Exercise 1

Solve by using the M.T.

$$T(n) = 8T\left(\frac{n}{2}\right) + 1000n^2$$

$a = 8$, $b = 2$, $f(n) = 1000n^2$ $\qquad n^{\log_2 8} = n^3$

Is $f(n) = O(n^{3-\varepsilon})$?

$cn^2 = O(n^{3-\varepsilon}) \Rightarrow$ Yes for any $0 < \varepsilon < 1$

$T(n) = \Theta(n^3)$

Polynomially smaller?

$$\lim_{n \to +\infty} \frac{cn^2}{n^{3-\varepsilon}} = \frac{+\infty}{+\infty}$$

$$\lim_{n \to +\infty} \frac{2cn}{(3-\varepsilon)n^{2-\varepsilon}} \to \emptyset$$

polynomially smaller OK!

Exercise 2

$$T(n) = 2T\left(\frac{n}{2}\right) + 10n$$

$a = 2 = b$ $\qquad n^{\lg 2} = n$ $\qquad f(n) = 10n$

is $f(n) = \Theta(n)$? $\rightarrow$ yes

$T(n) = \Theta(n \lg n)$

Exercise 3

$$T(n) = 2T\left(\frac{n}{2}\right) + 10n^2$$

$f(n) = n^2$ $\qquad n^{\lg 2} = n$

Is $f(n) = \Omega(n^{1+\varepsilon})$? $\rightarrow$ yes

check regularity condition

$$af\left(\frac{n}{b}\right) \leq cf(n)$$

$2\frac{n^2}{2} \leq c \cdot 10n^2 \Rightarrow n^2 \leq cn^2$, $c < 1 \Rightarrow$ yes

$T(n) = \Theta(n^2)$

Exercise 4

$$T(n) = 4T\left(\frac{n}{2}\right) + \frac{n^2}{\lg n}$$

$a = 4$ $\quad b = 2$ $\qquad n^{\lg 4} = n^2$ $\qquad f(n) = n^2/\lg n$

Is $f(n) = O(n^{2-\varepsilon})$?

We need to check is $f(n)$ grows at a smaller rate than n^2

$$\lim_{n\to+\infty} \frac{n^2/\lg n}{n^2} = \lim_{n\to\infty} \frac{1}{\lg n} \to \varnothing \rightsquigarrow OK$$

But, is it POLYNOMIALLY SMALLER?

$$\lim_{n\to\infty} \frac{\frac{n^2}{\lg n}}{n^{2-\varepsilon}} = \lim_{n\to+\infty} \frac{n^{\varepsilon}}{\lg n} = \frac{+\infty}{+\infty}$$

$$\stackrel{H}{=} \lim_{n\to+\infty} \frac{\varepsilon n^{\varepsilon-1}}{1/n} = +\infty \to \text{case 1 does not apply!}$$

Exercise 5

Consider the following recurrence

$$T(n) = \begin{cases} T_0, & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + w(n), & \text{otherwise} \end{cases}$$

where T_0 is an arbitrary constant and $w(n)$ is a non-negative and non-decreasing function.
Write the general solution and then specialize your formula in the following cases:

(a) $w(n) = a$, where a is a constant

(b) $w(n) = a \lg n$

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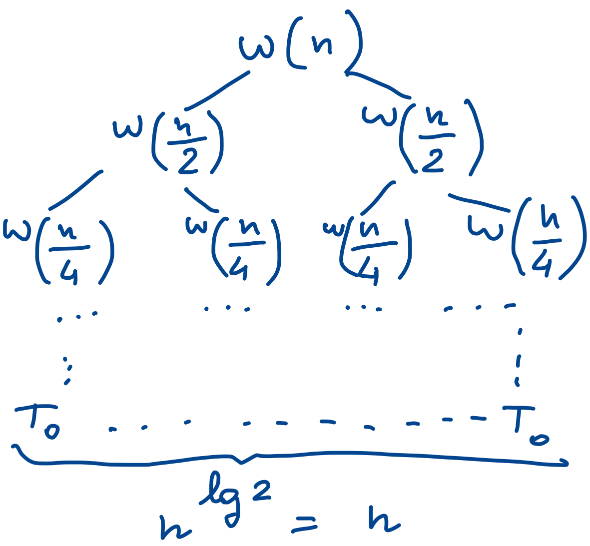
Let's write the general solution by drawing the recursion tree where we know that the initial problem of size n is cut down in half and divided into two subproblems:



$w(n)$

$2w\left(\frac{n}{2}\right)$

$4w\left(\frac{n}{4}\right)$

$2^i w\left(\frac{n}{2^i}\right)$

$i \in [0, \lg n - 1]$

General solution $T(n) = T_0\, n + \sum_{i=0}^{\lg n-1} 2^i\, w\left(\frac{n}{2^i}\right)$

(a) $w(n) = a$ (constant)

$$\sum_{i=0}^{\lg n-1} 2^i\, w\left(\frac{n}{2^i}\right) = a\sum_{i=0}^{\lg n-1} 2^i$$

$$= a\left(\frac{2^{\lg n}-1}{2-1}\right) = a(n-1)$$

→ as a is constant, so there is no way to reduce its size to $n/2^i$ then $w(n) = w\left(\frac{n}{2^i}\right) = a$

Thus, $T(n) = T_0\, n + a(n-1) = (a+T_0)\,n - a$

(b) $w(n) = a \lg n$

$$\sum_{i=0}^{\lg n-1} 2^i\, w(n) = \sum_{i=0}^{\lg n-1} 2^i\, a \lg\left(\frac{n}{2^i}\right) = a\sum_i 2^i \lg\left(\frac{n}{2^i}\right) = a\sum_i 2^i\left(\lg n - \lg 2^i\right)$$

$$= a\sum_i\left(\lg n - i\right) =$$

$$= a\left[\lg n + 2(\lg n-1) + 2^2(\lg n-2) + \dots + 2^{\lg n-2}\cdot 2 + 2^{\lg n-1}\right]$$

$$= a\left[\underbrace{1+1+\dots+1}_{\lg n} + \underbrace{2+2+\dots+2}_{\lg n-1} + \underbrace{2^2+2^2+\dots+2^2}_{\lg n-2} + \dots + \underbrace{2^{\lg n-2}+2^{\lg n-2}}_{2} + \underbrace{2^{\lg n-1}}_{1}\right]$$

(1) $1+2+2^2+\dots+2^{\lg n-2}+2^{\lg n-1} = \sum_{i=0}^{\lg n-1} 2^i$

(2) $1+2+2^2+\dots+2^{\lg n-2} = \sum_{i=0}^{\lg n-2} 2^i$

(3) $1+2+2^2+\dots+2^{\lg n-3} = \sum_{i=0}^{\lg n-3} 2^i$

$$a\left[\sum_{i=0}^{\lg n-1} 2^i + \sum_{i=0}^{\lg n-2} 2^i + \sum_{i=0}^{\lg n-3} 2^i + \dots + \sum_{i=0}^{1} 2^i + \sum_{i=0}^{0} 2^i\right]$$

since that $\sum_{i=0}^{\lg_2 n-1} 2^i = \frac{2^{\lg n}-1}{2-1} = 2^{\lg n}-1$

$$= a\left[\left(2^{\lg n}-1\right) + \left(2^{\lg n-1}-1\right) + \dots + \left(2^2-1\right) + 1\right]$$

$$= a\left[\sum_{i=1}^{\lg n} 2^i - \lg n\right]$$

→ as we have $\log n$ summations, each one subtracting 1 ⇒ $-\lg n$ and we sum $2+2^2+\dots+2^{\lg n-1}+2^{\lg n} = \sum_{i=1}^{\lg n} 2^i$

$$= a\left[\sum_{i=1}^{\lg n} 2^i - \lg n\right]$$

$$\begin{cases} \sum_{i=0}^{\lg n} 2^i = 2^{\lg n+1} - 1 \\ \sum_{i=1}^{\lg n} = 2^{\lg n+1} - 2 \end{cases}$$

$$= a\left[2^{\lg n+1} - 2 - \lg n\right]$$

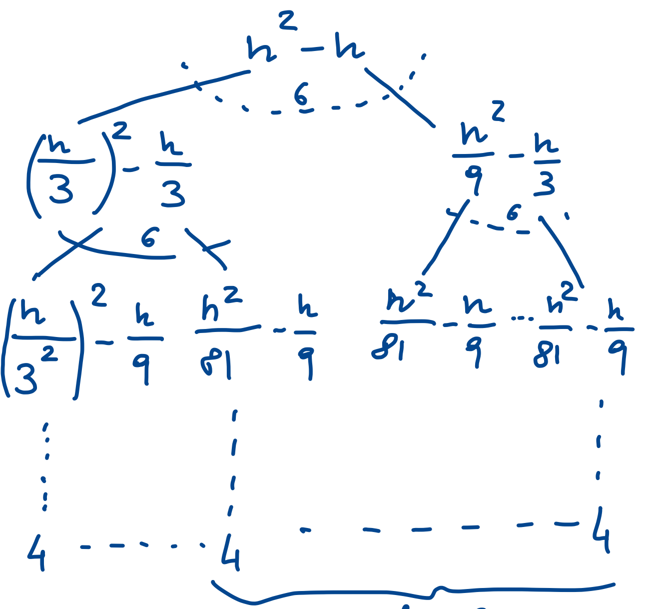
$$= a\left[2\cdot n - 2 - \lg n\right]$$

Thus, $T(n) = nT_0 + a(2n - \lg n - 2) = (T_0 + 2a)n - a\lg n - 2a$ $\square$

Exercise 6

Solve the following recurrence

$$T(n) = \begin{cases} 4, & \text{if } n = 1 \\ 6T\left(\frac{n}{3}\right) + n(n-1), & \text{otherwise} \end{cases}$$



$$n^{\log_3 6}$$

$$n^2 - n$$

$$6\left(\frac{n^2}{9} - \frac{n}{3}\right)$$

$$6^2\left(\frac{n^2}{81} - \frac{n}{9}\right)$$

$$6^j\left(\left(\frac{n}{3^j}\right)^2 - \frac{n}{3^j}\right)$$

$$T(n) = 4n^{\log_3 6} + \sum_{j=0}^{\log_3 n - 1} 6^j\left[\left(\frac{n}{3^j}\right)^2 - \frac{n}{3^j}\right] = 4n^{\log_3 6} + \sum_{j=0}^{\log_3 n - 1}\left[\frac{6^j n^2}{3^{2j}} - \frac{6^j n}{3^j}\right]$$

$$= 4n^{\log_3 6} + n^2\sum_{j=0}^{\log_3 n - 1}\left(\frac{6}{9}\right)^j - n\sum_{j=0}^{\log_3 n - 1} 2^j$$

$$= 4n^{\log_3 6} + 3n^2\left(1 - \left(\frac{2}{3}\right)^{\log_3 n}\right) - n\left(2^{\log_3 n} - 1\right)$$

$$= 4n^{\log_3 2 + 1} + 3n^2 - 3n^{2 + \log_3 2 - 1} - n^{1 + \log_3 2} + n$$

$$= 3n^2 + n$$