Proving mathematical statements with Lean

Lesson 6: Relations and functions

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Overview

1. Goals of today's meeting
2. Motivation
3. Exercises from sheet 4
4. Relations and functions
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1. Goals of today's meeting

- You learn how to implement functions in Lean!
- You will see the beauty of working with Lean first-hand.
- You will solve some of the exercises of the first part of sheet 4 with Lean.

2. Motivation

- Functions are used everywhere in mathematics. With Lean, you can prove if a function is injective or surjective in a clean way.
- Proving if a function is injective or surjective was one of the most annoying things to do in my opinion. I am positive that doing it with Lean will be more fun.

3. Exercises from sheet 4

Today, we will solve the following exercises from sheet 4 [1]:

Exercise 1 **(3pt)** . Define a relation R on $\mathbb{Z}$ as xRy if and only if $x^2 + y^2$ is even. Prove that R is an equivalence relation. ~~Describe its equivalence classes.~~

Exercise 3 **(4pt)** A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined by $f(m, n) := (m+n, 2m+n)$. Verify whether this function is injective and whether it is surjective.

4. Relations and functions

Definition (relation)

Let A be a set. A *relation* R is a subset of $A \times A$. For an element $(a, b) \in A \times A$, we write aRb if $(a, b) \in R$

Definition (function)

A *function* f from a set $\mathbb{D}$ to a set $\mathbb{W}$ assigns to each element $x \in \mathbb{D}$ a single element $f(x) = y \in \mathbb{W}$. $\mathbb{D}$ is called the *domain* and $\mathbb{W}$ the *codomain* of f. The *range* of a function is the set of elements y, such that $\exists x \in \mathbb{D}, y = f(x)$

Injective and surjective

Definition (injective)

A function $f : \mathbb{D} \mapsto \mathbb{W}$ is called *injective*, if

$$\forall x, y \in \mathbb{D}, (f(x) = f(y)) \Rightarrow (x = y)$$

Definition (surjective)

A function $f : \mathbb{D} \mapsto \mathbb{W}$ is called *surjective*, if

$$\forall y \in \mathbb{W}\ \exists x \in \mathbb{D}, f(x) = y$$

Bonus:

Definition (bijective)

A function $f : \mathbb{D} \mapsto \mathbb{W}$ is called *bijective*, if and only if f is *injective* and *surjective*

5. Voluntarily exercises for next week

- Solve the exercises from sheet 4. Note questions if there are any.

Thank you for your cooperation!!

References


Argentieri Fernando (2023)
HS 2023 - MAT 115 Foundation of Mathematics Problem sheet 4
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