Proving mathematical statements with Lean

Lesson 10: Natural numbers - Order relations and inequality world

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1. Goals of today's meeting

- Learn how to deal with order relations.
- Solve the inequality world.
- Understand a sophisticated Lean proof (if you want).

2. Motivation

- For the ones who are a bit tired, I have the inequality world, which is really fun.
- For the ones that are motivated, I have an interesting proof we can discuss.

3. Exercises from sheet 6

Today, we will solve the following exercise from sheet 6 in the natural number game [1]:

Exercise 3 (5pt) Define a relation R on $\mathbb{N}^* := \mathbb{N}_0 - \{0\}$ by: aRb means that a divides b, that is $b = ac$ for some $c \in \mathbb{N}^*$. Is R an order relation? If so, is it a weak order or a strict order? If not, is R a partial order on $\mathbb{N}^*$?

Exercise 4 (3pt) Let $X := \{1, 2, 6, 30, 210\}$ and define a relation S on X by: aSb means that a divides b. Is S an order relation? If so, is it a weak order or a strict order?

Exercise 5 (4pt) Let A be a set with a (strict) order relation S and B a set with (strict) order relation T. Define the lexicographic relation L on $A \times B$ by:

$(a, b)L(c, d)$ means: either aSc, or $a = c$ and bTd.

Is this an order relation? What is the connection between this and a dictionary?

4. Weak and partial order

Definition (Weak and partial order)

For a relation to be a *weak order relation* we need to check the three following statements:

- W01 (transitivity): $a \mathrel{R} b \land b \mathrel{R} c \Rightarrow a \mathrel{R} c$
- W02 (totality): either $a \mathrel{R} b \lor b \mathrel{R} a$ (or both)
- W03 (antisymmetry): If $a \mathrel{R} b \land b \mathrel{R} a$, then $a = b$

If we just have W01 and W03 we call it a *partial order relation*.

The division is a partial order on the natural numbers, if we do not consider a set where each number divides the next larger number. For example, $4 \nmid 5 \land 5 \nmid 4$.

In Lean, the instance of being a weak order relation is called: *IsLinearOrder* and for a partial order relation it is: *IsPartialOrder*.

5. Strict order

Definition (Strict order)

For a relation to be a *strict order relation* we need to check the two following statements:

- S01 (transitivity): $a \; S \; b \land b \; S \; c \Rightarrow a \; S \; c$
- S02 (trichotomy): either $a \; S \; b \lor b \; S \; a \lor a = b$

In the end, to see if we have a weak or a strict order relation we can just see if $a \; R \; b$ is defined for $b = a$. We have that $5 \leq 5$, but $5 \nless 5$. So $\leq$ is a weak and $<$ is a strict order relation.

In Lean, the instance of being a strict order relation is called: *IsStrictTotalOrder*.

6. Inequality world

Have fun solving this world. If you have a problem, read the hints on the left-hand side or ask me. If you would like to do something more challenging, have a look at the proof of exercise 6.5 with me.

7. Voluntarily exercises for next week

- Finish inequality world and/or the natural number game.
- Finish exercise sheet 6 on paper.
- Try to understand the proof of exercise 6.5 in Lean.
- Prepare questions for the last week.

Thank you for your cooperation!!

References

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HS 2023 - MAT 115 Foundation of Mathematics Problem sheet 6
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Kevin Buzzard, Jon Eugster (2023)
Natural Number Game
https://adam.math.hhu.de/ [29.11.2023]