EPFL

MATH-451: Numerical Approximation of PDEs

Black-Scholes Equation: European Put

Author:

Mattia Barbiere (387974)

Professor:

Annalisa Buffa

Spring Semester - 2025

Problem 0.1 (Black & Scholes Equation for European Put Option). *Consider the Black & Scholes equation for the value $u(S,t)$ of an European Put option:*

$$\begin{cases} \partial_t u - \frac{\sigma^2}{2} S^2 \partial_{SS} u - rS\partial_S u + ru = 0 & \text{in } \Omega \times (0,T], \\ u(S,0) = u_0(S) = \max\{K-S, 0\}, \end{cases}$$

where $\Omega = (S_{\min}, S_{\max})$, σ and r are strictly positive and bounded constants, together with the following boundary conditions:

$$\partial_S u(S_{\min}, t) = u(S_{\max}, t) = 0.$$

Definition 0.1 (Weighted Sobolev Space). *We introduce the weighted Sobolev space V:*

$$V = \left\{ v : v \in L^2(\Omega), S\frac{\partial v}{\partial S} \in L^2(\Omega), v(S_{\max}) = 0 \right\}.$$

Endowed with the inner product and norm:

$$(v,w)_V = \int_\Omega \left(v(S)w(S) + S^2 \frac{\partial v}{\partial S}(S) \frac{\partial w}{\partial S}(S) \right) dS, \quad \|v\|_V = (v,v)_V^{1/2}.$$

The seminorm

$$|v|_V^2 = \int_\Omega \left(S\frac{\partial v}{\partial S} \right)^2 dS$$

is a norm in the Hilbert space V and we have the following Poincaré inequality:

$$\|v\|_{L^2(\Omega)} \le 2|v|_V, \quad \forall v \in V.$$

1 Question A

Definition 1.1. *For $u \in C^0((0,T]; L^2(\Omega))$ and $v \in V$ define the following bilinear form*

$$a(u,v) := \frac{\sigma^2}{2} \int_\Omega S^2 \, \partial_S v \; \partial_S u \; dS + (\sigma^2 - r) \int_\Omega Sv \, \partial_S u \; dS + \int_\Omega ruv \, dS.$$

For any $u^* \in V$ we also have $u^*(\cdot, t) = u^* \in C^0((0,T]; L^2(\Omega))$ and thus with a slight abuse of notation also $a(u^*, v)$ is well defined.

Proposition 1.1. *The variational form of the problem defined in Problem 0.1 is*

$$\begin{aligned} &(\, \partial_t u \,, v) + a(u,v) = \\ &= \int_\Omega v \, \partial_t u \; dS + \frac{\sigma^2}{2} \int_\Omega S^2 \, \partial_S v \; \partial_S u \; dS + (\sigma^2 - r) \int_\Omega Sv \, \partial_S u \; dS + \int_\Omega ruv \, dS = 0 \end{aligned}$$

Proof. Take $v \in V$. Starting from Problem 0.1 we multiply by the test function v and integrate over Ω,

$$\int_\Omega v \, \partial_t u \; dS - \int_\Omega \frac{\sigma^2}{2} S^2 v \, \partial_{SS} u \; dS - \int_\Omega rSv \, \partial_S u \; dS + \int_\Omega ruv \, dS = 0.$$

We now apply Theorem A.1 to the second term resulting in

$$\int_\Omega v \, \partial_t u \; dS - \frac{\sigma^2}{2} \left[\left[S^2 v \, \partial_S u \right] \big|_{\partial\Omega} - \int_\Omega \partial_S (S^2 v) \; \partial_S u \; dS \right] - \int_\Omega rSv \, \partial_S u \; dS + \int_\Omega ruv \, dS = 0.$$

On the boundary

$$(S_{min})^2 v(S_{min})\,\partial_S u\,(S_{min},t) = 0 \text{ because of the boundary condition } \partial_S u\,(S_{min},t) = 0$$

$$(S_{max})^2 v(S_{max})\,\partial_S u\,(S_{max},t) = 0 \text{ because of the boundary condition } v(S_{max}) = 0$$

allowing us to eliminate the boundary term. Next using the product rule of the differential gives

$$\int_\Omega v\,\partial_t u\;dS + \frac{\sigma^2}{2}\int_\Omega \left[2Sv\,\partial_S u\, + S^2\,\partial_S v\;\partial_S u\right]\,dS - \int_\Omega rSv\,\partial_S u\;dS + \int_\Omega ruv\,dS = 0.$$

Next using the inner product $(\,\partial_t u\,,v) = \int_\Omega v\,\partial_t u\;dS$ and Definition 1.1 we can rewrite this as

$$\begin{aligned}&(\,\partial_t u\,,v) + a(u,v) =\\ &= \int_\Omega v\,\partial_t u\;dS + \frac{\sigma^2}{2}\int_\Omega S^2\,\partial_S v\;\partial_S u\;dS + (\sigma^2 - r)\int_\Omega Sv\,\partial_S u\;dS + \int_\Omega ruv\,dS = 0\end{aligned}$$

□

Proposition 1.2. *For any $t \in (0,T]$ and any $v \in V$ we have that*

$$a(v,v) \geq \frac{\sigma^2}{4}|v|_V - \alpha\|v\|^2_{L^2(\Omega)}$$

where $\alpha > 0$ is defined as

$$\alpha = \begin{cases}\frac{(\sigma^2-r)^2}{\sigma^2}, & \text{if} \quad r \neq \sigma^2\\ \varepsilon, & \text{if} \quad r = \sigma^2\end{cases}$$

with ε being any positive real number. In particular, $\varepsilon > 0$ can be chosen arbitrarily small.

Proof. Recalling Definition 1.1 we have that

$$a(v,v) = \frac{\sigma^2}{2}\int_\Omega S^2\,\partial_S v\;\partial_S v\;dS + (\sigma^2 - r)\int_\Omega Sv\,\partial_S v\;dS + \int_\Omega rvv\,dS.$$

By definition of $\|\cdot\|^2_{L^2(\Omega)}$ and $|\cdot|^2_V$ we have that

$$a(v,v) = \frac{\sigma^2}{2}|v|^2_V + (\sigma^2 - r)\int_\Omega Sv\,\partial_S v\;dS + r\|v\|^2_{L^2(\Omega)}.$$

Using properties of the absolute value we can bound this by below

$$a(v,v) \geq \frac{\sigma^2}{2}|v|^2_V - \left|(\sigma^2 - r)\int_\Omega Sv\,\partial_S v\;dS\right| + r\|v\|^2_{L^2(\Omega)} \geq \frac{\sigma^2}{2}|v|^2_V - \left|(\sigma^2 - r)\int_\Omega Sv\,\partial_S v\;dS\right|. \tag{1}$$

Let us consider two cases. If $r = \sigma^2$ then for any choice of $\varepsilon > 0$ we must have

$$a(v,v) \geq \frac{\sigma^2}{2}|v|^2_V \geq \frac{\sigma^2}{4}|v|^2_V \geq \frac{\sigma^2}{4}|v|^2_V - \varepsilon\|v\|^2_{L^2(\Omega)}. \tag{2}$$

We can thus choose $\alpha = \varepsilon$ to get the desired inequality.

Conversely, if $r \neq \sigma^2$ then looking at the second term from Equation (1) we have

$$\begin{aligned}
-\left|(\sigma^2 - r)\int_\Omega Sv\,\partial_S v\; dS\right| &\geq -\left|(\sigma^2 - r)\right| \int_\Omega |Sv\,\partial_S v\,|\; dS && \text{by the triangle inequality}\\
&\geq -\left|(\sigma^2 - r)\right| \|S\,\partial_S v\,\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} && \text{by Cauchy–Schwarz inequality}\\
&\geq -\left|(\sigma^2 - r)\right| \left(\frac{\|S\,\partial_S v\,\|^2_{L^2(\Omega)}}{2\delta} + \frac{\delta\|v\|^2_{L^2(\Omega)}}{2}\right) && \text{by Young's inequality, } \delta > 0\\
&\geq -\frac{\sigma^2}{4}|v|^2_V - \frac{(\sigma^2 - r)^2\|v\|^2_{L^2(\Omega)}}{\sigma^2}
\end{aligned}$$

where the last inequality came from choosing $\delta = 2|(\sigma^2 - r)|/\sigma^2 > 0$. Plugging this back into Equation (1) give us the following lower bound for $a(v, v)$

$$a(v, v) \geq \frac{\sigma^2}{4}|v|^2_V - \frac{(\sigma^2 - r)^2}{\sigma^2}\|v\|^2_{L^2(\Omega)}$$

By choosing $\alpha = (\sigma^2 - r)^2/\sigma^2 > 0$ we are left with the requested inequality. $\square$

2 Question B

Before answering this question observe a useful lemma.

Lemma 2.1. *For any integrable functions f, g we have that*

$$\int_\Omega (f - g)f\, dS = \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} - \|g\|^2_{L^2(\Omega)} + \|f - g\|^2_{L^2(\Omega)}\right).$$

Moreover,

$$\int_\Omega (f - g)f\, dS \geq \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} - \|g\|^2_{L^2(\Omega)}\right)$$

Proof. Starting from the right hand side we have that

$$\begin{aligned}
&\frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} - \|g\|^2_{L^2(\Omega)} + \|f - g\|^2_{L^2(\Omega)}\right) =\\
&= \frac{1}{2}\left(\int_\Omega f^2\, dS - \int_\Omega g^2\, dS + \int_\Omega f^2\, dS + \int_\Omega g^2\, dS - 2\int_\Omega fg\, dS\right) =\\
&= \int_\Omega f^2\, dS - \int_\Omega fg\, dS = \int_\Omega (f - g)f\, dS.
\end{aligned}$$

The second part is immediate after noticing that $\|f - g\|^2_{L^2(\Omega)} \geq 0$. $\square$

Lemma 2.2. *A semi-discretization of the problem defined in Problem 0.1 is the following*

$$\begin{aligned}
&\int_\Omega \left[\frac{u^j - u^{j-1}}{\Delta t} v\right] dS + \\
&+ \frac{\sigma^2}{2}\int_\Omega S^2 \left(\partial_S\left(\theta u^j + (1-\theta)u^{j-1}\right)\right) \partial_S v \; dS + \\
&+ (\sigma^2 - r)\int_\Omega S v\, \partial_S \left(\theta u^j + (1-\theta)u^{j-1}\right) \; dS + \\
&+ r \int_\Omega \left(\theta u^j + (1-\theta)u^{j-1}\right) v \, dS = \\
&= 0.
\end{aligned}$$

Proof. The result is immediate after recalling the variational form of the problem as shown in Proposition 1.1, using finite difference for approximating the derivative and replacing u with $\theta u^j + (1-\theta)u^{j-1}$. □

With the goal of applying the Lax-Milgram lemma Lemma A.2 let us make the following observations.

Definition 2.1. *For $v, w \in V$ and $\Delta t \in \mathbb{R}$, define the bilinear form $b(v,w)$ as*

$$b(v,w) := \int_\Omega v\, w\, dS + \Delta t\, a(v,w)$$

Lemma 2.3. *Assume $0 < \Delta t \leq \frac{1}{2\alpha}$ where α is defined in Proposition 1.2. Then for any $v, w \in V$ the bilinear form $b(\cdot,\cdot)$ defined in Definition 2.1 satisfies*

$$\begin{aligned}
&b(v,w) \leq M\|v\|_V \|w\|_V && \textit{(Continuity)}\\
&b(v,v) \geq \beta \|v\|_V^2 && \textit{(Coercivity)}
\end{aligned}$$

for some $M, \beta > 0$.

Proof. By definition of $b(v,w)$ we have that

$$\begin{aligned}
b(v,w) &= \int_\Omega v\, w\, dS + \Delta t\, a(v,w)\\
&= \int_\Omega v\, w\, dS + \Delta t \left[\frac{\sigma^2}{2}\int_\Omega S^2\, \partial_S w\; \partial_S v \; dS + (\sigma^2 - r)\int_\Omega S w\, \partial_S v \; dS + \int_\Omega r v w \, dS\right].
\end{aligned}$$

Using Hölder's inequality multiple times and the definition of the seminorm, this can all be bounded by

$$\begin{aligned}
b(v,w) &\leq \|v\|^2_{L^2(\Omega)}\|w\|^2_{L^2(\Omega)} + \Delta t \left[\frac{\sigma^2}{2}|v|_V^2 |w|_V^2 + (\sigma^2 - r)|v|_V^2 \|w\|^2_{L^2(\Omega)} + r\|v\|^2_{L^2(\Omega)}\|w\|^2_{L^2(\Omega)}\right]\\
&\leq (1 + \Delta t\, r)\|v\|^2_{L^2(\Omega)}\|w\|^2_{L^2(\Omega)} + \Delta t \left[\frac{\sigma^2}{2}|v|_V^2 |w|_V^2 + \left|\sigma^2 - r\right| |v|_V^2 \|w\|^2_{L^2(\Omega)}\right].
\end{aligned}$$

As $v, w \in V$, by the definition of the space V we have that $|v|_V^2, |w|_V^2 < +\infty$ and $\|v\|^2_{L^2(\Omega)}, \|w\|^2_{L^2(\Omega)} < +\infty$ thus, as the above quantity is finite, there must exist a $M > 0$ such that

$$b(v,w) \leq M\|v\|_V \|w\|_V.$$

For coercivity $b(v,v)$ can be bounded by below by

$$b(v,v) = \|v\|^2_{L^2(\Omega)} + \Delta t\, a(v,v) \geq \|v\|^2_{L^2(\Omega)} + \Delta t \left(\frac{\sigma^2}{4}|v|^2_V - \alpha\|v\|^2_{L^2(\Omega)} \right) = (1-\Delta t\,\alpha)\,\|v\|^2_{L^2(\Omega)} + \Delta t\,\frac{\sigma^2}{4}|v|^2_V$$

where the second inequality comes Proposition 1.2. By assumption we have that $\Delta t \leq \frac{1}{2\alpha}$ and hence $(1-\Delta t\,\alpha) \geq \frac{1}{2}$ giving

$$b(v,v) \geq \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \Delta t\,\frac{\sigma^2}{4}|v|^2_V \geq \min\left(\frac{1}{2}, \Delta t\,\frac{\sigma^2}{4}\right)\left(\|v\|^2_{L^2(\Omega)} + |v|^2_V\right) = \beta\|v\|^2_V$$

by taking $\beta = \min\left(\frac{1}{2}, \Delta t\,\frac{\sigma^2}{4}\right) > 0$. □

Proposition 2.4. *The semi-discrete problem in Lemma 2.2 with $\theta = 1$ (i.e implicit Euler) is well posed for any time step $0 < \Delta t \leq \frac{1}{2\alpha}$.*

Proof. Using $\theta = 1$ in the result of Lemma 2.2 leaves us with

$$\int_\Omega \left[\frac{u^j - u^{j-1}}{\Delta t} v\right] dS + \frac{\sigma^2}{2}\int_\Omega S^2\,\partial_S\left(u^j\right)\ \partial_S v\ dS + (\sigma^2 - r)\int_\Omega Sv\,\partial_S\left(u^j\right)\ dS + r\int_\Omega u^j v\,dS = 0.$$

Using Definition 1.1 we can simplify this as

$$\int_\Omega \left[\frac{u^j - u^{j-1}}{\Delta t} v\right]\ dS + a(u^j, v) = 0.$$

Moving terms around

$$\int_\Omega u^j v\,dS + \Delta t\,a(u^j,v) = \int_\Omega u^{j-1} v\,dS.$$

We now perform induction over j to prove that the problem is well posed.
Base case: $j = 1$. In this case

$$\int_\Omega u^1 v\,dS + \Delta t\,a(u^1,v) = \int_\Omega u^0 v\,dS.$$

and as u^0 is known this is equivalent to

$$b(u^1, v) = F(v) \tag{3}$$

by Definition 2.1 and $F(v) = \int_\Omega u^0 v\,dS$ which is bounded and continuous. Invoking the Lax-Milgram lemma (see Lemma A.2) and Lemma 2.3 we conclude that there exists a unique u^1 solving Equation (3).
Induction case: For a general value of j, assume that for $j-1$ the problem is well posed. Then this implies that u^{j-1} exists and is unique. For this reason we can write the problem as

$$b(u^j, v) = \int_\Omega u^j v\,dS + \Delta t\,a(u^j,v) = \int_\Omega u^{j-1} v\,dS = F(v)$$

as u^{j-1} is fixed. The conclusion comes once again by the Lax-Milgram lemma Lemma A.2 and Lemma 2.3, proving that the problem is well posed for all j. □

Proposition 2.5. *Given the semi-discrete problem in Lemma 2.2 with $\theta = 1$ (i.e implicit Euler), for any time step $0 < \Delta t \leq \frac{1}{2\alpha}$ the following inequality holds*

$$(1-2\alpha\Delta t)^N\|u^N\|^2_{L^2} + \frac{\Delta t}{2}\sigma^2\sum_{j=1}^{N-1}(1-2\alpha\Delta t)^{j-1}|u^j|^2_V \leq \|u^0\|^2_{L^2}.$$

Proof. Using $\theta = 1$ and $v = u^j$ in the result of Lemma 2.2 leaves us with

$$\int_\Omega \left[\frac{u^j - u^{j-1}}{\Delta t} u^j\right] dS + \frac{\sigma^2}{2}\int_\Omega S^2\, \partial_S\left(u^j\right)\ \partial_S u^j\ dS + (\sigma^2 - r)\int_\Omega S u^j\, \partial_S\left(u^j\right)\ dS + r\int_\Omega (u^j)^2\, dS = 0.$$

which can be rewritten as

$$\int_\Omega \left[\frac{u^j - u^{j-1}}{\Delta t} u^j\right] dS + a(u^j, u^j) = 0. \tag{4}$$

We start by multiplying Equation (4) by $\Delta t > 0$

$$\int_\Omega \left[(u^j - u^{j-1})u^j\right]\ dS + \Delta t\, a(u^j, u^j) = 0.$$

Next we use the inequality in Lemma 2.1 using $f = u^j$ and $g = u^{j-1}$

$$\frac{1}{2}\left(\|u^j\|^2_{L^2(\Omega)} - \|u^{j-1}\|^2_{L^2(\Omega)}\right) + \Delta t\, a(u^j, u^j) \leq 0.$$

After multiplying by 2 we can invoke Proposition 1.2 and we are left with

$$\|u^j\|^2_{L^2(\Omega)} - \|u^{j-1}\|^2_{L^2(\Omega)} + 2\Delta t\left(\frac{\sigma^2}{4}|u^j|^2_V - \alpha\|u^j\|^2_{L^2(\Omega)}\right) \leq 0.$$

After moving some terms around this can be rewritten as

$$(1 - 2\alpha\Delta t)\|u^j\|^2_{L^2(\Omega)} - \|u^{j-1}\|^2_{L^2(\Omega)} + \Delta t\,\frac{\sigma^2}{2}|u^j|^2_V \leq 0.$$

We multiply everything by $(1 - 2\alpha\Delta t)^{j-1}$

$$(1 - 2\alpha\Delta t)^j\|u^j\|^2_{L^2(\Omega)} - (1 - 2\alpha\Delta t)^{j-1}\|u^{j-1}\|^2_{L^2(\Omega)} + \Delta t\,\frac{\sigma^2}{2}(1 - 2\alpha\Delta t)^{j-1}|u^j|^2_V \leq 0.$$

As this inequality is valid for all $j \in \{1, \ldots, N\}$ we can take the sum over j and we observe that the first two terms form a telescoping sum

$$\sum_{j=1}^N \left[(1 - 2\alpha\Delta t)^j\|u^j\|^2_{L^2(\Omega)} - (1 - 2\alpha\Delta t)^{j-1}\|u^{j-1}\|^2_{L^2(\Omega)}\right] + \Delta t\,\frac{\sigma^2}{2}\sum_{j=1}^N (1 - 2\alpha\Delta t)^{j-1}|u^j|^2_V$$

$$= (1 - 2\alpha\Delta t)^N\|u^N\|^2_{L^2(\Omega)} - \|u^0\|^2_{L^2(\Omega)} + \Delta t\,\frac{\sigma^2}{2}\sum_{j=1}^N (1 - 2\alpha\Delta t)^{j-1}|u^j|^2_V \leq 0.$$

We conclude by moving $\|u^0\|^2_{L^2(\Omega)}$ to the right hand side. □

3 Question C

This section the finite element method is employed to solve the Black-Scholes equation. In Section 3.1 a constructed solution with a non-zero right-hand side is solved while Section 3.2 solves the PDE explained in Problem 0.1.

3.1 Constructed solution

For the constructed problem the parameters are taken to be: $S_{min} = 3, S_{max} = 10, r = 0.04, \sigma = 0.2$, and $T = 1$. Let us construct a solution by adapting the right-hand side and the initial conditions. The constructed solution is chosen as

$$u(S,t) = [\cos(\phi(S)) - 1]\, e^{-\sin(t)} \quad \text{where} \quad \phi(S) = 0.4(S-3)(S-10). \tag{5}$$

A visualization of this function is given in Figure 1 for some values of $t \in [0,1]$. This function is the analytical solution to the following problem.

Problem 3.1 (Constructed Black & Scholes Equation)**.** *Take $u(S,t)$ and ϕ as given in Equation* (5)*. Let $\Omega = (3,10)$, $\sigma = 0.2$ and $r = 0.04$. The function $u(S,t)$ for $S \in \Omega$ and $t \in [0,1]$ satisfies the Black & Scholes equation:*

$$\begin{cases} \partial_t u - \frac{\sigma^2}{2} S^2 \partial_{SS} u - rS\partial_S u + ru = f(S,t) & \text{in } \Omega \times (0,T], \\ u(S,0) = u_0(S) = \cos(\phi(S)) - 1, \end{cases}$$

where right-hand side $f(S,t)$ is given by

$$\begin{aligned} f(S,t) &= u(S,t)\,(-\cos(t)) + \\ &+ e^{-\sin(t)} \left\{ -\frac{1}{2}\sigma^2 S^2 \left[-0.16(2S-13)^2 \cos(\phi(S)) - 0.8\sin(\phi(S)) \right] - rS\left[-0.4(2S-13)\sin(\phi(S)) \right] \right\} + \\ &+ ru(S,t). \end{aligned} \tag{6}$$

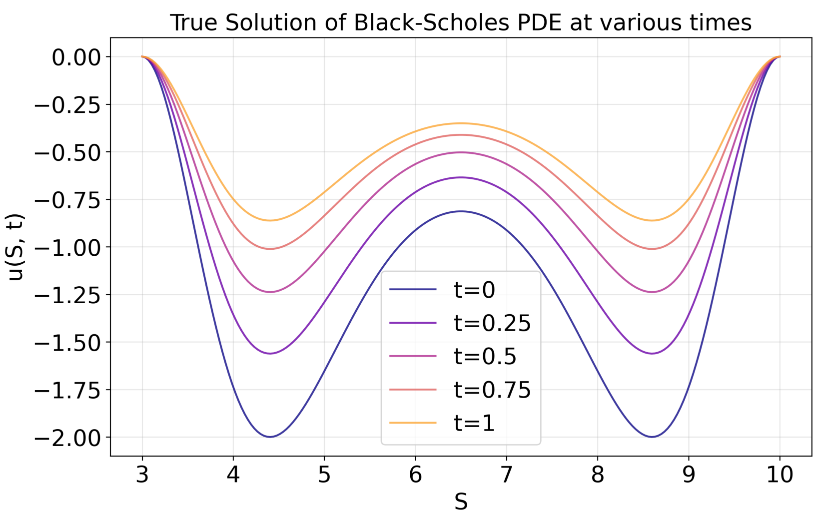

Figure 1: Analytical solution $u(S,t)$ given by Equation (5) as a function of S for values of $t \in \{0, 0.25, 0.5, 0.75, 1\}$.

The right-hand side and initial conditions were constructed starting from $u(S,t)$, however the boundary conditions given by Problem 0.1 were imposed *a priori* thus, to make sure Problem 3.1 is valid, we need to verify that $u(S,t)$ agrees with the given boundary conditions.

Lemma 3.1. *The solution $u(S,t)$ proposed in Equation* (5) *agrees with the boundary conditions stated in Problem 0.1.*

Proof. By constructed we have that $\phi(3) = \phi(10) = 0$. Thus $u(10,t) = [\cos(0) - 1]\, e^{-\sin(t)} = 0$. Furthermore, the derivative with respect to S of $u(S,t)$ evaluated at $S = 3$ is

$$\partial_S u\,(3,t) = [-0.4(6-13)\sin(\phi(3))]\, e^{-\sin(t)} = 2.8\sin(0)e^{-\sin(t)} = 0$$

which proves that $u(S,t)$ satisfies the boundary conditions. $\square$

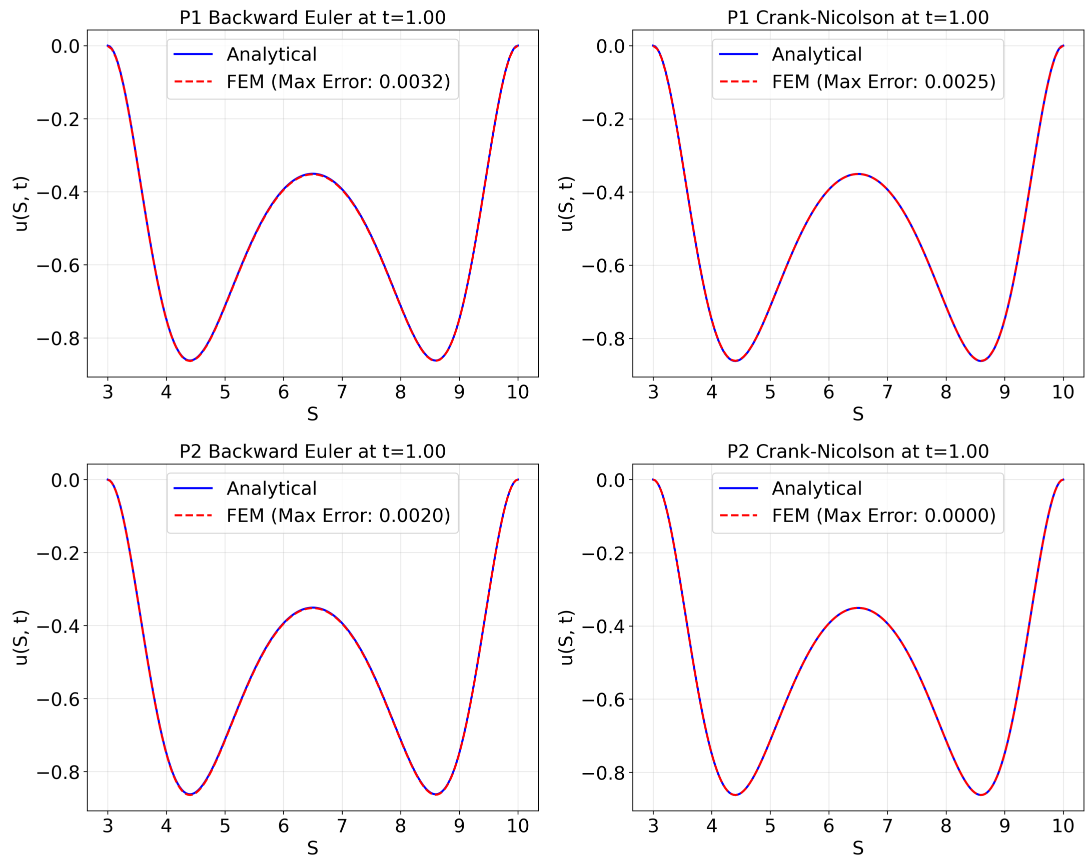

Figure 2: Analytical solution (blue) versus interpolated finite element solution (dashed red) for various methods at time $t = 1$ when tasked to solve Problem 3.1. The PDE was solved with $h = 0.07$ and $\Delta t = 3.33 \cdot 10^{-3}$ using the Gauss-Legendre quadrature with 10 points. First and second row correspond to P1 and P2 finite elements respectively. First and second column correspond to backwards Euler and Crank-Nicolson's method respectively. In the legend the maximum errors is presented for all four plots.

As a initial visual check Figure 2 illustrates the analytical versus the (interpolated) solution given by the finite element solver. A visual inspection shows that all methods provide a reasonable approximation of the solution. There is, nonetheless, some visual deviation for all methods except P2 Crank-Nicolson which provides the best approximation.

Remark 3.1. *To produce Figure 2 the solution points given by the finite element solver where interpolated using the* `scipy.interpolate.interp1d` *function. Nonetheless, all errors throughout the report are computed without interpolation to avoid any interpolation errors contaminating the true errors of the methods.*

3.1.1 Convergence study of the constructed solution

The final experiment on the artificial solution was to perform a convergence study. For the results to be meaningful, it is important that the time discretization error does not dominate. To ensure this, the convergence study was performed using element size h and time interval Δt as shown in Table 1. Performing the experiment gives Figure 3. The plot clearly illustrates that both Crank-Nicolson and backwards Euler have a convergence order of 2 when using P1 finite elements and order 3 when working with P2 finite elements.

	Backwards Euler	Crank-Nicolson
P1	$\Delta t \approx h^2$	$\Delta t \approx h$
P2	$\Delta t \approx h^3$	$\Delta t \approx h^{3/2}$

Table 1: Chosen orders of Δt as a function of h for the four methods. The order of Δt was chosen in such a way that the time discretization error does not dominate when performing the convergence study. The orders of Δt were inspired from [1] and [2], and also validated experimentally.

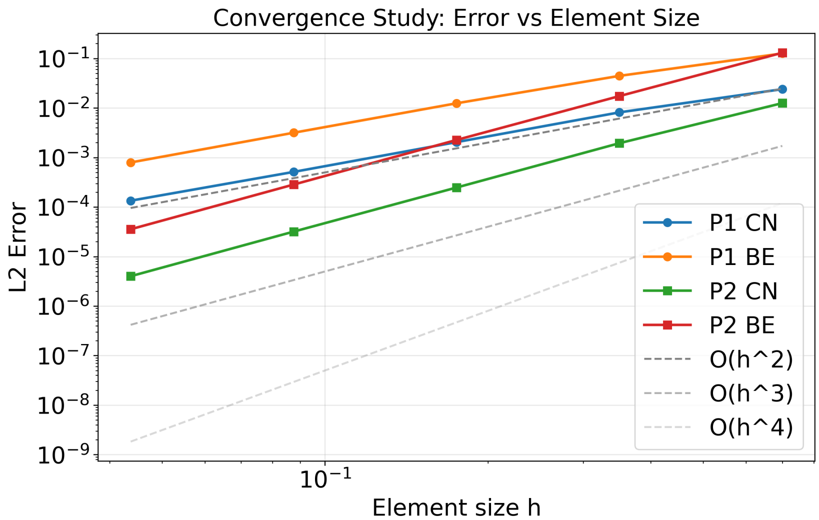

Figure 3: Convergence study for all four methods when solving Problem 3.1. The Gauss-Legendre quadrature with 5 points was used when integrating. The L^2 error for all the methods was computed at the final time $T = 1$.

3.2 True solution

With the results presented in the previous section, the next step is to perform the same study on the analytical solution of Problem 0.1. The parameters are chosen as: $S_{min} = 0, S_{max} = 300, K = 100, r = 0.04, \sigma = 0.2$, and $T = 5$. To begin, Figure 4 illustrates the analytical solution when compared to the interpolated solution of the finite element solver. Visually it is clear that all the methods manage to capture the analytical solution. It is, however, important to point out that the maximum errors are indeed large than before. This will also be evident in the convergence study.

3.2.1 Convergence study of the true solution

Recalling Table 1, in this section a very similar convergence study is performed. The results are shown in Figure 5.

The first striking observation is that the solver struggles a lot more to solve this problem than Problem 3.1. There are many reasons why this might be. First the space Ω and the final time T are much larger, leaving way for numerical errors to appear and propagate. Secondly, and more importantly, is the lack of smoothness in the initial condition of Problem 0.1. Indeed this initial function has a problematic point at $S = K$. This leads to an overall suboptimal outcome of the finite element solver because some regularity assumptions fail. There are nonetheless some possible ways to mitigate this. The first candidate is to use higher order finite elements. This is likely to help by providing a better approximation and reducing any errors when using P1 or P2 elements. Secondly, other methods beside Crank-Nicolson and backwards Euler might prove to be more beneficial in this case, giving a possibly better and more stable convergence. Finally, [2] provides important insights into other potential improvements, suggesting more advanced techniques and approaches.

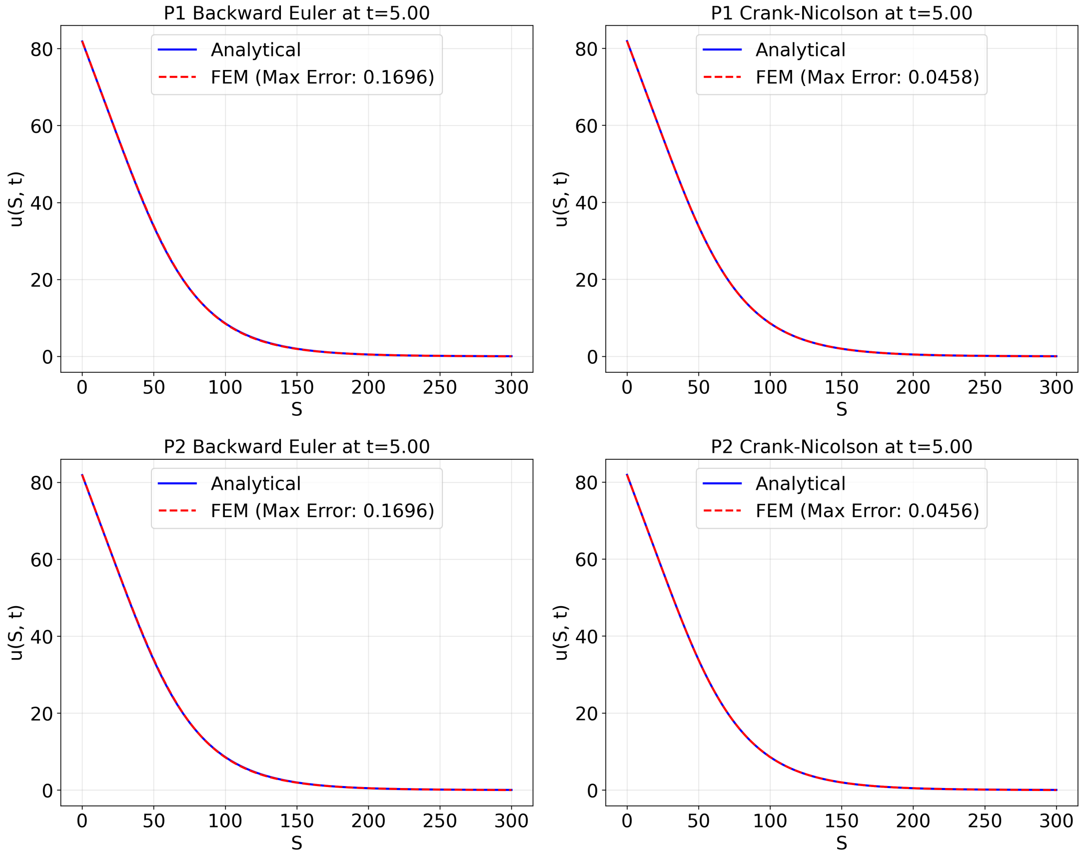

Figure 4: Analytical solution (blue) versus interpolated finite element solution (dashed red) for various methods at time $t = 5$ when tasked to solve Problem 0.1. The PDE was solved with $h = 0.5$ and $\Delta t = 0.5$ using the Gauss-Legendre quadrature with 10 points. First and second row correspond to P1 and P2 finite elements respectively. First and second column correspond to backwards Euler and Crank-Nicolson's method respectively. In legend the maximum errors is presented for all four plots.

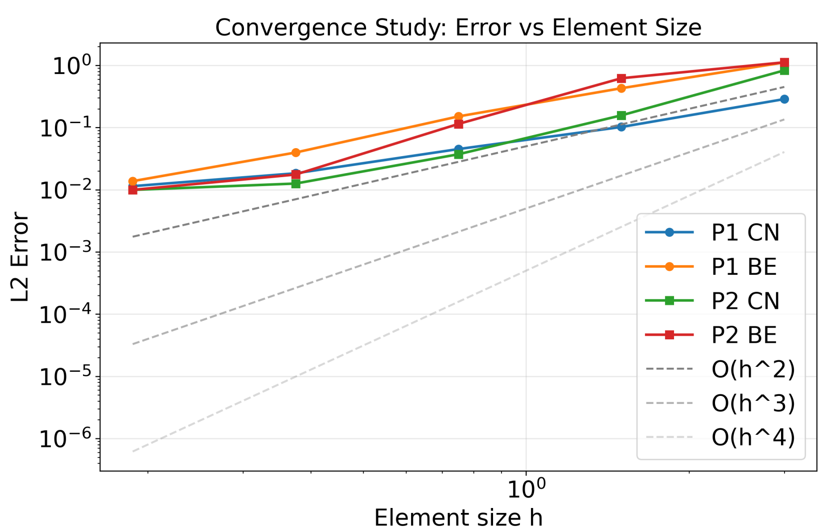

Figure 5: Convergence study for all four methods when solving Problem 0.1. The Gauss-Legendre quadrature with 10 points was used when integrating. The L^2 error for all the methods was computed at the final time $T = 5$.

References

[1] Brenner, Susanne, and L. Ridgway Scott. *The Mathematical Theory of Finite Element Methods.* 1st ed. vol. 15. New York, NY: Springer, 1994.

[2] Achdou, Y. and Pironneau, O., 2005. *Computational methods for option pricing* (Vol. 30). Siam

A Useful results

Theorem A.1 (Integration by parts formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with $\partial\Omega \subset C^1$. For $u, v \in C^1(\bar{\Omega})$,*

$$\int_\Omega u_{x_i}(x)v(x)\,dx = -\int_\Omega u(x)v_{x_i}(x)\,dx + \int_{\partial\Omega} u(y)v(y)\nu_i(y)\,dS(y), \qquad i = 1, \cdots, n.$$

or, in vectorial form,

$$\int_\Omega \nabla u(x)v(x)\,dx = -\int_\Omega u(x)\nabla v(x)\,dx + \int_{\partial\Omega} u(y)v(y)\nu(y)\,dS(y)$$

All previous identities are valid also if the boundary $\partial\Omega$ is only Lipschitz continuous (since Lipschitz functions are differentiable everywhere but a set of points of Lebesgue measure zero).

Lemma A.2 (Lax-Milgram Lemma)**.** *Given a Hilbert space V and a bilinear form $b(\cdot,\cdot)$ on V such that*

$$b(u,v) \leq M\|u\|_V\|v\|_V \qquad \textit{(Continuity)}$$

$$b(u,u) \geq \beta\|u\|_V^2 \qquad \textit{(Coercivity)}$$

for some $M, \beta > 0$ then, given a bounded functional F on V, the problem

$$b(u,v) = F(v)$$

admits a unique solution. Moreover, $\|u\|_V \leq \frac{1}{\beta}\|F\|_{V'}$.