

# Proving mathematical statements with Lean

## Lesson 5: induction and some relations

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# Overview

1. Goals of today's meeting
2. Motivation
3. Exercises from sheet 3
4. Relations
5. Voluntarily exercises for next week

# 1. Goals of today's meeting

- Solve exercise 3.2 with strong induction on paper like Lean would do it.
- Understanding the three characteristics of a relation.
- See how the three different characteristics of a relation are implemented in Lean 4.

## 2. Motivation

- You start to apply your Lean knowledge on paper.
- You get to see how nicely and elegant one can prove statements about relations using Lean 4.

### 3. Exercises from sheet 3

Today, we will solve the following exercises from sheet 3 [1]:

#### Exercise 4 (2pt)

1. Define a relation on  $\mathbb{Z}$  as  $xRy$  if  $|x - y| < 1$ . Is  $R$  reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?
2. Define a relation  $R$  on  $\mathbb{Z}$  as  $xRy$  if and only if  $|x - y| \leq 1$ . Say whether  $R$  is reflexive. Is it symmetric? Transitive?

**Exercise 5 (2pt)** Suppose  $A \neq \emptyset$ . Since  $\emptyset \subseteq A \times A$ , the set  $R = \emptyset$  is a relation on  $A$ . Is  $R$  reflexive? Symmetric? Transitive? If a property does not hold, say why.

#### Exercise 6 (3pt)

1. Suppose  $R$  is a symmetric and transitive relation on a set  $A$ , and there is an element  $a \in A$  for which  $aRx$  for every  $x \in A$ . Prove that  $R$  is reflexive.
2. Prove or disprove: If a relation is symmetric and transitive, then it is also reflexive.

## 4. Relations

### Definition (relation)

Let  $A$  be a set. A relation  $R$  is a subset of  $A \times A$ . For an element  $(a, b) \in A \times A$ , we write  $aRb$  if  $(a, b) \in R$

# Relations

## Definition (reflexive relation)

A relation  $R$  is called *reflexive*, if

$$\forall a \in A, aRa$$

## Definition (symmetric relation)

A relation  $R$  is called *symmetric*, if

$$\forall a, b \in A, aRb \Rightarrow bRa$$

## Definition (transitive relation)

A relation  $R$  is called *transitive*, if

$$\forall a, b, c \in A, aRb \wedge bRc \Rightarrow aRc$$

## 5. Voluntarily exercises for next week

- Finish the exercises from sheet 3 we could not finish today in Lean.



*Thank you for your cooperation!!*

# References



Argentieri Fernando (2023)

HS 2023 - MAT 115 Foundation of Mathematics Problem sheet 3

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