2.1 Households

The representative household solves the following optimization problem:

$$\begin{aligned}
&\max_{c_t, k_{t+1}, h_t, \tilde{m}_t} E_0 \sum_{t=0}^{\infty} \beta^t \left(ln(c_t) + \left[A \frac{ln(1-h_0)}{h_0} \right] h_t \right) \\
&\text{subject to} \\
&\tilde{p}_t c_t = \frac{\tilde{m_{t-1}} + (g_t - 1)}{g_t} \\
&k_{t+1} + \frac{\tilde{m}_t}{\tilde{p}_t} = r_t k_t + (1-\delta)k_t + w_t h_t
\end{aligned} \tag{7}$$

Notice that we rewrote the utility function in terms consumption c_t and of hours worked h_t, in the following way:

$$\begin{aligned}
u(c_t, l_t) &= ln(c_t) + \pi_t A ln(1-h_0) + (1-\pi_t) A ln(1) \\
&= ln(c_t) + A \frac{ln(1-h_0)}{h_t} h_t
\end{aligned} \tag{8}$$

The Lagrangian function is:

$$\begin{aligned}
L = \max_{c_t, k_{t+1}, h_t, \tilde{m}_t} E_0 \sum_{t=0}^{\infty} \beta^t \left[ln(c_t) + B h_t \right] + \lambda_t^1 \left(\frac{\tilde{m_{t-1}} + (g_t - 1)}{g_t} - \tilde{p}_t c_t \right) \\
+ \lambda_t^2 \left(r_t k_t + (1-\delta)k_t + w_t h_t - k_{t+1} - \frac{\tilde{m}_t}{\tilde{p}_t} \right)
\end{aligned} \tag{9}$$

Where $B = A \frac{ln(1-h_0)}{h_0}$, and the **F.O.C.s** are:

$$
\begin{aligned}
&\frac{\partial L}{\partial c_t} = \beta^t \frac{1}{c_t} - \lambda_t^1 \tilde{p_t} = 0 \\
&\frac{\partial L}{\partial k_{t+1}} = \lambda_{t+1}^2 (r_{t+1} + 1 - \delta) - \lambda_t^2 = 0 \\
&\frac{\partial L}{\partial h_t} = \beta^t B + \lambda t^2 w_t = 0 \\
&\frac{\partial L}{\partial \tilde{m_t}} = \lambda_{t+1}^1 \frac{1}{g_{t+1}} - \lambda_t^2 \frac{1}{\tilde{p_t}} = 0 \\
&\frac{\partial L}{\partial \lambda_t^1} = 0; \quad \tilde{p_t} c_t = \frac{\tilde{m_{t-1}} + (g_t - 1)}{g_t} \\
&\frac{\partial L}{\partial \lambda_t^2} = 0; \quad k_{t+1} + \frac{\tilde{m_t}}{\tilde{p_t}} = r_t k_t + (1-\delta) k_t + w_t h_t \\
&\lim_{t\to\infty} E_t[\lambda_t^1 k_{t+1}] = 0 \\
&\lim_{t\to\infty} E_t[\lambda_t^2 k_{t+1}] = 0
\end{aligned}
\tag{10}
$$

From which we get the equilibrium conditions:

$$
\begin{aligned}
&\beta \frac{w_t}{w_{t+1}} (r_{t+1} + 1 - \delta) = 1 \\
&\beta \frac{1}{g_{t+1} c_{t+1} \tilde{p_{t+1}}} = -\frac{B}{w_t \tilde{p_t}} \\
&\tilde{p_t} c_t = \frac{\tilde{m_{t-1}} + (g_t - 1)}{g_t} \\
&k_{t+1} + \frac{\tilde{m_t}}{\tilde{p_t}} = r_t k_t + (1-\delta) k_t + w_t h_t
\end{aligned}
\tag{11}
$$

Where in the system, we removed the transversality condition to the money supply (g_t).

2.2 Firms

The representative firm maximizes the profits, solving the following optimization problem:

$$
\max_{K_t, H_t} \ \Pi_t = z_t K_t^{\theta} H_t^{1-\theta} - w_t H_t - r_t K_t \tag{12}
$$

From the **F.O.C.s**, we retrieve the equilibrium conditions:

$$
\begin{aligned}
w_t &= (1-\theta) z_t \left(\frac{K_t}{H_t}\right)^{\theta} \\
r_t &= (\theta) z_t \left(\frac{K_t}{H_t}\right)^{\theta - 1}
\end{aligned}
\tag{13}
$$

Where in the system, we removed the transversality condition to the technology shock (z_t). Capital letters are used to distinguish per capita variables that a competitive household takes as parametric from individual-specific variables that are chosen by the household.

2.3 Market clearing

The clearing in the good market implies $y_t = c_t + i_t$, but since we have only one market, we don't need to introduce any additional equation in the system. The aggregation conditions for an equilibrium are: $K_t = k_t$, $H_t = h_t$, $C_t = c_t$ and $\tilde{M_t} = \tilde{m_t} = 1$.

3 Equilibrium

The equilibrium conditions of the model are the following:

$$\begin{aligned}
&\beta \frac{w_t}{w_{t+1}}(r_{t+1} + 1 - \delta) = 1 \\
&\beta \frac{1}{g_{t+1}c_{t+1}\tilde{p_{t+1}}} = -\frac{B}{w_t \tilde{p_t}} \\
&\tilde{p_t} c_t = \frac{(g_t - 1)}{g_t} \\
&k_{t+1} + \frac{1}{\tilde{p_t}} = r_t k_t + (1-\delta)k_t + w_t h_t \\
&w_t = (1-\theta) z_t \left(\frac{k_t}{h_t}\right)^{\theta} \\
&r_t = (\theta) z_t \left(\frac{k_t}{h_t}\right)^{\theta - 1} \\
&y_t = z_t K_t^{\theta} H_t^{1-\theta} \\
&i_t = y_t - c_t \\
&ln(z_{t+1}) = \gamma ln(z_t) + \epsilon_{t+1}^{z} \\
&ln(g_{t+1}) = \alpha ln(g_t) + \epsilon_{t+1}^{z}
\end{aligned} \tag{14}$$

Notice that we have **3 pre-determined state variables**, k_t,z_t, g_t, **7 jumping variables**, c_t, h_t, p_t, w_t, r_t, y_t, i_t and 10 equations.

4 Steady State

The steady state of the system is obtained when all variables are constant over time:

$$
\begin{aligned}
&\frac{1}{\beta} = (1-\delta) + \bar{r} \rightarrow \bar{r} = \frac{1}{\beta} - (1-\delta) \\
&\frac{\beta}{\bar{g}\bar{c}} = -\frac{B}{\bar{\bar{w}}} \rightarrow \bar{c} = -\frac{\beta\bar{w}}{\bar{g}B} \\
&\bar{p}\bar{c} = 1 \rightarrow \bar{p} = \frac{1}{\bar{c}} \\
&\frac{1}{\bar{p}} = (\bar{r}-\delta)\bar{k} + \bar{w}\bar{h} \rightarrow \bar{c} = (\bar{r}-\delta)\bar{k} + (1-\theta)\left(\frac{\bar{r}}{\theta}\right)^{\frac{\theta}{\theta-1}}\left(\frac{\bar{r}}{\theta}\right)^{\frac{1}{1-\theta}}\bar{k} \rightarrow \bar{k} = \frac{\bar{c}}{\frac{\bar{r}}{\theta}-\delta} \\
&\bar{w} = (1-\theta)\left(\frac{\bar{k}}{\bar{h}}\right)^{\theta} \rightarrow \bar{w} = (1-\theta)\left(\frac{\bar{r}}{\theta}\right)^{\frac{\theta}{\theta-1}} \\
&\frac{\bar{r}}{\theta} = \left(\frac{\bar{k}}{\bar{h}}\right)^{\theta-1} \rightarrow \bar{h} = \left(\frac{\bar{r}}{\theta}\right)^{\frac{1}{1-\theta}}\bar{k} \\
&\bar{y} = \bar{k}^{\theta}\bar{h}^{1-\theta} \\
&\bar{i} = \bar{y} - \bar{c} \\
&log(\bar{z}) = 0 \rightarrow \bar{z} = 1
\end{aligned}
\tag{15}
$$

While we use different values for $\bar{g}$.

5 Log-Linearization

Now we can proceed to log-linearize the equilibrium conditions obtained in (14): (where $\hat{m}_t = 0$ because $\tilde{m}_t = 1$, and therefore, $\log(\tilde{m}_t) = \log(\bar{m}) + \hat{m}_t$, and we obtain: $0 = \hat{m}_t$).

$$
\begin{aligned}
&(1) \\
&log(\beta) + log(w_t) - log(w_{t+1}) + log(r_{t+1} + 1 - \delta) = log(1) \\
&log(\beta) + log(\bar{w}) + \hat{w}_t - log(\bar{w}) - \hat{w_{t+1}} + log(\bar{r} + 1 - \delta) + \frac{1}{\bar{r}+1-\delta}(r_{t+1} - \bar{r}) = 0 \\
&log(\beta) + \hat{w}_t - \hat{w_{t+1}} + log\left(\frac{1}{\beta}\right) + \beta\bar{r}\hat{r_{t+1}} \\
&\hat{w_{t+1}} - \beta\bar{r}\hat{r_{t+1}} = \hat{w}_t
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&(2) \\
&log(\frac{\beta}{\bar{g}\bar{c}\bar{p}}) - \hat{g_{t+1}} - \hat{c_{t+1}} - \hat{p_{t+1}} = log(\frac{-B}{\bar{w}\bar{p}}) - \hat{w_t}) - \hat{p}_t \\
&\hat{p_{t+1}} + \hat{g_{t+1}} + \hat{c_{t+1}} = \hat{w}_t + \hat{p}_t
\end{aligned}
$$

(3)

$$ln(p_t) + ln(c_t) = ln(\tilde{m_{t-1}} + g_t - 1) - ln(g_t)$$

$$ln(\bar{p}\bar{c}) + \hat{p_t} + \hat{c_t} = ln\left(\frac{\bar{m} + \bar{g} - 1}{\bar{g}}\right) + \frac{1}{\bar{m} + \bar{g} - 1}\left((\tilde{m_{t-1}} - \bar{m}) + (g_t - \bar{g})\right) - \hat{g_t}$$

$$\hat{p_t} + \hat{c_t} = \frac{1}{\bar{g}}\bar{m}\hat{m_{t-1}} + \hat{g_t} - \hat{g_t}$$

$$0 = \hat{p_t} + \hat{c_t}$$

(4)

$$\frac{1}{\bar{k} + \frac{\bar{m}}{\bar{p}}}\left((k_{t+1} - \bar{k}) + \frac{1}{\bar{p}}(\tilde{m_t} - \bar{m}) - \frac{\bar{m}}{\bar{p}^{-2}}(\tilde{p_t} - \bar{p})\right) = \frac{1}{(1-\delta+\bar{r})\bar{k} + \bar{w}\bar{h}}((1-\delta+\bar{r})(k_t - \bar{k}) + \bar{k}(r_t - \bar{r})$$

$$+ \bar{w}(h_t - \bar{h}) + \bar{h}(w_t - \bar{w}))$$

$$\bar{k}\hat{k_{t+1}} + \frac{\bar{m}}{\bar{p}}(\hat{m_t} - \hat{p_t}) = (1-\delta+\bar{r})\bar{k}\hat{k_t} + \bar{k}\bar{r}\hat{r_t} + \bar{w}\bar{h}\hat{h_t} + \bar{h}\bar{w}\hat{w_t}$$

$$\bar{k}\hat{k_{t+1}} = \frac{1}{\bar{p}}\hat{p_t} + (1-\delta+\bar{r})\bar{k}\hat{k_t} + \bar{k}\bar{r}\hat{r_t} + \bar{w}\bar{h}\hat{h_t} + \bar{w}\bar{h}\hat{w_t}$$

(5)

$$ln(\bar{w}) + \hat{w_t} = ln((1-\theta)\bar{z}) + \hat{z_t} + ln\left(\frac{\bar{k}}{\bar{h}}\right)^{\theta} + \theta(\hat{k_t} - \hat{h_t})$$

$$0 = -\hat{w_t} + \hat{z_t} + \theta(\hat{k_t} - \hat{h_t})$$

(6)

$$0 = -\hat{r_t} = \hat{z_t} + (\theta - 1)(k_t - \hat{h_t})$$

(7)

$$ln(y_t) = ln(z_t k_t^{\theta} h_t^{(1-\theta)})$$

$$\hat{y_t} = \hat{z_t} + \theta\hat{k_t} + (1-\theta)\hat{h_t}$$

(8)

$$ln(i_t) = ln(y_t - c_t)$$

$$\hat{i_t} = \left(\frac{\bar{y}}{\bar{y} - \bar{c}}\right)(\hat{y_t} - \hat{c_t})$$

(9)

$$\hat{z_{t+1}} = \gamma\hat{z_t} + \epsilon^z_{t+1}$$

$$\hat{z_{t+1}} = \gamma\hat{z_t}$$

(10)

$$\hat{g_{t+1}} = \alpha\hat{g_t} + \epsilon^g_{t+1}$$

$$\hat{g_{t+1}} = \alpha\hat{g_t}$$

(17)

Notice that the log-linearized equation in (7)-(8) are not included in the state space, but they are used to compute the value of the output and investment.

6 State space

Now we rewrite the log-linearized equations as a system, focusing on the state variables (k_t, z_t, g_t), and control variables $(w_t, r_t, p_t, c_t, h_t)$:

$$\begin{aligned}
\hat{w_{t+1}} - \beta\bar{r}\hat{r_{t+1}} &= \hat{w}_t \\
\hat{p_{t+1}} + \hat{g_{t+1}} + \hat{c_{t+1}} &= \hat{w}_t + \hat{p}_t \\
0 &= \hat{p}_t + \hat{c}_t \\
\bar{k}\hat{k_{t+1}} &= \frac{1}{\bar{p}}\hat{p}_t + (1-\delta+\bar{r})\bar{k}\hat{k}_t + \bar{k}\bar{r}\hat{r}_t + \bar{w}\bar{h}\hat{h}_t + \bar{w}\bar{h}\hat{w}_t \\
0 &= -\hat{w}_t + \hat{z}_t + \theta(\hat{k}_t - \hat{h}_t) \\
0 &= -\hat{r}_t = \hat{z}_t + (\theta-1)(k_t - \hat{h}_t) \\
\hat{z_{t+1}} &= \gamma\hat{z}_t \\
\hat{g_{t+1}} &= \alpha\hat{g}_t
\end{aligned} \tag{18}$$

Where in matrix form:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & -\beta\bar{r} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{k_{t+1}} \\ \hat{z_{t+1}} \\ \hat{g_{t+1}} \\ \hat{w_{t+1}} \\ \hat{r_{t+1}} \\ \hat{p_{t+1}} \\ \hat{c_{t+1}} \\ \hat{h_{t+1}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ (1-\delta+\bar{r})\bar{k} & 0 & 0 & \bar{w}\bar{h} & \bar{r}\bar{k} & \frac{1}{\bar{p}} & 0 & \bar{w}\bar{h} \\ \theta & 1 & 0 & -1 & 0 & 0 & 0 & -\theta \\ \theta-1 & 1 & 0 & 0 & -1 & 0 & 0 & -(\theta-1) \\ 0 & \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \\ \hat{g}_t \\ \hat{w}_t \\ \hat{r}_t \\ \hat{p}_t \\ \hat{c}_t \\ \hat{h}_t \end{bmatrix} \tag{19}$$

Which in the compact form is: (x_t contains the state variables, while y_t the jumping variables).

$$A \begin{bmatrix} E_t x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = B \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{20}$$

Since the key Blanchard-Khan condition is not satisfied (invertibility of A) I used the approach based by Klein(2000) that uses the **generalized Schur decomposition** to solve the model.
The code on this same repository shows the construction of the state space, the computation of the policy and transition functions, and the simulation of the different economies. The last table on the jupyter notebook tries to reproduce the results of the paper, which are in line apart for the variable "price", which can be due to the different solution methods used. However, the implications are the same, that is, higher volatility and lower correlation with output in an economy with unpredictable money growth (or inflation).