Proving mathematical statements with Lean

Lesson 7: Relations, functions and cardinality

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Overview

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1. Goals of today's meeting

- You understand the meaning of sets having the same cardinality.
- You know why it is hard to implement such proofs in Lean.

2. Motivation

- This is one of the "hardest" topics you will encounter in this course. After understanding this, you already understand a lot!
- Even though it is a bit hard to implement in Lean, you should be able to do it on paper using the Lean way of proving mathematical statements.

3. Exercises from sheet 4

Today, we will solve the following exercises from sheet 4 [1]:

Exercise 4 (3pt)

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2 + 3$. Find $f([3,5])$ and $f^{-1}([12,19])$.

2. Given a function $f : A \to B$, $X, Y \subseteq A$, prove that $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

Exercise 6 **(3pt)** Show that the "relation" defined by "equal cardinality" for sets is "an equivalence relation". Thus we can define the *cardinality* of a set as *its equivalence class*. (Hint: You can use that the composition of bijections stays bijective.)

Remark: Here we do not worry about where the "relation" is actually defined on, since the set of all sets is not really a set. Here it is enough that you verify the reflexivity, symmetry, and transitivity.

4. Intervals in Lean

Definition (Interval)

A *interval* in Lean is noted as *Ijk a b*, where *j* and *k* are either *o* for open, *c* for closed or *i* for infinite and *a*, *b* are the lower and upper bound.

Problem: At the moment, I have not figured out a suitable way to define a function from e.g. $\mathbb{R}$ to an interval. The problem is, that $\mathbb{R}$ is a *type* in Lean but the interval is a *set*. You do not have to understand this and I will try to make it work in some way.

5. Cardinality

Definition (Cardinality)

The *cardinality* of a set A is defined as the number of elements in A and denoted as $|A|$. The cardinality can be *finite*, *countably infinite* or *uncountably infinite*.

Definition (Equal cardinality of sets)

Two sets A and B have *equal cardinality*, if there exists a *bijection* $f : A \to B$. We say that a set is countably infinite, if there exists a bijection $f : \mathbb{N} \to A$ and it is uncountably infinite if there is not.

6. Voluntarily exercises for next week

- Finish exercise sheet 4 on paper.
- Try to solve the Lean exercises of sheet 4. Write questions if you cannot do it.
- If you don't know a tactic, either use *apply?* or try to find the tactic on this website: `https://leanprover-community.github.io/mathlib4_docs/Mathlib/Algebra/Ring/Defs.html`
- I also uploaded a helpful cheat-sheet onto GitHub.

Thank you for your cooperation!!

References

Argentieri Fernando (2023)
HS 2023 - MAT 115 Foundation of Mathematics Problem sheet 4
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