Proving mathematical statements with Lean

Lesson 8: Natural numbers and Peano axioms

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Overview

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1. Goals of today's meeting

- You get to know the natural number game.
- You learn how to build up the natural numbers from scratch.
- You finish the first two worlds of the natural number game.

2. Motivation

- After you did some hard work proving statements about relations and functions, it is time to lean back and enjoy implementing the natural numbers.
- You will start doing the natural number game, which is really addicting and fun.

3. Peano axioms

Please recall the definition of the Peano's axioms from your lecture notes.

Definition (Addition)

With Peano's axioms (more precisely, his axiom of induction) one can define two fundamental rules for addition:

$$n + 0 = n \text{ for } \forall n \in \mathbb{N} \text{ (base case)} \tag{1}$$

and

$$n + S(m) = S(n + m) \text{ for } \forall n, m \in \mathbb{N} \text{ (induction step).} \tag{2}$$

With these two rules, try to prove the following theorem:

Theorem (zero_add)

For all $n \in \mathbb{N}$, we have $0 + n = n$.

Hint: You should use induction to do this.

Proof of theorem (zero_add)

Proof by induction:

Proof.

- base case (k = 0):

$$0 + 0 = 0 \text{ (by equation (1))}$$

- induction hypothesis:

$$0 + n = n \text{ for } \forall n \in \mathbb{N}$$

- induction step (k = S(n)):

$$\begin{aligned} 0 + S(n) &= S(0 + n) \text{ for } \forall n \in \mathbb{N} \quad \text{(by equation (2))} \\ &= S(n) \text{ for } \forall n \in \mathbb{N} \quad \text{(by induction hypothesis)} \end{aligned}$$

□

4. Tutorial world

- Did you struggle while proving the theorem zero_add? At which steps?
- We will now have a look at the first world of the natural number games together. This leads us to prove the theorem zero_add by interacting with Lean.

5. Addition world

- Try to solve level 1 of addition world (5 mins).
- Struggle? Have a look at the menu on the left-hand side or ask me.
- Let us look at level 1 altogether.
- Afterwards, try to solve the rest by yourself. If you get stuck have a look at the menu on the left-hand side or ask me.

6. Voluntarily exercises for next week

- Finish addition world.
- Start doing implication world. Please do not start with multiplication world yet.
- Finish exercise sheet 5.

Thank you for your cooperation!!

References


Kevin Buzzard, Jon Eugster (2023)
Natural Number Game
https://adam.math.hhu.de/ [29.11.2023]