HOMEWORK 2

Consider the linearized shallow water equations:

$$\begin{cases} \eta_t(x,t) + q_x(x,t) = 0 & (x,t) \in (0,L) \times (0,T), \\ q_t(x,t) + gH\eta_x(x,t) = 0 & (x,t) \in (0,L) \times (0,T), \end{cases} \tag{1}$$

where $\eta(x,t)$ is the free surface elevation, $q(x,t) = Hu(x,t)$ is the discharge with $u(x,t)$ the horizontal velocity, H is the mean depth, g is the gravity force, and $c = gH$ is the wave speed. Together with (1) we consider the following periodic boundary conditions

$$\eta(0,t) = \eta(L,t), \quad q(0,t) = q(L,t).$$

1. Show that problem (1) is equivalent to

$$\eta_{tt}(x,t) = gH\eta_{xx}(x,t) \quad (x,t) \in (0,L) \times (0,T)$$

   with suitable boundary conditions.

2. Show that the energy
$$E(t) = \int_0^L \left(\frac{1}{2}gH\eta^2 + \frac{1}{2}\frac{q^2}{H} \right) dx$$
   is conserved under periodic boundary conditions.

3. Write the weak formulation of problem (1) and the corresponding spectral element discretization. Show that the Galerkin discretization leads to the following system

$$M\dot{\boldsymbol{U}} = -S\boldsymbol{U}. \tag{2}$$

4. Implement the solver in MATLAB and verify your implementation by considering:

$$L = 1, \quad H = 1, \quad and \quad g = 9.81,$$

   with initial conditions $\eta(x,0) = e^{-50(x-0.5)^2}$ and $q(x,0) = c\eta(x,0)$, and final time $T = 0.5$. Compute and report the L^2 norm $\|\eta - \eta_h\|_{L^2(0,L)}$ for different choices of the discretization parameter h (mesh size) and polynomial degrees. Consider the exact solution $\eta(x,t) = \eta(x - ct, 0)$. Use a time integrator of the Newmark family for (2).

5. (*Optional.*) Study the effects on the solution of

   – reflecting wall boundary conditions:

$$q = 0 \quad at \quad x = 0, L$$

   – friction:

$$q_t(x,t) + gH\eta_x(x,t) = -\gamma q(x,t),$$

   for a positive γ.