All pairs shortest path $G(V,E)$ $G=(V,E)$

Floyd-Warshall Algorithm → $\Theta(V^3)$

√ we can have negative weights but NO negative cycles

① We consider intermediate vertices of a path

$$p_{1,\ell} = \langle v_1, v_2, \dots, v_\ell \rangle \rightsquigarrow \langle v_2, v_3, \dots, v_{\ell-1} \rangle$$

Lemma 1

Given a weighted directed graph $G=(V,E)$ with $w: E \rightarrow \mathbb{R}$, let $p = \langle v_0, v_1, \dots, v_k \rangle$ be the SHORTEST PATH (SP) from v_0 to v_k and for any $i,j \mid 0 \leq i \leq j \leq k$ let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ be the subpath of p from v_i to v_j.

Then, p_{ij} is a SP from v_i to v_j.

Proof.

If we can find a SP p'_{ij} then it means that also p has a shortest path that uses p'_{ij}

② Given $V = \{\overset{v_1}{1}, \overset{v_2}{2}, \dots, \overset{v_n}{n}\}$ consider a subset $\{1, 2, \dots, k\}$ for some $k < n$

For any $i, j \in V$ consider all paths from i to j whose intermediate vertices are drawn from $\{1, \dots k\}$ and let p be the SP among them.

$\{1, 2, \dots k-1\}$

ⓐ if k is not an intermediate node of $p \Rightarrow$ all intermediate nodes are $\{1, 2, \dots, k-1\}$ thus, the SP from i to j with intermediates in $\{1, 2, \dots k-1\}$ is also the SP with interm. in $\{1, 2, \dots k\}$

ⓑ if k is intermediate

$$p = i \overset{p_1}{\rightsquigarrow} k \overset{p_2}{\rightsquigarrow} j$$

by lemma 1, p_1 is the SP with intermediates in $\{1 \dots k\}$

p_2 is the SP from k to j

RECURSIVE SOLUTION

$d_{ij}^{(k)} \rightarrow$ weight of the SP from i to j with intermediates in $\{1, 2, \dots k\}$

$d_{ij}^{(0)} = P_{ij} = i \longrightarrow j$ without intermediate

$d_{ij}^{(0)} = w_{ij} = w(i,j)$

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\left(d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \end{cases}$$

$\boxed{D^{(\underline{n})} = d_{ij}^{(\underline{n})}}$ gives final answer $d_{ij}^{(n)} = \delta(i,j)\ \forall i,j \in V$

W $$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ w(i,j) & \text{if } i \neq j \wedge (i,j) \in E \\ +\infty & \text{if } i \neq j \wedge (i,j) \notin E \end{cases}$$

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FLOYD-WARSHALL (W)
  n = rows[W]
  $D^{(0)} \leftarrow W$
  for k = 1 to n do   ←
    for i = 1 to n do
      for j = 1 to n do
        $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
  return $D^{(n)}$
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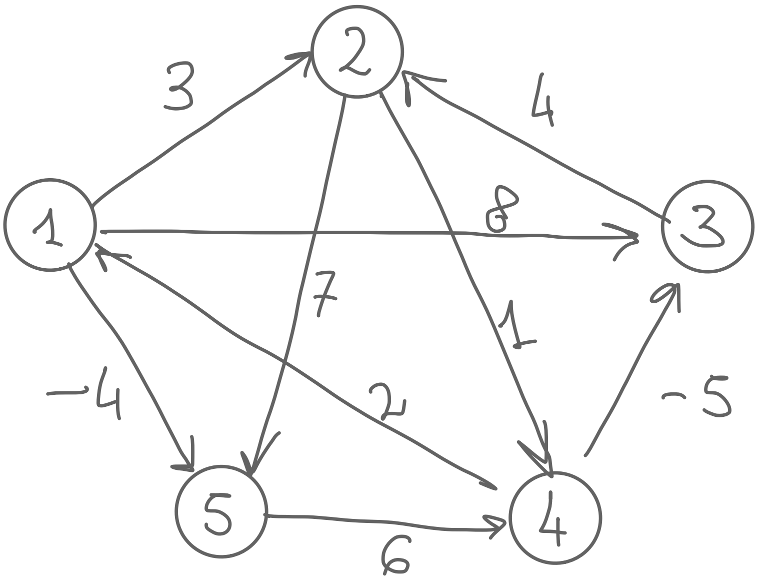
$\Theta(V^3)$

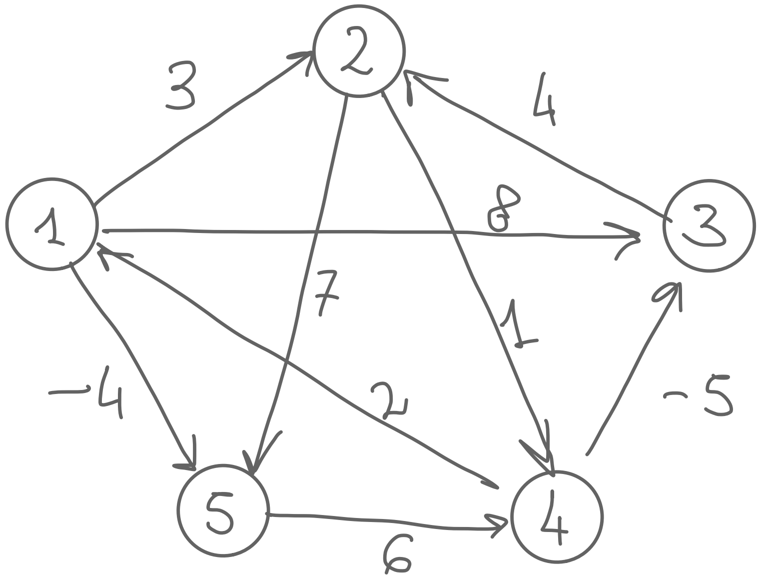
FLOYD-WARSHALL



$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$$\underset{\substack{\shortparallel \\ W}}{D^{(0)}} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 3 & 8 & \infty & -4 \\ 2 & \infty & 0 & \infty & 1 & 7 \\ 3 & \infty & 4 & 0 & \infty & \infty \\ 4 & 2 & \infty & -5 & 0 & \infty \\ 5 & \infty & \infty & \infty & 6 & 0 \end{array}$$

$$\Pi^{(0)} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & N & 1 & 1 & N & 1 \\ 2 & N & N & N & 2 & 2 \\ 3 & N & 3 & N & N & N \\ 4 & N & 2 & 4 & N & N \\ 5 & N & N & N & 5 & N \end{array}$$



$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)} \,,\, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right)$$

$$\begin{matrix} D^{(1)} \\ \shortparallel \\ W \end{matrix} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 3 & 8 & \infty & -4 \\ 2 & \infty & 0 & \infty & 1 & 7 \\ 3 & \infty & 4 & 0 & \infty & \infty \\ 4 & 2 & 5 & -5 & 0 & -2 \\ 5 & \infty & \infty & \infty & 6 & 0 \end{array}$$

$k=1$

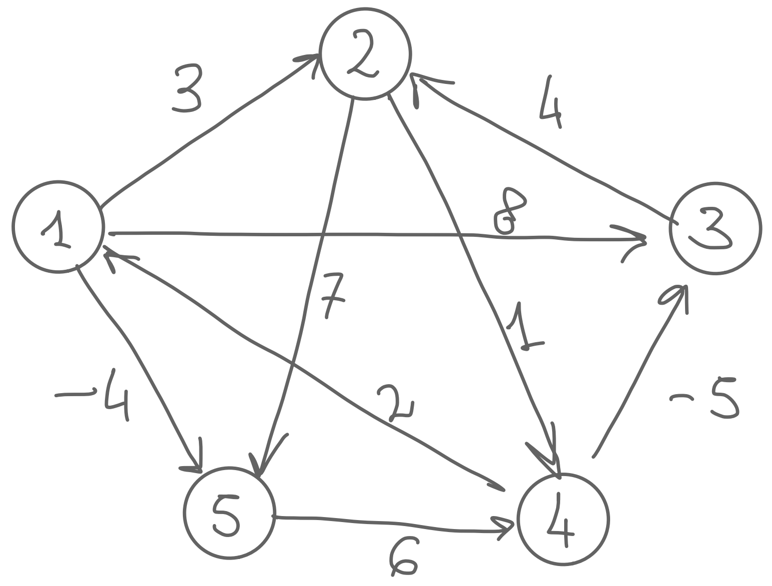
$i=4$ $\quad k=1$

$j=2$

$d_{4,2}$

$d_{4,1}^{(1)} + d_{1,2}$

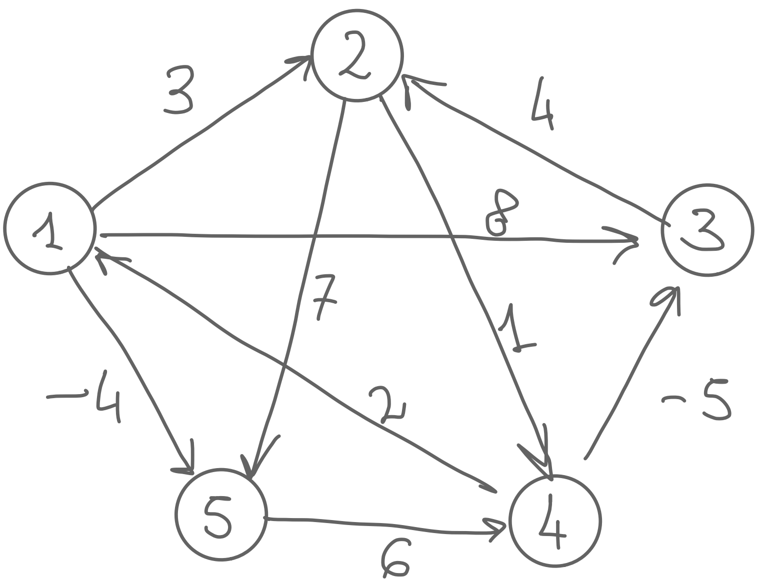
$$\Pi^{(1)} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & N & 1 & 1 & N & 1 \\ 2 & N & N & N & 2 & 2 \\ 3 & N & 3 & N & N & N \\ 4 & N & 1 & 4 & N & 1 \\ 5 & N & N & N & 5 & N \end{array}$$



$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$$\underset{\overset{\shortparallel}{W}}{D^{(2)}} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 3 & 8 & 4 & -4 \\ 2 & \infty & 0 & \infty & 1 & 7 \\ 3 & \infty & 4 & 0 & 5 & 11 \\ 4 & 2 & 5 & -5 & 0 & -2 \\ 5 & \infty & \infty & \infty & 6 & 0 \end{array}$$

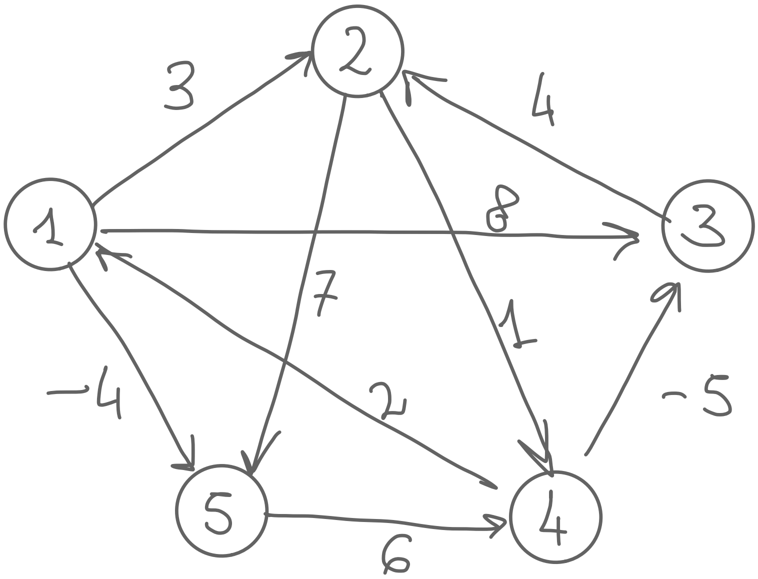
$$\Pi^{(2)} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & N & 1 & 1 & 2 & 1 \\ 2 & N & N & N & 2 & 2 \\ 3 & N & 3 & N & 2 & 2 \\ 4 & N & 1 & 4 & N & 1 \\ 5 & N & N & N & 5 & N \end{array}$$



$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)} , d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right)$$

$$\underset{\overset{\shortparallel}{W}}{D^{(3)}} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 3 & 8 & 4 & -4 \\ 2 & \infty & 0 & \infty & 1 & 7 \\ 3 & \infty & 4 & 0 & 5 & 11 \\ 4 & 2 & -1 & -5 & 0 & -2 \\ 5 & \infty & \infty & \infty & 6 & 0 \end{array}$$

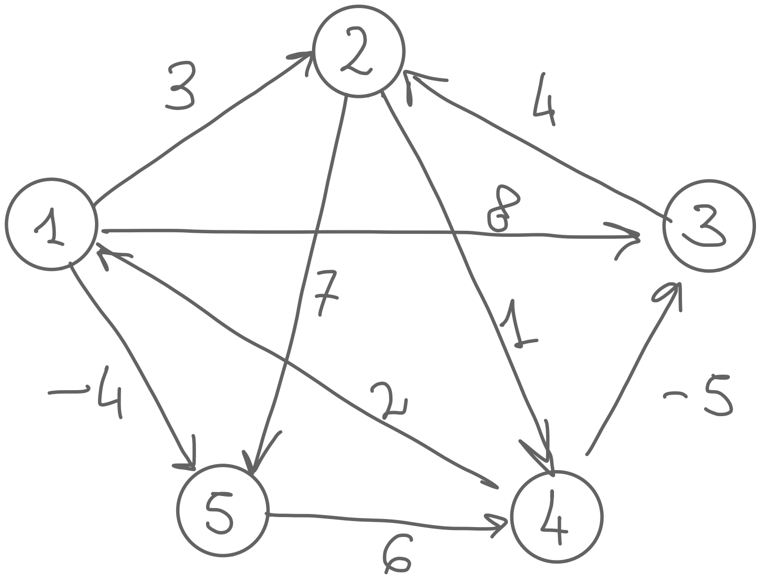
$$\Pi^{(3)} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & N & 1 & 1 & 2 & 1 \\ 2 & N & N & N & 2 & 2 \\ 3 & N & 3 & N & 2 & 2 \\ 4 & N & 3 & 4 & N & 1 \\ 5 & N & N & N & 5 & N \end{array}$$



$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)} , d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right)$$

$$\underset{\underset{W}{\shortparallel}}{D^{(4)}} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 3 & -1 & 4 & -4 \\ 2 & 3 & 0 & -4 & 1 & -1 \\ 3 & 7 & 4 & 0 & 5 & 3 \\ 4 & 2 & -1 & -5 & 0 & -2 \\ 5 & 8 & 5 & 1 & 6 & 0 \end{array}$$

$$\Pi^{(4)} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & N & 1 & 4 & 2 & 1 \\ 2 & 4 & N & 4 & 2 & 1 \\ 3 & 4 & 3 & N & 2 & 1 \\ 4 & 4 & 3 & 4 & N & 1 \\ 5 & 4 & 3 & 4 & 5 & N \end{array}$$



$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)} \, , \, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right)$$

$$\underset{\overset{\shortparallel}{W}}{D}^{(5)} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & -3 & 2 & -4 \\ 2 & 3 & 0 & -4 & 1 & -1 \\ 3 & 7 & 4 & 0 & 5 & 3 \\ 4 & 2 & -1 & -5 & 0 & -2 \\ 5 & 8 & 5 & 1 & 6 & 0 \end{array}$$

$$\Pi^{(5)} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & N & 3 & 4 & 5 & 1 \\ 2 & 4 & N & 4 & 2 & 1 \\ 3 & 4 & 3 & N & 2 & 1 \\ 4 & 4 & 3 & 4 & N & 1 \\ 5 & 4 & 3 & 4 & 5 & N \end{array}$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \end{cases}$$

$$\Pi^{(k)}$$