Regularization in PINNs

Various regularization methods including Gaussian process smoothing, ReLoBRaLo, and DN-PINNs have been successfully used to enhance PINN training performance through improved accuracy, efficiency, and robustness.

Abstract

Regularization methods applied to physics-informed neural networks (PINNs) include nine distinct approaches that enhance training for forward and, in some cases, inverse partial differential equation problems. Gaussian process smoothing—used in two studies—improves robustness for problems such as the nonlinear Schrödinger, Burgers', and heat equations, yielding performance close to that for error-free data. ReLoBRaLo adapts loss scalings iteratively to boost both accuracy and computational efficiency, while dynamically normalized PINNs (DN-PINNs) reduce relative L2 error by roughly a factor of 100 compared to baseline models. Curriculum regularization paired with a sequence-to-sequence formulation achieves error reductions of one to two orders of magnitude, and inverse Dirichlet weighting increases convergence order from 1 to 4. Other approaches—self-adaptive PINNs, physics-driven regularization, augmented Lagrangian relaxation, and Bayesian regularization—contribute gains in interpretability, boundary error reduction, and uncertainty quantification.

Paper search

Using your research question "Which regularization methods have been used successfully to train PINNs?", we searched across over 126 million academic papers from the Semantic Scholar corpus. We retrieved the 50 papers most relevant to the query.

Screening

We screened in papers that met these criteria:

- **PINN Regularization Focus**: Does the study explicitly investigate regularization techniques for Physics-Informed Neural Networks (PINNs)?
- **Empirical Evidence**: Does the study present empirical results with quantitative performance metrics (such as loss convergence, prediction accuracy, or error measures) for the PINN regularization methods?
- **Technical Detail**: Does the study provide sufficient technical details about the regularization methods to enable replication?
- **PINN Specificity**: Does the study focus on regularization specifically for physics-informed neural networks rather than only traditional neural networks?
- **Analysis Depth**: Does the study include substantial analysis of the regularization methods rather than just brief mentions?
- **Study Type**: Is the study either (a) primary research on PINN regularization or (b) a systematic review/meta-analysis of PINN regularization techniques?

We considered all screening questions together and made a holistic judgement about whether to screen in each paper.

Data extraction

We asked a large language model to extract each data column below from each paper. We gave the model the extraction instructions shown below for each column.

- **Specific Regularization Technique Used**:

  Identify and precisely describe the regularization method used in the PINN study. Look in the methods or experimental design sections. If multiple techniques are used, list all of them. Be specific about the technique's name, implementation, and key characteristics. Examples might include:

- Bayesian regularization
- Loss balancing approaches
- Soft attention mechanisms
- Adaptive weighting strategies

If the technique is novel, note that and describe its key innovative features. If the technique is a variation of an existing method, specify how it differs from the original approach.

- **Regularization Implementation Details**:

  Describe how the regularization method was technically implemented in the neural network. Look for details in the methodology section about:

- Specific mathematical formulation
- How the regularization was integrated into the loss function
- Any computational or algorithmic modifications
- Hyperparameters used in the regularization approach

If mathematical equations are provided, extract them verbatim. If implementation details are complex, summarize the key steps concisely.

- **Performance Impact of Regularization**:

  Extract quantitative and qualitative evidence of how the regularization method improved PINN performance. Look in results and discussion sections for:

- Comparative performance metrics
- Reduction in overfitting
- Improvement in prediction accuracy
- Uncertainty quantification capabilities

Report numerical improvements if available (e.g., percentage reduction in error, improvement in L2 norm). If multiple performance metrics are reported, include all relevant ones.

- **Problem Domain and PDE Type**:

  Identify the specific type of partial differential equation (PDE) used to evaluate the regularization method. Extract:

- Name of the PDE (e.g., Burgers' equation, Helmholtz equation)
- Whether it was a forward or inverse problem
- Complexity characteristics of the PDE (e.g., linear/nonlinear, stiff/non-stiff)

If multiple PDEs were used for validation, list all of them and note any differences in results across different problem types.

- **Computational Experimental Setup**:

  Describe the computational environment and experimental configuration:

- Neural network architecture (layers, nodes)
- Training approach (optimization algorithm, learning rates)
- Number and type of collocation points
- Computing resources used (if specified)

Be precise about technical details. If multiple experimental configurations were tested, summarize the key variations.

Results

Characteristics of Included Studies

Study	Study Focus	Regularization Method	Problem Type	Performance Metrics	Full text retrieved
Bajaj et al., 2021a	Improving Physics-Informed Neural Networks (PINNs) robustness	Gaussian process (GP) based smoothing	Forward problem: Nonlinear Schrödinger equation, Burgers' equation, 2D heat equation	Validation Mean Squared Error (MSE), qualitative assessment of robustness	Yes
Bajaj et al., 2021b	Enhancing PINN performance and robustness	Gaussian Process (GP) based smoothing	Forward problem: Time-dependent Schrödinger equation, Burgers' equation	No mention found in abstract	No
Bischof and Kraus, 2021	Multi-objective loss balancing in PINNs	ReLoBRaLo (Relative Loss Balancing with Random Lookback)	Forward and Inverse problems: Burgers' equation, Kirchhoff's plate bending equation, Helmholtz's equation	Accuracy, computational efficiency	Yes

Study	Study Focus	Regularization Method	Problem Type	Performance Metrics	Full text retrieved
Deguchi and Asai, 2023	Dynamic normalization of gradients in PINNs	Dynamically Normalized Physics-Informed Neural Networks (DN-PINNs)	Forward problems: 1D Burgers' equation, 1D Allen-Cahn equation, 2D Poisson equation	Relative L2 error, gradient distribution	Yes
Krishnapriyan et al., 2021	Addressing failure modes in PINNs	Curriculum regularization, sequence-to-sequence learning	Forward problem: PDEs with convection, reaction, and diffusion operators	Error reduction (1-2 orders of magnitude)	No
Maddu et al., 2021	Reliable PINN training	Inverse Dirichlet weighting	Forward problem: Generalized incompressible Navier-Stokes equation	Accuracy, convergence order, power spectrum recovery	Yes
McClenny and Braga-Neto, 2020	Self-adaptive PINNs using soft attention	Self-Adaptive Physics-Informed Neural Networks (SA-PINNs)	Forward problems: Allen-Cahn equation, Burgers' equation, Helmholtz equation	L2 error, training efficiency	Yes
Nabian and Meidani, 2018	Physics-informed regularization of deep neural networks	Physics-driven regularization	Forward problems: Burgers' equation, Navier-Stokes equations	Generalization accuracy, interpretability, relative L2 norm	Yes
Son et al., 2022	Augmented Lagrangian relaxation for PINNs	Augmented Lagrangian relaxation method (AL-PINNs)	Forward problems: Helmholtz equation, viscous Burgers equation, Klein-Gordon equation	Relative error, boundary error reduction	Yes

Study	Study Focus	Regularization Method	Problem Type	Performance Metrics	Full text retrieved
Yang et al., 2020	Bayesian PINNs for forward and inverse PDE problems	Bayesian regularization	Forward and inverse problems: Nonlinear PDEs (not specified)	Prediction accuracy, uncertainty quantification	No

Based on our analysis of the included studies:

- We found a diverse range of study focuses, with improving robustness being mentioned in 2 out of 10 studies. Other focuses included performance enhancement, loss balancing, gradient normalization, addressing failure modes, reliable training, self-adaptation, regularization, Lagrangian relaxation, and Bayesian approaches.
- Each study used a different regularization method, with Gaussian process (GP) based smoothing being the only method used in more than one study (2 out of 10). Other methods included ReLoBRaLo, DN-PINNs, curriculum regularization, sequence-to-sequence learning, inverse Dirichlet weighting, SA-PINNs, physics-driven regularization, AL-PINNs, and Bayesian regularization.
- All 10 studies addressed forward problems, while 2 out of 10 also included inverse problems.
- We found a variety of performance metrics used across studies. Accuracy was mentioned in 4 out of 10 studies, followed by L2 error in 3 out of 10. Other metrics included computational efficiency, robustness, MSE, gradient distribution, error reduction, convergence order, power spectrum recovery, interpretability, relative error, boundary error reduction, and uncertainty quantification. We didn't find mention of specified performance metrics in the abstract of one study where full text was not retrieved.

Thematic Analysis

Core Regularization Approaches

1. Probabilistic Frameworks
    - Gaussian process (GP) based smoothing (Bajaj et al., 2021a, 2021b)
        - Enhances PINN robustness against noise and errors in measurements
    - Bayesian physics-informed neural network (B-PINN) framework (Yang et al., 2020)
        - Combines Bayesian neural networks with PINNs
        - Allows for uncertainty quantification
2. Optimization-based Techniques
    - ReLoBRaLo (Bischof and Kraus, 2021)
        - Self-adaptive loss balancing scheme
        - Adapts scalings at every iteration
    - AL-PINNs (Son et al., 2022)
        - Uses augmented Lagrangian relaxation method

    - Treats initial and boundary conditions as constraints
- Inverse Dirichlet weighting (Maddu et al., 2021)
    - Uses weights based on the variance of loss gradients
    - Aims to prevent vanishing gradients and improve training stability

3. Physics-Informed Regularization

    - Physics-driven regularization (Nabian and Meidani, 2018)
        - Incorporates known governing physical laws into the training process
        - Penalizes divergence from these laws

Implementation Strategies

1. Adaptive Weighting Schemes

    - DN-PINNs (Deguchi and Asai, 2023)
        - Dynamically adjusts weights of loss components based on gradient norms
    - SA-PINNs (McClenny and Braga-Neto, 2020)
        - Uses fully trainable adaptation weights applied to individual training points

2. Sequential Training Approaches

    - Curriculum regularization (Krishnapriyan et al., 2021)
        - Complexity of PDE regularization increases progressively during training
    - Sequence-to-sequence learning (Krishnapriyan et al., 2021)
        - Reformulates the problem as a sequence-to-sequence learning task

Performance and Reliability

1. Convergence Characteristics

    - Inverse Dirichlet weighting (Maddu et al., 2021)
        - Achieves a convergence order of r = 4, compared to r = 1 for other methods
    - ReLoBRaLo (Bischof and Kraus, 2021)
        - Reports faster convergence compared to existing methods

2. Error Reduction Capabilities

    - Curriculum regularization (Krishnapriyan et al., 2021)
        - Claims up to 1-2 orders of magnitude lower error compared to regular PINN training
    - DN-PINNs (Deguchi and Asai, 2023)
        - Reports relative L2 error approximately 100 times smaller than baseline PINNs

3. Robustness to Noise

    - GP-smoothed PINNs (Bajaj et al., 2021a, 2021b)
        - Perform almost as well as error-free PINNs, significantly better than corrupted data without smoothing
    - B-PINNs (Yang et al., 2020)
        - Provide more accurate predictions in scenarios with large noise

Comparative Analysis

Method Type	Key Advantages	Limitations	Best Use Cases
Gaussian Process Smoothing (Bajaj et al., 2021a, 2021b)	Improves robustness against noise/errors, enhances uncertainty quantification	Computational complexity may increase	Noisy or error-prone data scenarios
ReLoBRaLo (Bischof and Kraus, 2021)	Outperforms existing methods in accuracy and efficiency, adapts scalings at every iteration	May require careful tuning of hyperparameters	Multi-objective optimization problems
DN-PINNs (Deguchi and Asai, 2023)	Significantly reduces relative L2 error, improves gradient distribution	Computational cost increase of ~30%	Problems with imbalanced gradients
Curriculum Regularization (Krishnapriyan et al., 2021)	Achieves 1-2 orders of magnitude lower error	May require careful design of curriculum	Complex PDEs with multiple scales
Inverse Dirichlet Weighting (Maddu et al., 2021)	Significantly improves accuracy and convergence, protects against catastrophic forgetting	May require careful implementation in optimizers	Multi-scale problems, sequential training
SA-PINNs (McClenny and Braga-Neto, 2020)	Outperforms baseline methods in L2 error reduction, adapts to difficult regions	May increase computational complexity	Stiff PDEs with sharp transitions
Physics-Driven Regularization (Nabian and Meidani, 2018)	Improves generalization accuracy and interpretability	Requires knowledge of governing physical laws	Problems with well-understood physics
AL-PINNs (Son et al., 2022)	Yields smaller relative errors, effectively reduces boundary errors	May require tuning of penalty parameters	Problems with complex boundary conditions
B-PINNs (Yang et al., 2020)	Provides uncertainty quantification, improves predictions in noisy scenarios	May increase computational complexity	Inverse problems, uncertainty quantification

Based on our analysis of the included studies:

- We found 9 unique methods for improving Physics-Informed Neural Networks (PINNs) or related approaches.
- Key advantages of these methods:

    - 4 methods focused on error reduction
    - 3 methods improved accuracy
    - 2 methods each enhanced uncertainty quantification and robustness
    - Other advantages included improved efficiency, gradient distribution, convergence, protection against forgetting, adaptability, and interpretability

- Common limitations across methods:
    - 4 methods potentially increased computational complexity
    - 2 methods required careful parameter tuning
    - Other limitations included implementation complexity, curriculum design challenges, and the need for domain knowledge

- Best use cases varied widely:
    - We found applications for noisy data, multi-objective optimization, imbalanced gradients, complex PDEs, multi-scale problems, sequential training, stiff PDEs, problems with well-understood physics, complex boundary conditions, inverse problems, and uncertainty quantification
    - Each use case was mentioned for one method, suggesting these methods are often tailored to specific problem types

- We didn't find any method that clearly outperformed all others across all criteria, indicating that the choice of method may depend on the specific problem and computational constraints.

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