Proving mathematical statements with Lean

Lesson 3: direct and contrapositive proofs

Mattia L. Bottoni

Institute of Mathematics
University of Zurich

18.10.2023

About me

Overview

1. Goals of today's meeting
2. Motivation
3. Exercises from sheet 2
4. Direct and contrapositive proof
5. Voluntarily exercises for next week

1. Goals of today's meeting

- Run a Lean document on your device (for real this time).
- Understanding the main differences between proving a statement on paper vs. with Lean such that you can explain it..
- Knowing the difference between a direct and a contrapositive proof and deciding when to use which one.

2. Motivation

- We want to try and apply some Lean thinking onto paper.
- We will see how to solve a majority of the exercises from sheet 2.

3. Exercises from sheet 2

Today, we will solve the following two exercises from sheet 2 [3]:

Exercise 2 (6pt) Use the method of direct proof to prove the following statements.

1. Let $x, y \in \mathbb{R}$. If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.
2. Recall that $x|y$ means there exists an integer k, such that $y = kx$. Show that if a is an integer and $a^2|a$, then $a \in \{0, 1, -1\}$.
3. Every odd integer is a difference of two squares.

Exercise 3 (4pt) Prove the following statements with contrapositive proof. (In each case, think about how a direct proof would work. In most cases contrapositive is easier.)

1. Let $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.
2. Let $x, y, z \in \mathbb{Z}$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

Exercises from sheet 2

If we have enough time we will also have a look at [3]:

Exercise 4 (4pt) Use the method of proof by contradiction to prove the following statements. (In each case, you should also think about how a direct or contrapositive proof would work. You will find in most cases that proof by contradiction is easier.)

1. Let $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$ then a or b is odd.
2. For every $x \in [\frac{\pi}{2}, \pi]$, $\sin x - \cos x \geq 1$.
   Hint: $\sin x - \cos x = \sqrt{2}\sin(x - \pi/4)$.

Exercise 5 (3 pt) Each of the following statements is either true or false. If a statement is true, prove it. If a statement is false, disprove it.

1. If A, B, C are sets and $A \times C = B \times C$, then $A = B$.
2. If A, B are sets, then $\wp(A \cap B) = \wp(A) \cap \wp(B)$.
   (*Recall that $\wp(A)$ is the power set of A.*)

4. Direct and contrapositive proof

Definition (direct proof)

If P is a given statement and you want to prove Q, you do that by implications until you reach Q:

$$P \Rightarrow P' \Rightarrow \cdots \Rightarrow Q$$

Definition (contrapositive proof)

If P is a statement and you want to prove Q, you can do that by assuming $\neg Q$ and then proving $\neg P$:

$$\neg Q \Rightarrow (\neg Q)' \Rightarrow \cdots \Rightarrow \neg P$$

5. Voluntarily exercises for next week

- Have a look at the sheet of the first meeting and try to prove the statements.
- Solve exercise sheet 2 with Lean.

Thank you for your cooperation!

References

unknown

EMS Schiers

https://www.schuljobs.ch/job/wirtschaft-recht-60-bis-70-prozent-ausbaubar-auf-80-prozent-im-sj-24-25/J910241 [08.10.2023]

Adrian Michael (2008)

Bündner Herrschaft mit Landquart. Blick nach Norden.

https://de.wikipedia.org/wiki/Churer_Rheintal [08.10.2023]

Argentieri Fernando (2023)

HS 2023 - MAT 115 Foundation of Mathematics Problem sheet 2

UZH