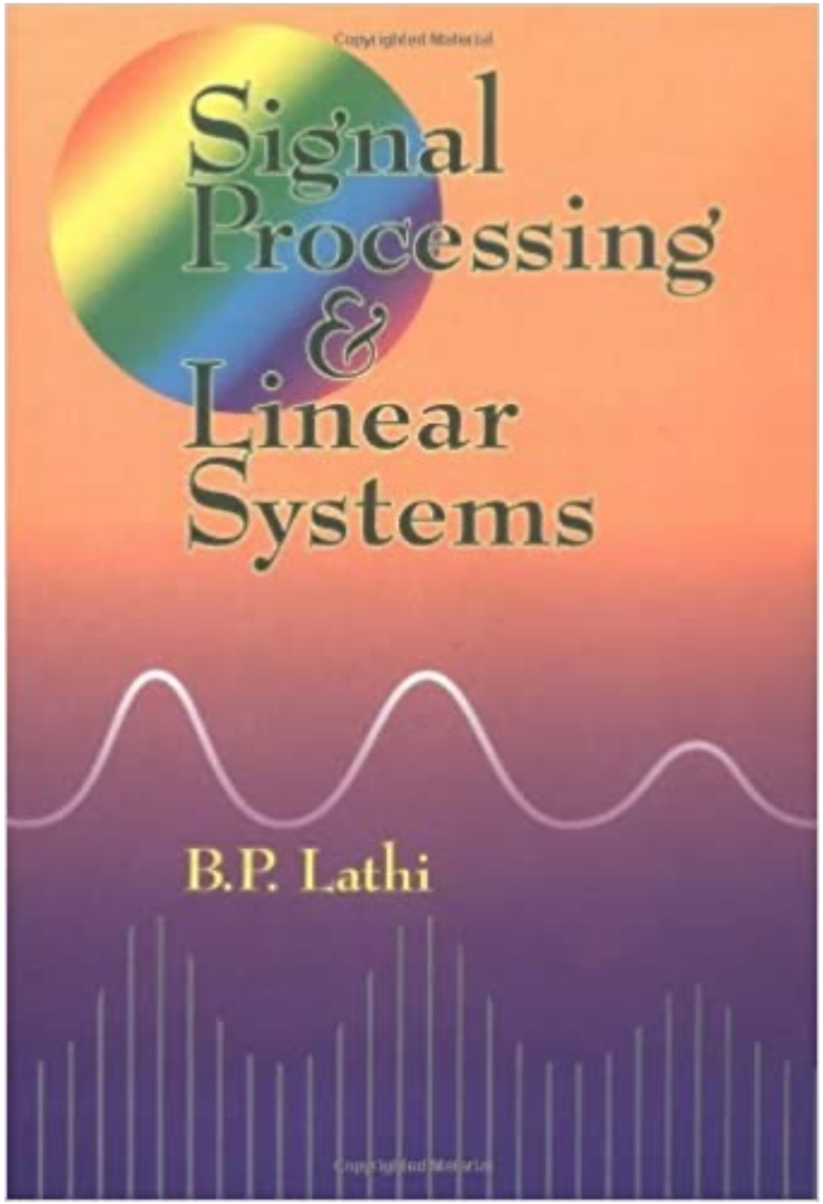
Introduction to Signals and Systems

Lathi Chapt. 1

Textbook

Signal Processing and Linear Systems, B.P. Lathi, CRC Press



Signals&Systems

Input signal

System

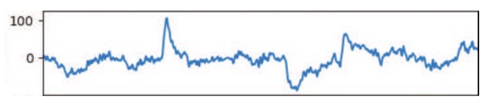
Output signal

Linear time invariant systems (LTIS)

LTIS perform any kind of processing on the input data to generate output data

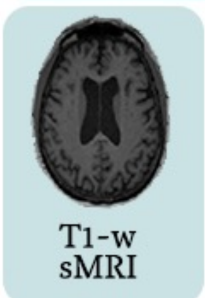
1D signal

Raw EEG

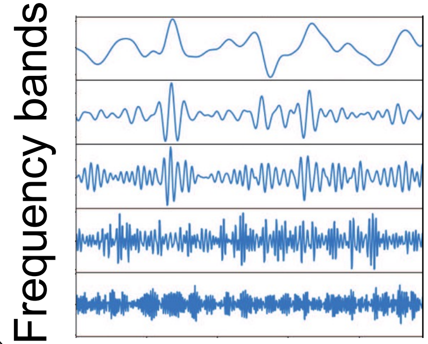


2D signal

Raw T1-MRI



Filtered EEG



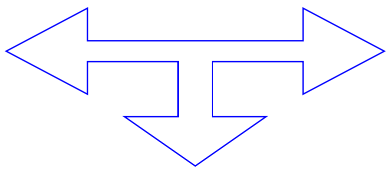
Contents

Signals

- Signal classification and representation
  - Types of signals
  - Sampling theory
  - Quantization
- Signal analysis
  - Fourier Transform
    - Continuous time, Fourier series, Discrete Time Fourier Transforms, Discrete FT, Windowed FT
  - Spectral Analysis

Systems

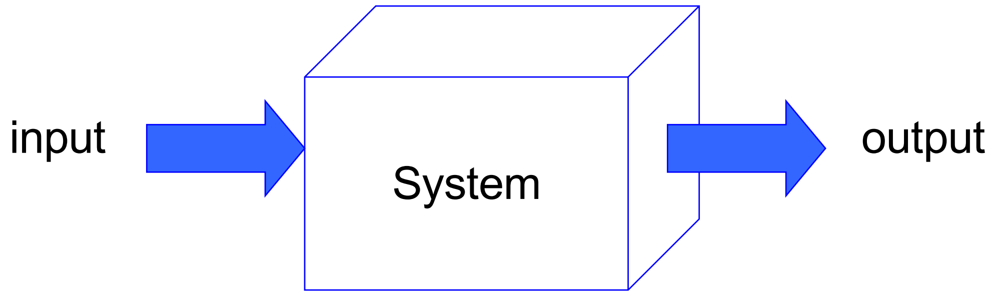
- Linear Time-Invariant Systems
  - Time and frequency domain analysis
  - Impulse response
  - Stability criteria
- Digital filters
  - Finite Impulse Response (FIR)



Applications in the domain of Bioinformatics

What is a system?

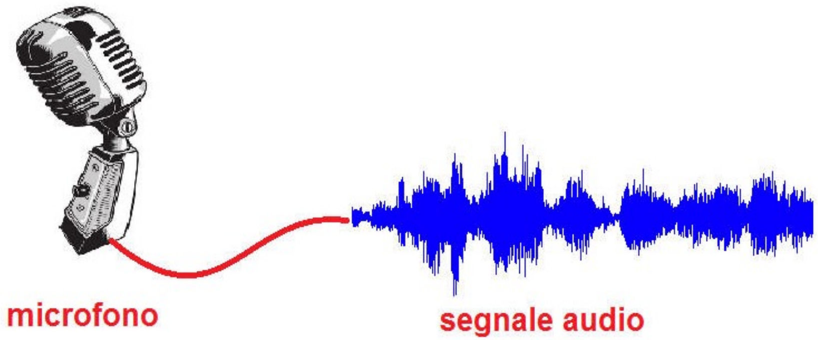
- Systems process signals to
  - Extract information
  - Enable transmission over channels with limited capacity (JPEG, JPEG2000, MPEG coding)
  - Improve security over networks (encryption, watermarking)
  - Support the formulation of diagnosis and treatment planning (medical imaging)
  - …….



The function linking the output of the system with the input signal is called impulse response h(t) in time domain and transfer function $H(\omega)$ in frequency domain

What is a signal?

- A signal is a piece of information of data
  - A signal represents any kind of physical variable subject to variations
  - Both the independent variable and the physical variable can be either scalars or vectors
    - Independent variable: time (t), space (x, $\mathbf{x}=[x_1,x_2]$, $\mathbf{x}=[x_1,x_2,x_3]$)
    - Signals examples
      - Electrochardiography signal (EEG) 1D, voice 1D, music 1D
      - Images (2D), video sequences (2D+time), volumetric data (3D)



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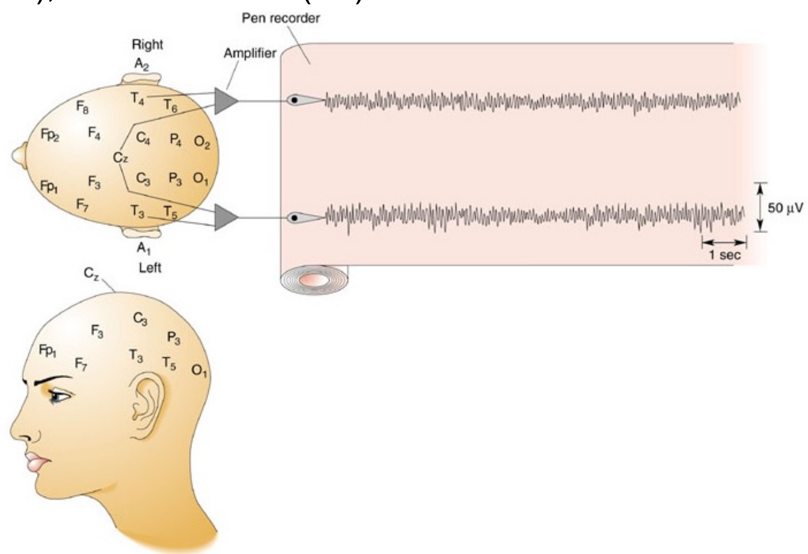
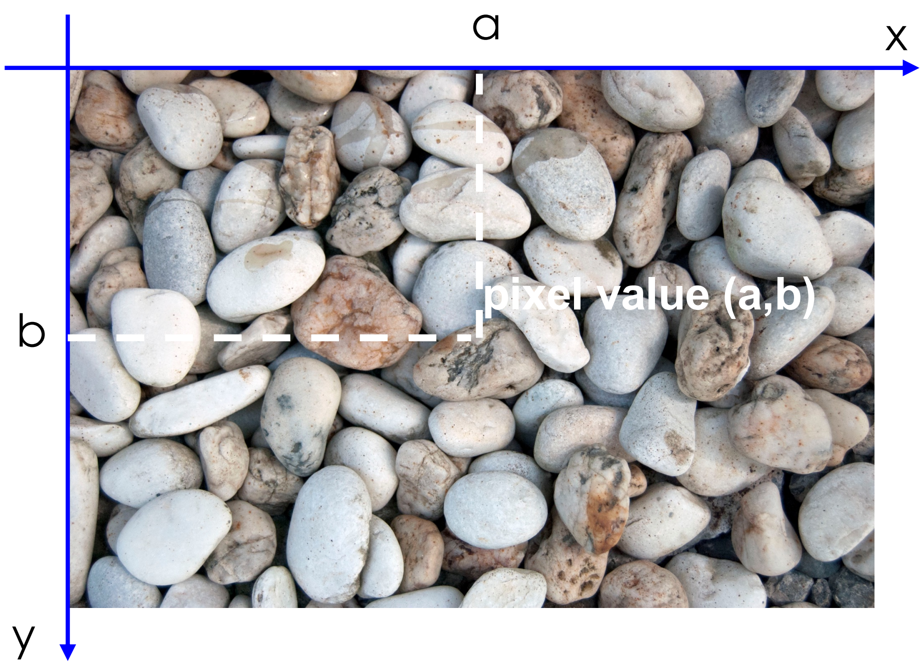


Images are 2D signals



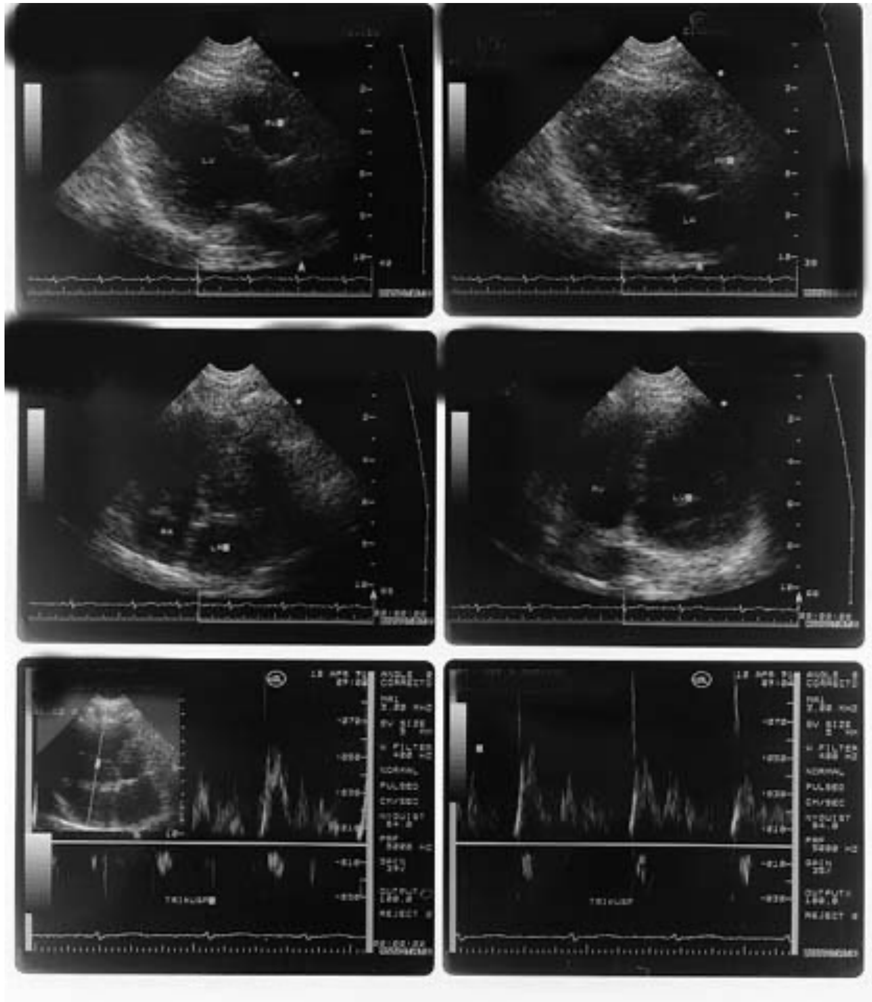
Images are 2D signals

independent variable = spatial

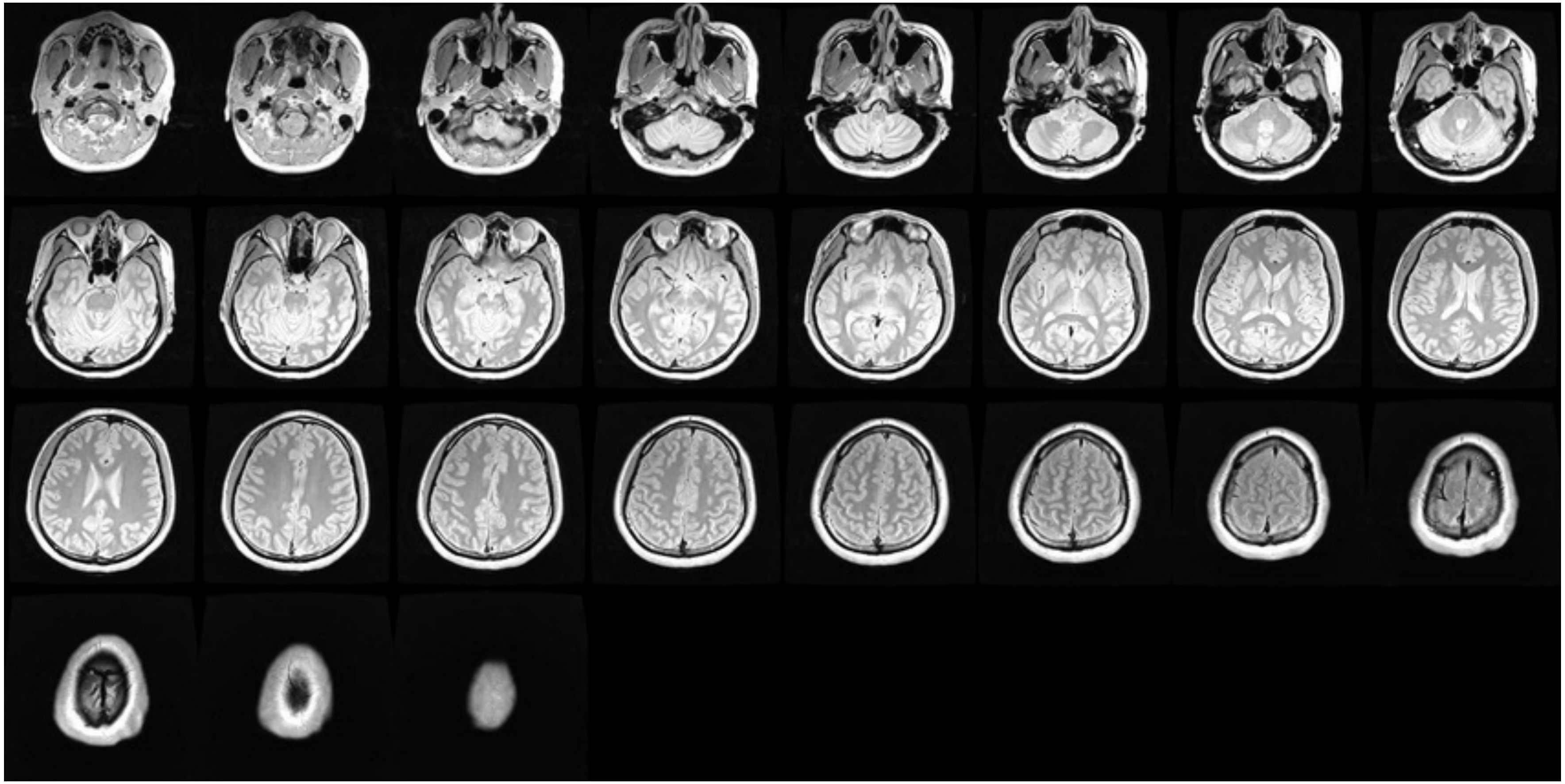
coordinates (x,y)

dependent variable = pixel value

Videos are 2D+time signals



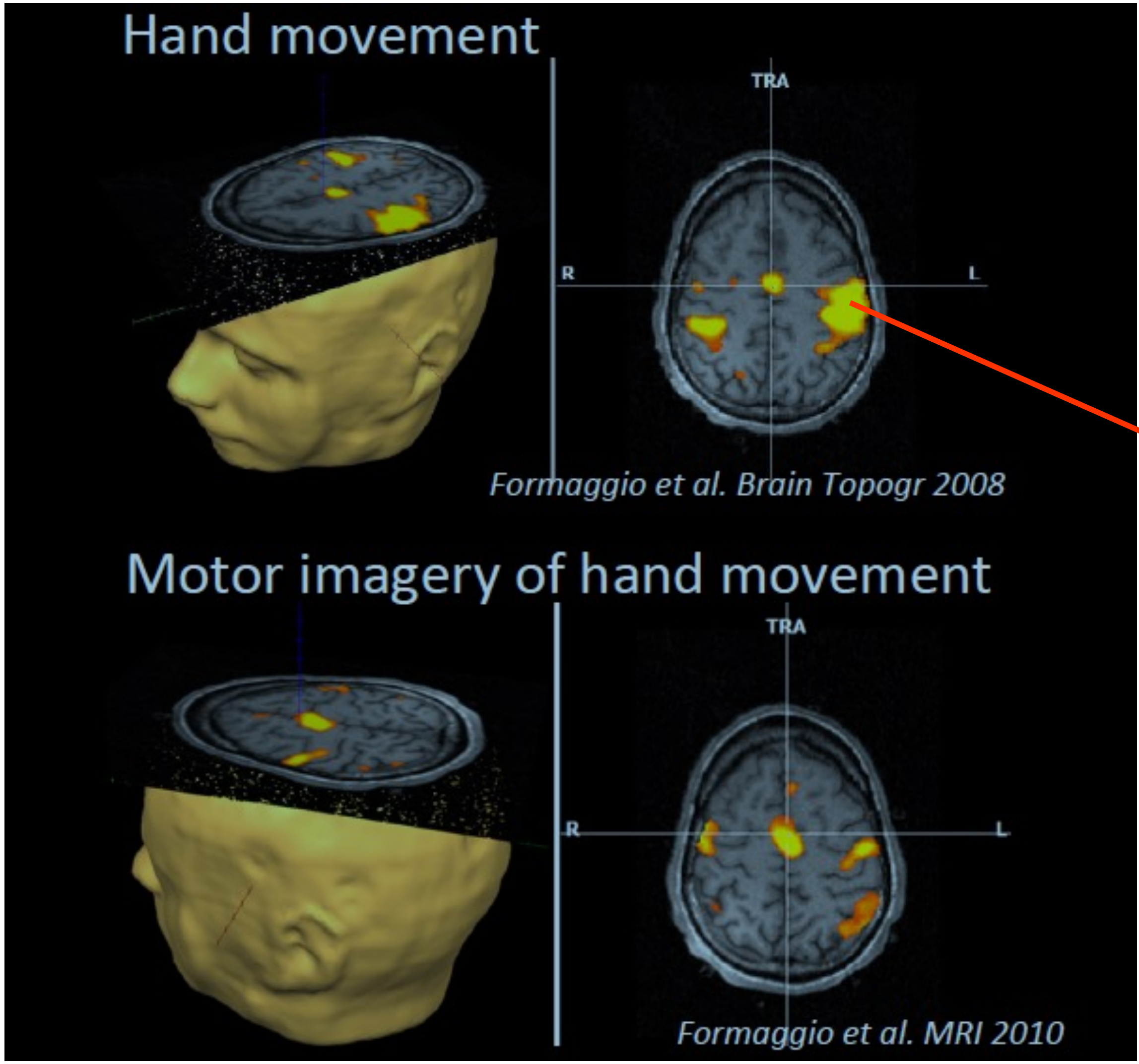
Volumetric data are 3D signals



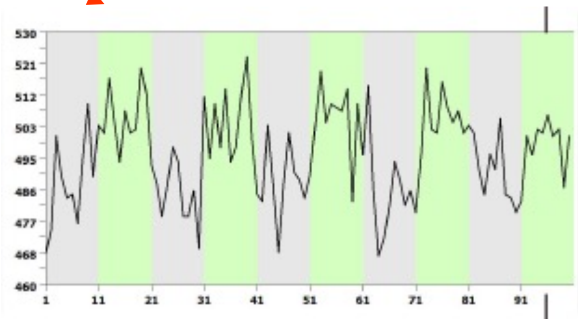
Structural MRI

3D+time signals

Functional MRI (fMRI)



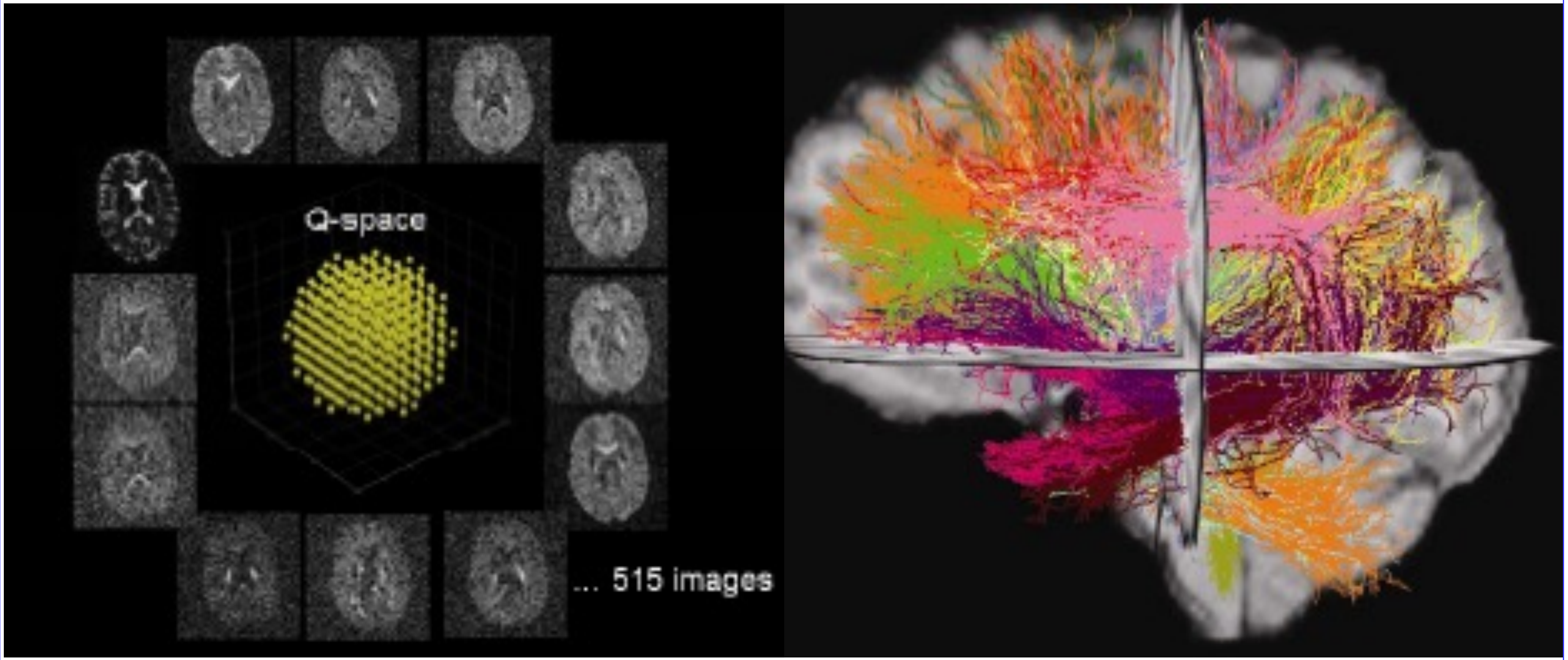
One signal for each voxel



time

Multidimensional signals

Diffusion MRI (dMRI)



n signals per each voxel

Take-home message

- Signals live in a space with *n dimensions*
- Understanding signals is the basis for analysing images, video sequences, volumetric data as well as signals living in higher dimensional spaces

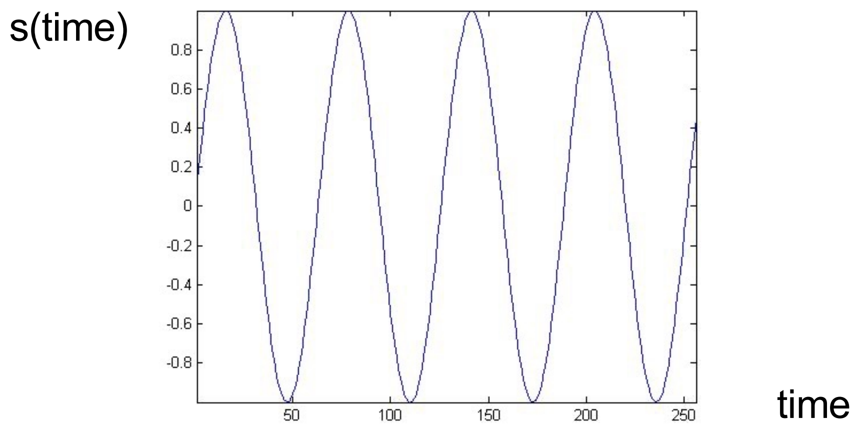
Taxonomy of signals

Classification of signals

- Continuous time – Discrete time
- Analog – Digital (numerical)
- Periodic – Aperiodic
- Energy – Power
- Deterministic – Random (probabilistic)
- Note
  - Such classes are not disjoint, so there are digital signals that are periodic of power type and others that are aperiodic of power type etc.
  - Any combination of single features from the different classes is possible

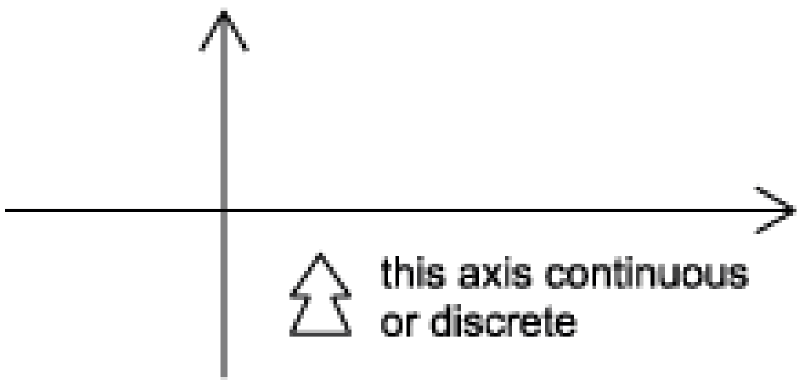
Continuous time – discrete time

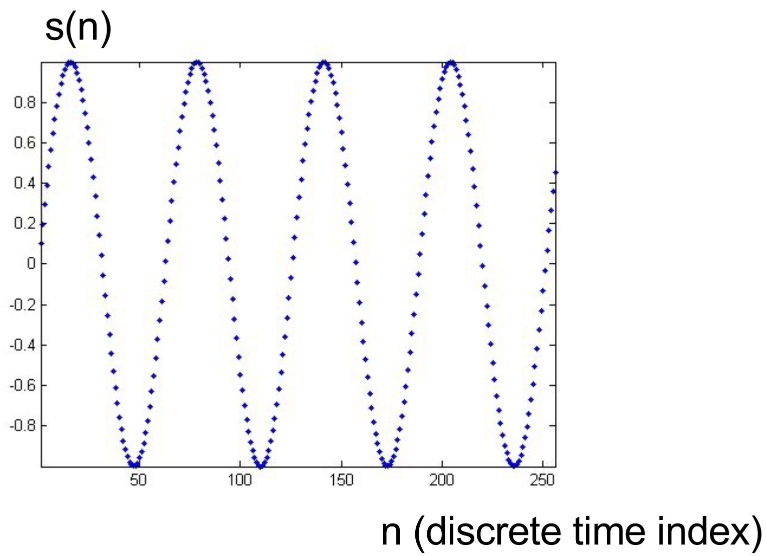
- Continuous time signal: a signal that is specified for every real value of the independent variable
  - The independent variable is continuous, that is it takes any value on the real axis
  - The domain of the function representing the signal has the cardinality of real numbers
    - Signal ↔ f=f(t)
    - Independent variable ↔ time (t), position (x) $t \in \mathbb{R}$
    - For continuous-time signals:



Continuous time – discrete time

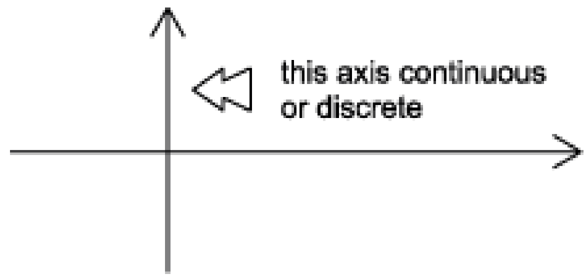
- Discrete time signal: a signal that is specified only for set of values of the independent variable having the cardinality of Z
  - It is usually generated by *sampling* a continuous time signal so it will only have values at *equally spaced* intervals along the time axis
  - The domain of the function representing the signal has the cardinality of integer numbers
    - Signal ↔ f=f[n], also called "sequence"
    - Independent variable ↔ n
    - For discrete-time functions: $t \in \mathbf{Z}$





Analog - Digital

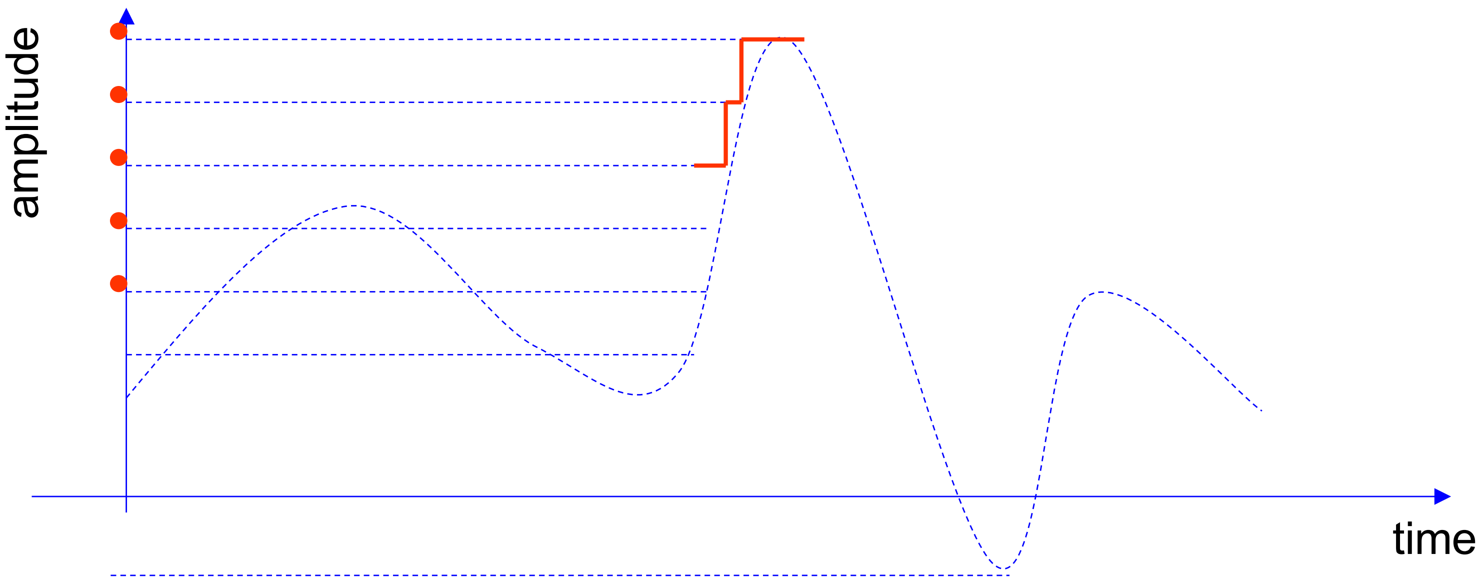
- **Analog signal**: signal whose amplitude can take on any value in a continuous range
  - The codomain of the function f(t) (or f(x)) has the cardinality of real numbers
    - The difference between analog and digital is similar to the difference between continuous-time and discrete-time. In this case, however, the difference is with respect to the codomain of the function (y-axis)
  - Analog corresponds to a “continuous” y-axis, while digital corresponds to a “discrete” y-axis



- An analog signal can be both continuous time and discrete time

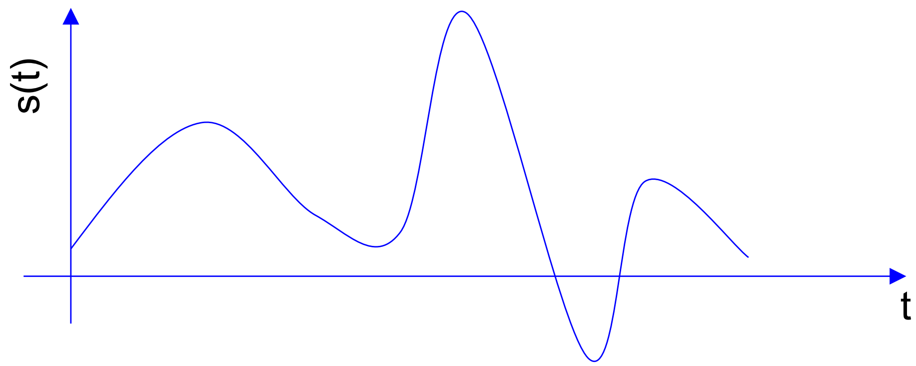
Analog - Digital

- **Digital signal**: a signal whose amplitude can take on only a finite number of values (thus it is *quantized*)
  - The amplitude of the function f() can take only a finite number of values
  - A digital signal whose amplitude can take only M different values is said to be M-ary
    - Binary signals are a special case for M=2



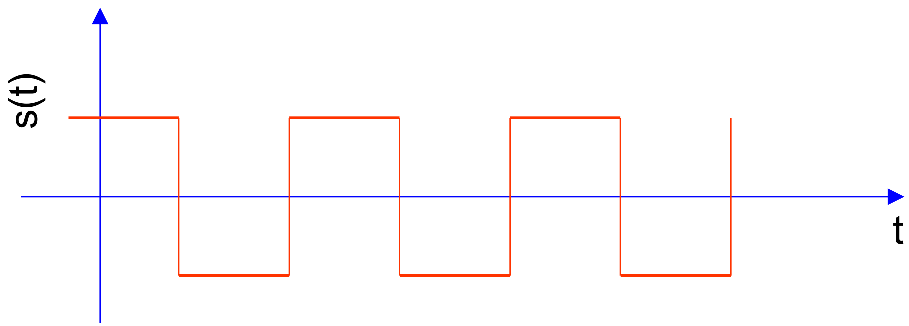
Example

– Continuous time analog signal



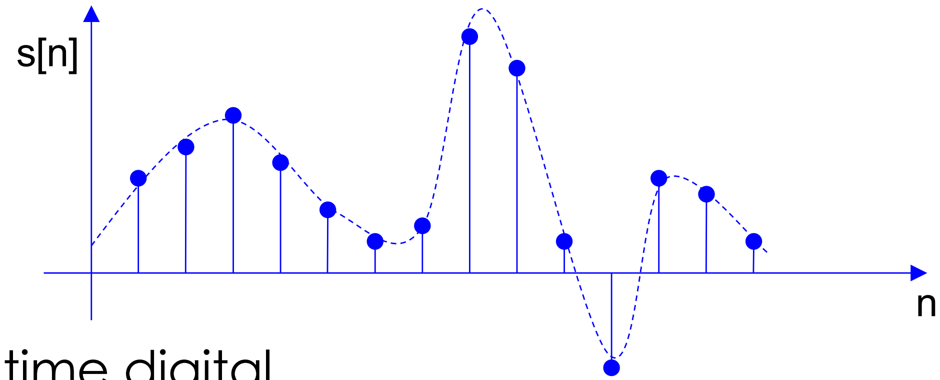
– Continuous time digital (or quantized) signal

- binary sequence, where the values of the function can only be one or zero.

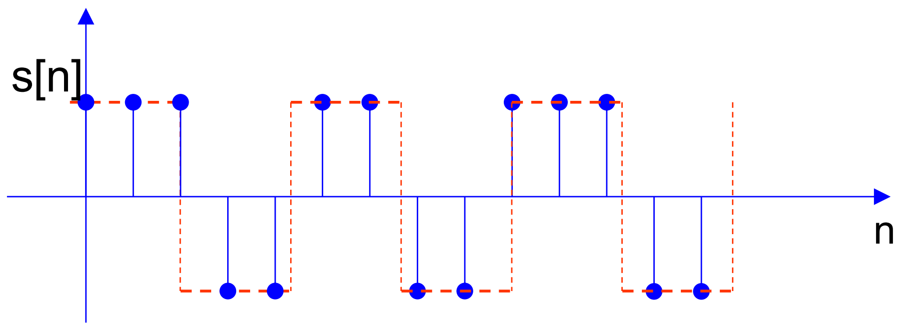


Example

- Discrete time analog



- Discrete time digital
  - binary sequence, where the values of the function can only be one or zero.



Summary

Signal amplitude/ Time or space	Real	Integer
Real	Analog Continuous-time	Digital Continuous-time
Integer	Analog Discrete-time	Digital Discrete time

The definition used here is as in the Lathi textbook.

Periodic - Aperiodic

- A signal f(t) is *periodic* if there exists a positive constant T_0 such that

$$f(t+T_0)=f(t) \qquad \forall t$$

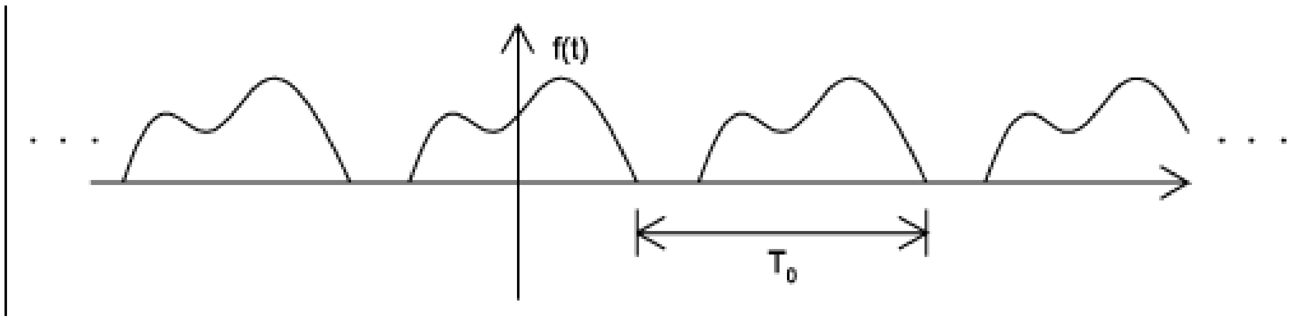
  - The *smallest* value of T_0 which satisfies such relation is said the *period* of the function f(t)
  - A periodic signal remains unchanged when *time-shifted* of integer multiples of the period
  - Therefore, by definition, it starts at minus infinity and lasts forever

$$-\infty \leq t \leq +\infty \text{ for } t \in R$$
$$-\infty \leq n \leq +\infty \text{ for } n \in Z$$

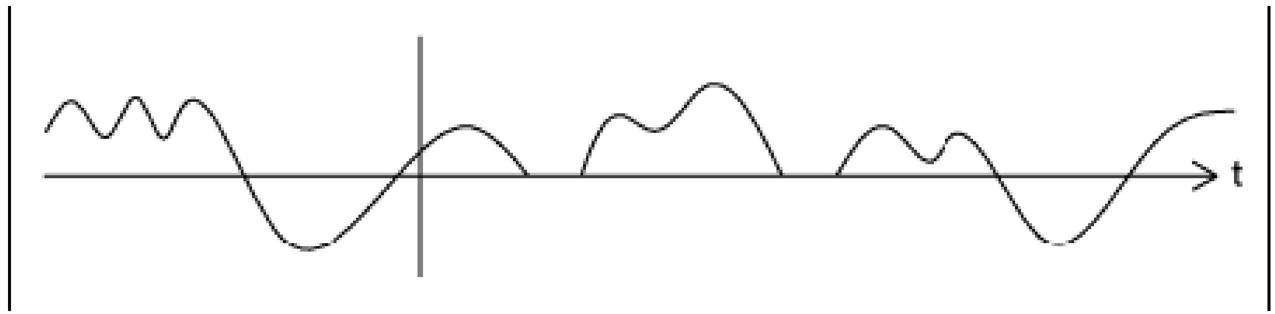
  - Periodic signals can be generated by *periodic extension*

Examples

- Periodic signal with period T_0

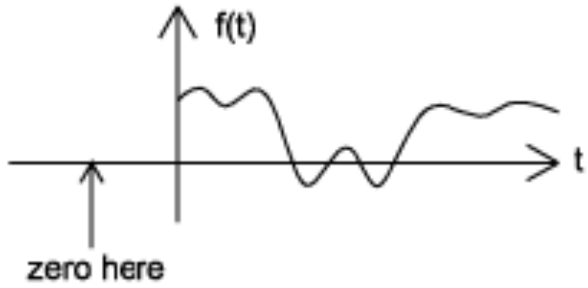


- Aperiodic signal

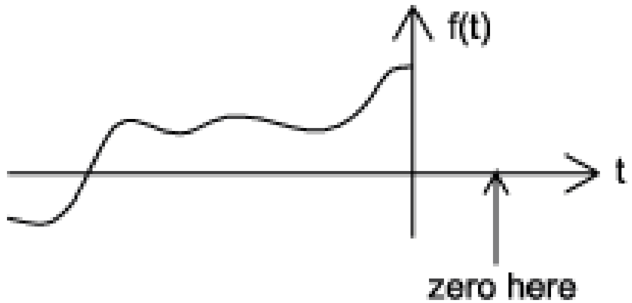


Causal and non-Causal signals

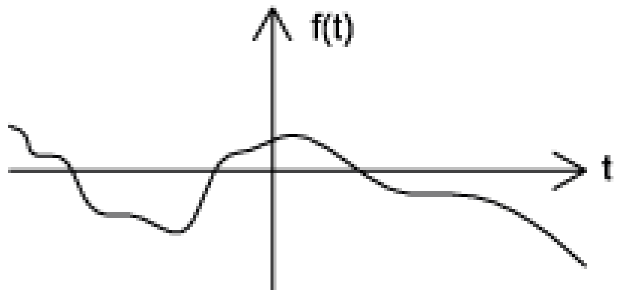
- ***Causal*** signals are signals that are *zero for all negative time (or spatial positions)*, while



- ***Anticausal*** are signals that are zero for all *positive time (or spatial positions)*.



- ***Noncausal*** signals are signals that have nonzero values in both positive and negative time



Causal and non-causal signals

- Causal signals

$$f(t) = 0 \qquad t < 0$$

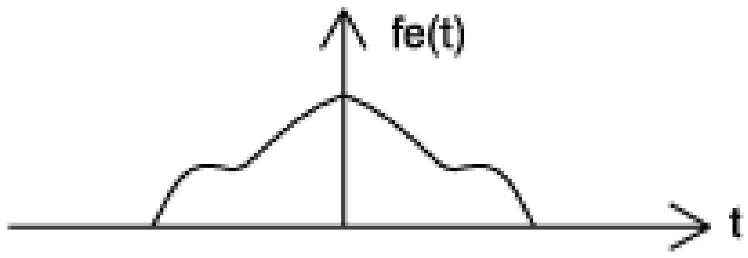
- Anticausals signals

$$f(t) = 0 \qquad t \geq 0$$

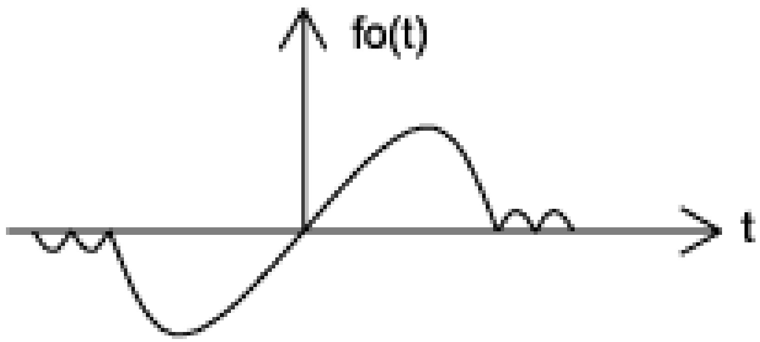
- Non-causal signals

Even and Odd signals

- An even signal is any signal f such that f (t) = f (-t). Even signals can be easily spotted as they are symmetric around the vertical axis.



- An odd signal, on the other hand, is a signal f such that f (t)= - (f (-t))



Decomposition in even and odd components

- Any signal can be written as a combination of an even and an odd signal
  - Even and odd components

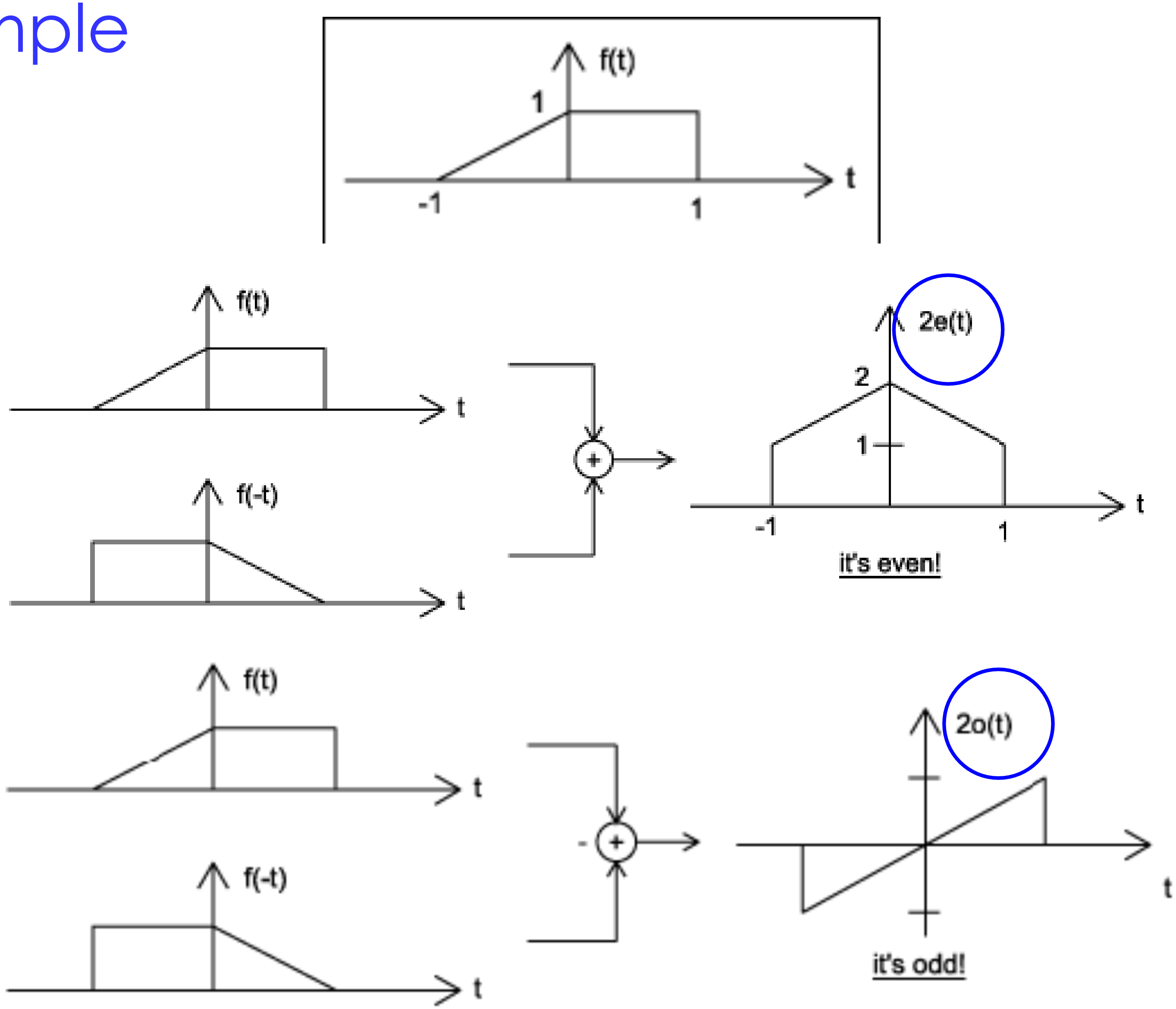
$$f(t) = \frac{1}{2}\left(f(t) + f(-t)\right) + \frac{1}{2}\left(f(t) - f(-t)\right)$$

$$f_e(t) = \frac{1}{2}\left(f(t) + f(-t)\right) \quad \text{even component}$$

$$f_o(t) = \frac{1}{2}\left(f(t) - f(-t)\right) \quad \text{odd component}$$

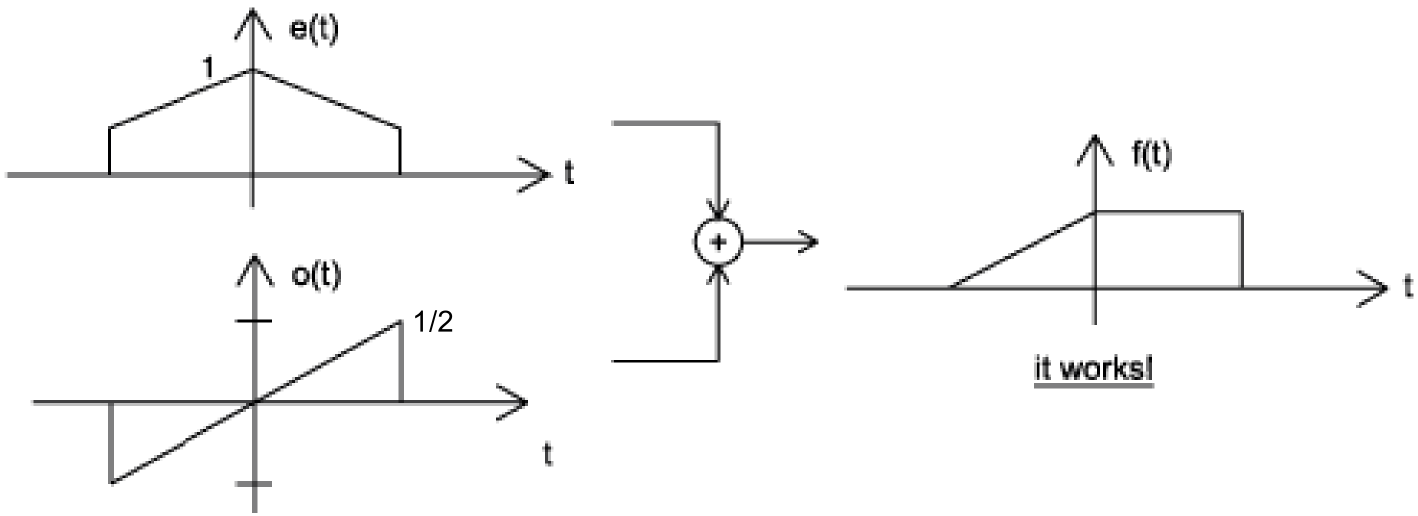
$$f(t) = f_e(t) + f_o(t)$$

Example



Example

- Proof

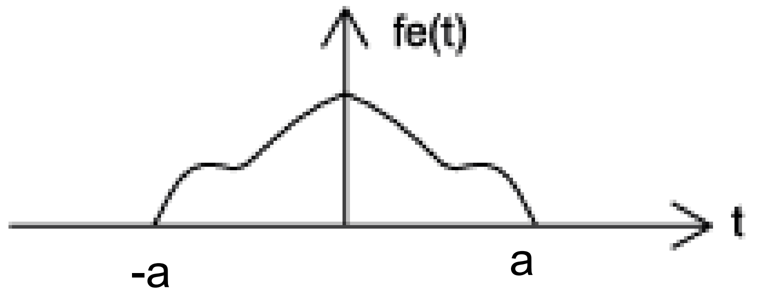


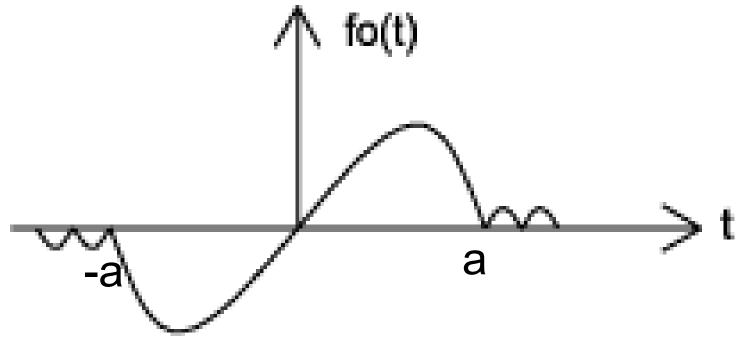
Some properties of even and odd functions

- even function x odd function = odd function
- odd function x odd function = even function
- even function x even function = even function
- Area

$$\int_{-a}^{a} f_e(t)\,dt = 2\int_{0}^{a} f_e(t)\,dt$$

$$\int_{-a}^{a} f_o(t)\,dt = 0$$





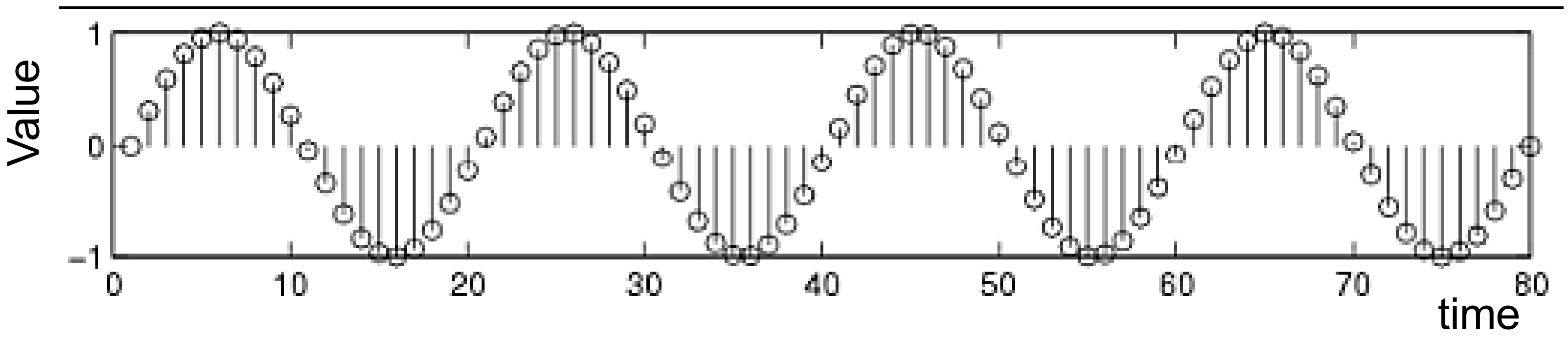
Deterministic – Random

- Deterministic signal: a signal whose *physical description* in known completely
- A deterministic signal is a signal in which each value of the signal is known completely, either in a mathematical or in a graphical form
- Because of this the future values of the signal can be calculated with complete confidence.
  - There is *no uncertainty* about the values
- Example: sinusoid

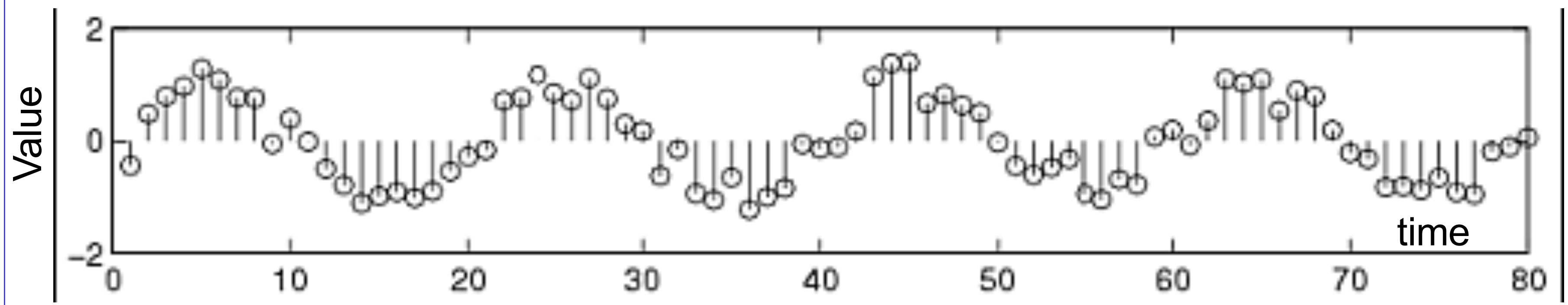
- Random signals are realization of *random processes*
- There is ***uncertainty*** in some parameters of the signal
  - Amplitude, frequency, phase….
- Some descriptors of the random variables can be derived
  - mean value and mean squared value
- The signal values are known within an uncertainty interval
- The future exact values of a random signal cannot be predicted
- ***All measured signals are random***
  - EEG, ECG, audio, scanned images

Example

- Deterministic signal



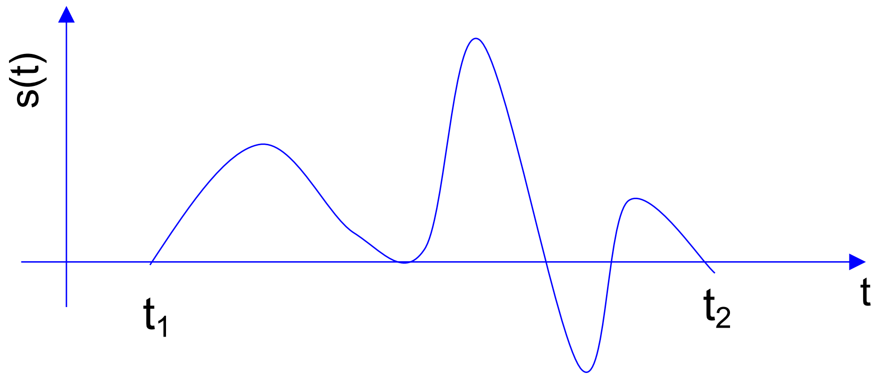
- Signal with random values (realization of a random process). This is called *noisy signal*



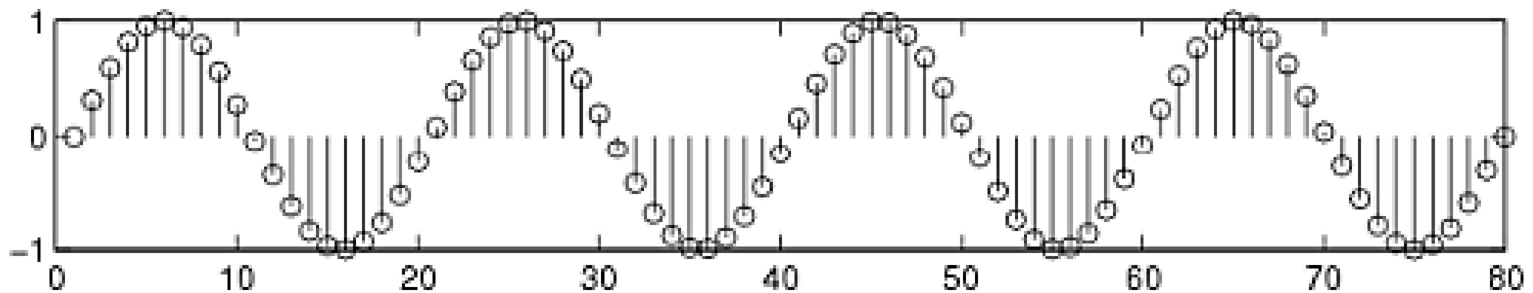
Finite and Infinite length signals

- A finite length signal has a finite *domain*

$$f = f(t), \forall t : t_1 \le t \le t_2$$
$$t_1 > -\infty, t_2 < +\infty$$

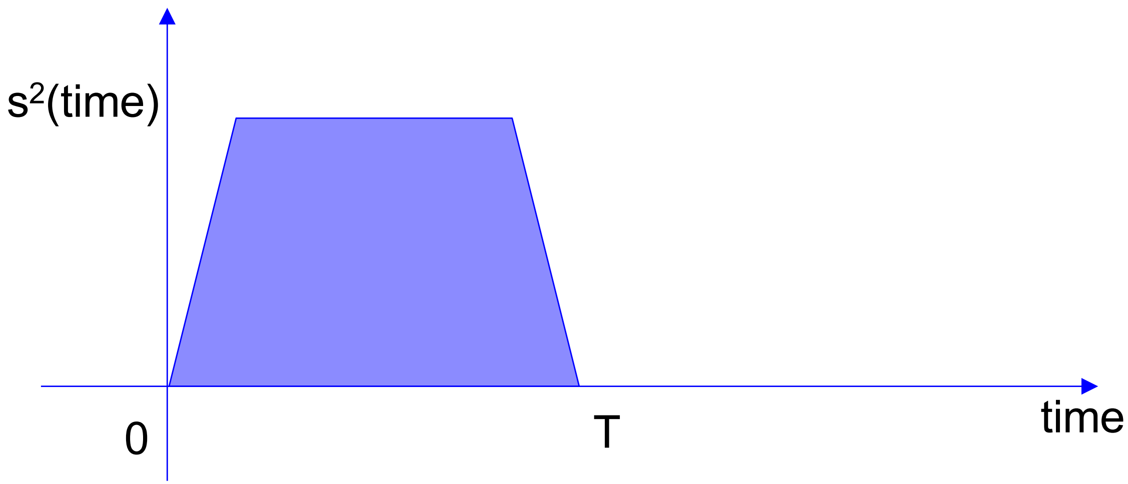


- An infinite length signal has an infinite domain
  - A continuous time sinusoid f(t)=sin(ωt) is an infinite length signal
  - A discrete time sinusoid is an infinite length signal



Size of a signal: Norms

- "Size" indicates largeness or strength
- We will use the mathematical concept of the norm to quantify this notion for both continuous-time and discrete-time signals
- The **energy** is represented by the **area** under the curve of the **squared** signal



Energy

- Signal energy

$$E_f = \int_{-\infty}^{+\infty} f^2(t)dt$$

$$E_f = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt$$

- Generalized energy : L_p norm
  - For p=2 we get the square root of the energy as defined above (L_2 norm)

$$\|f(t)\| = \left(\int \left(|f(t)| \right)^p dt \right)^{1/p}$$

$$1 \leq p < +\infty$$

Power

- **Power**
  - The power is the **time average** (mean) of the squared signal values, that is the *mean-squared* value of f(t)

$$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f^2(t)\,dt$$

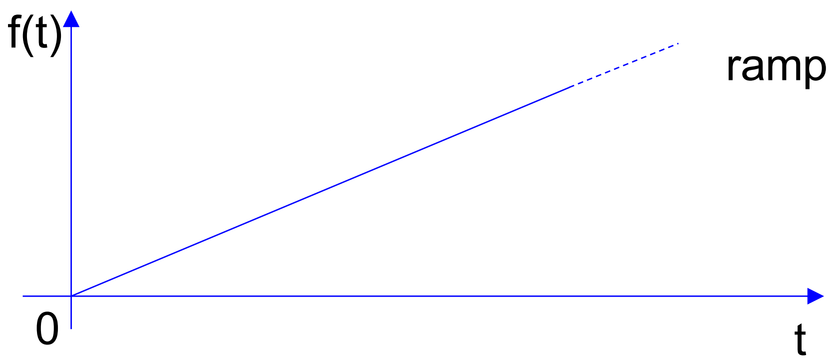
$$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} \left| f(t) \right|^2 dt$$

Power - Energy

- The square root of the power is the **root mean square** (*rms*) value
  - This is a very important quantity as it is the most widespread measure of similarity/dissimilarity among signals
  - It is the basis for the definition of the **Signal to Noise Ratio (SNR)**

$$SNR = 20\log_{10}\left(\sqrt{\frac{P_{signal}}{P_{noise}}}\right)$$

  - It is such that a constant signal whose amplitude is =rms holds the same power content of the signal itself
- There exists (ideal) signals for which neither the energy nor the power are finite

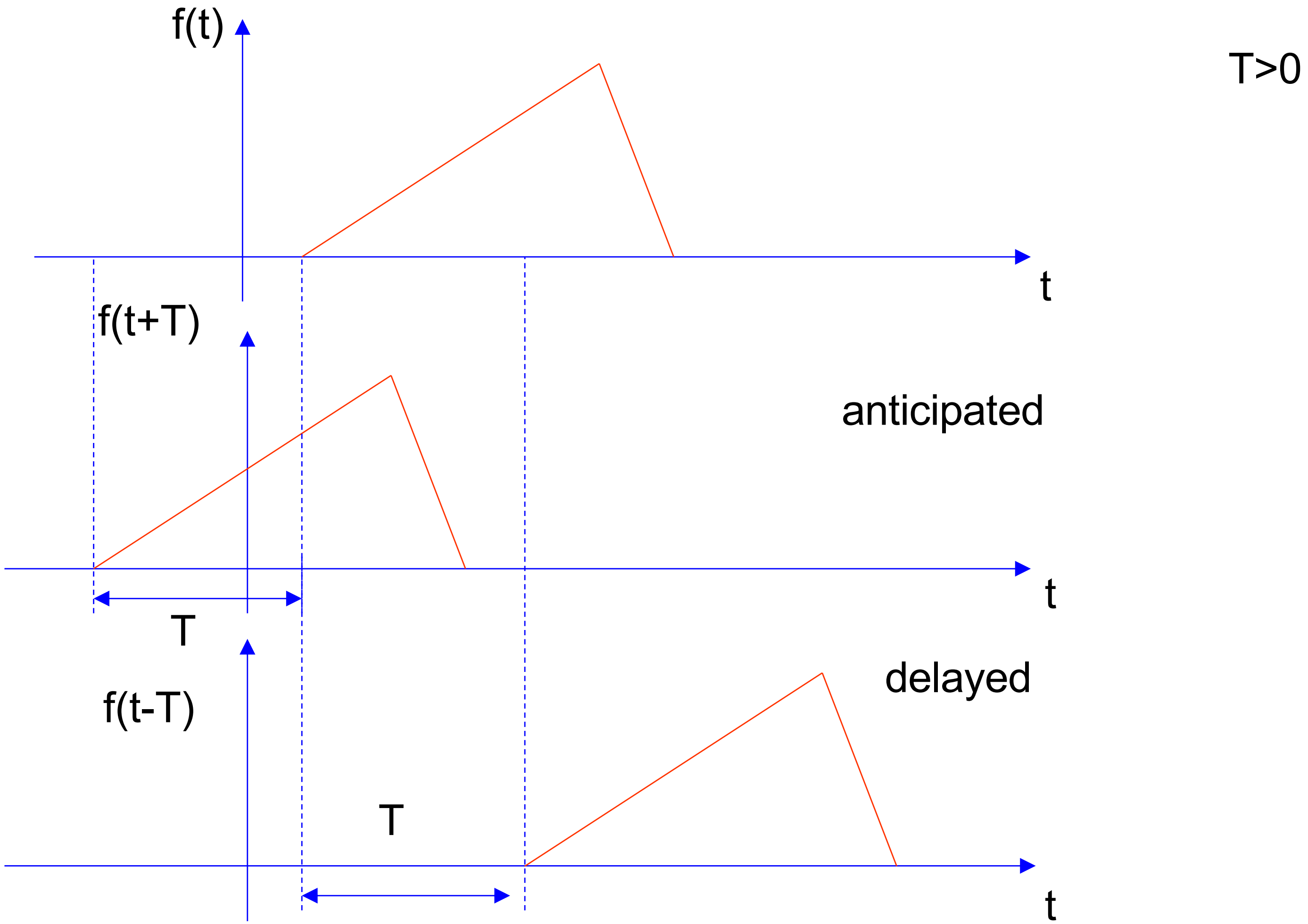


Energy and Power signals

- A signal with finite energy is an **energy signal**
  - Necessary condition for a signal to be of energy type is that the amplitude goes to zero as the independent variable tends to infinity
- A signal with finite and different from zero power is a **power signal**
  - The mean of an entity averaged over an infinite interval exists if either the entity is periodic or it has some statistical regularity
- **A power signal has infinite energy and an energy signal has zero power**
  - There exist signals that are neither power nor energy, such as the ramp
- All **measured signals** have ***finite energy*** and thus are energy signals
  - It is impossible to generate a real power signal because this would have infinite duration and infinite energy, which is not doable.

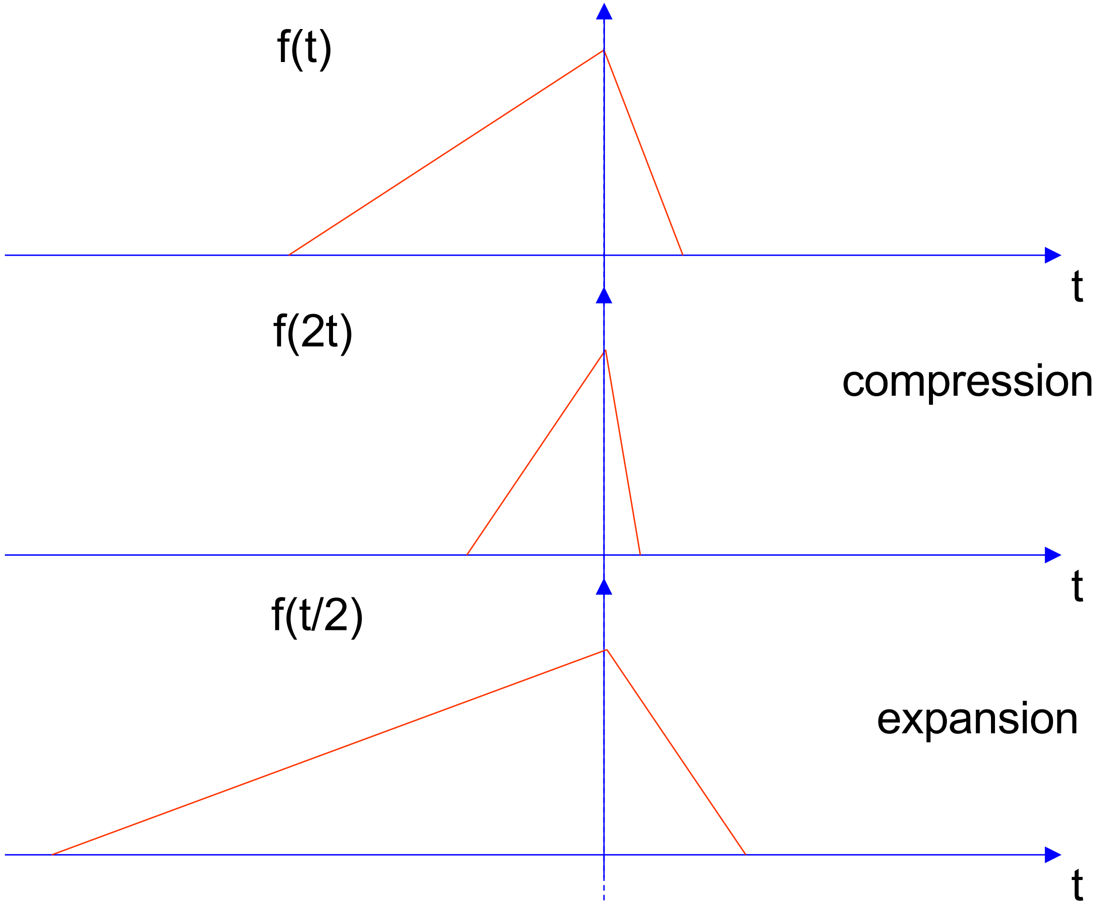
Useful signal operations: shifting, scaling, inversion

- **Shifting**: consider a signal f(t) and the same signal delayed/anticipated by T seconds



Useful signal operations: shifting, scaling, inversion

- (Time) Scaling: compression or expansion of a signal in time



compression $\varphi(t) = f(2t)$

expansion $\varphi(t) = f(t/2)$

Useful signal operations: shifting, scaling, inversion

- Scaling: generalization

$$a > 1$$

$$\varphi(t) = f(at) \;\rightarrow\; \text{compressed version}$$

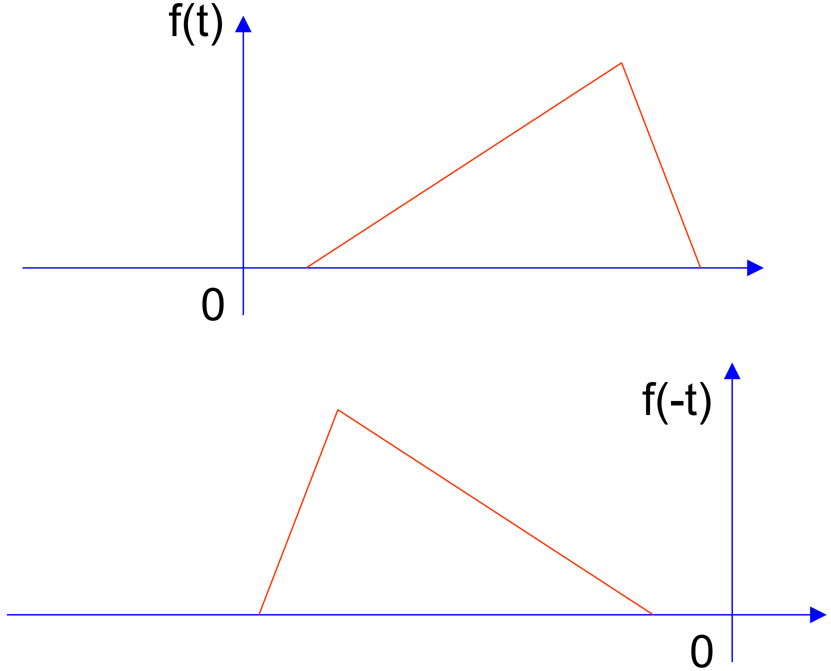
$$\varphi(t) = f\left(\frac{t}{a}\right) \rightarrow \text{dilated (or expanded) version}$$

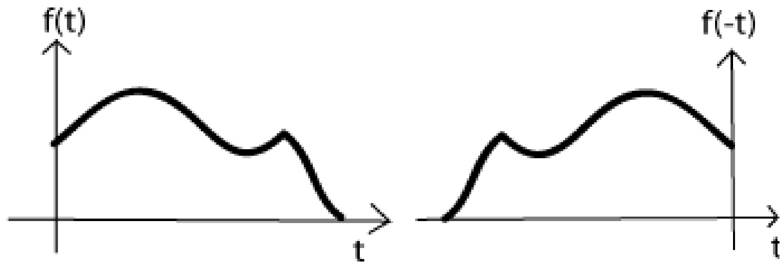
$$\text{Viceversa for } a < 1$$

Useful signal operations: shifting, scaling, inversion

- (Time) inversion: mirror image of f(t) about the vertical axis

$$\varphi(t) = f(-t)$$





Shifting, scaling, inversion

- Combined operations: f(t) → f(at-b)
- Two possible sequences of operations
  - Time shift f(t) by to obtain f(t-b). Now time scale the shifted signal f(t-b) by a to obtain f(at-b).
  - Time scale f(t) by a to obtain f(at). Now time shift f(at) by b/a to obtain f(at-b).
  - Note that you have to replace t by (t-b/a) to obtain f(at-b) from f(at) when replacing t by the translated argument (namely t-b/a))

Popular (and useful) signals

Unit step and exponential signals

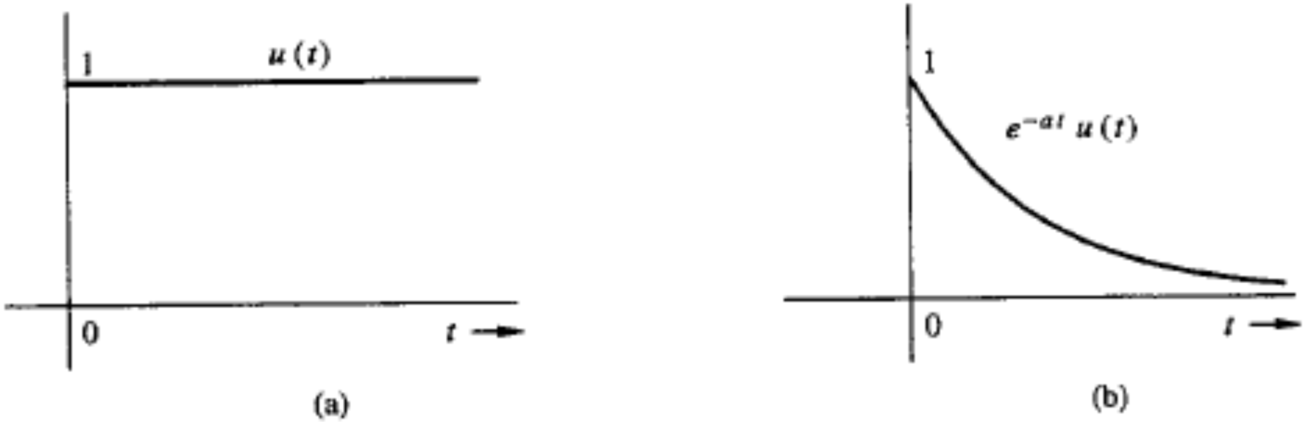


Fig. 1.14 (a) Unit step function $u(t)$ (b) exponential $e^{-at}u(t)$.

Box signal can be built using unit step signals

- The unit step function
  - Useful for representing causal signals

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

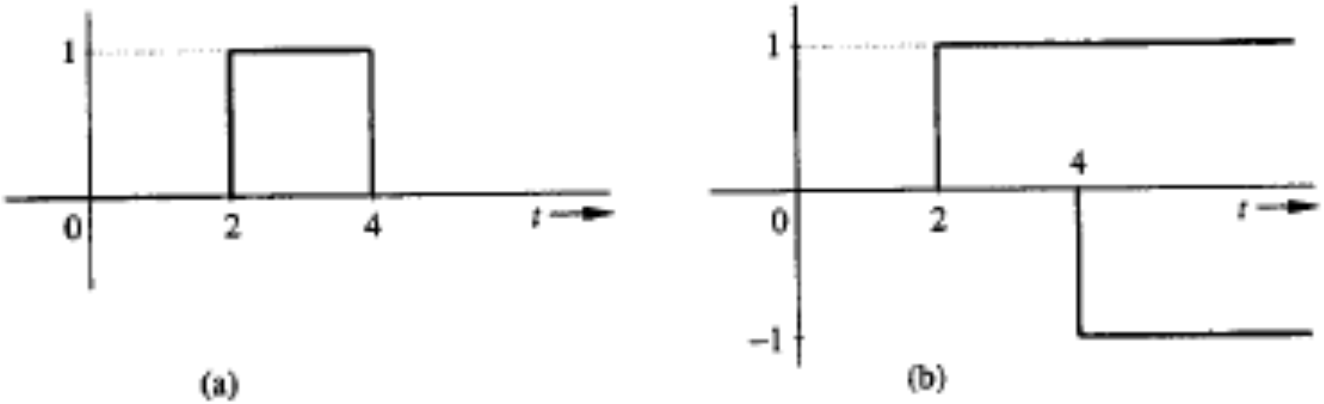
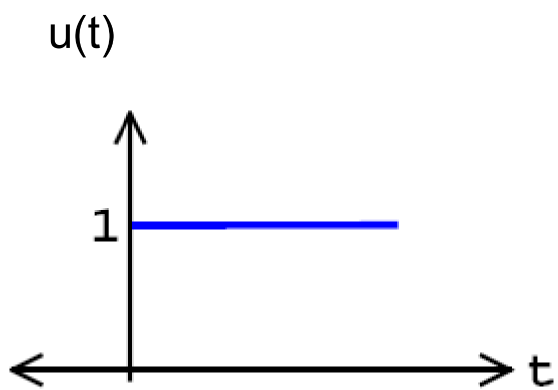


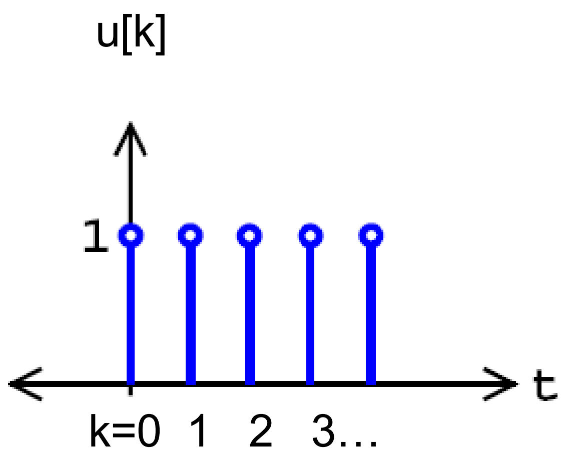
Fig. 1.15 Representation of a rectangular pulse by step functions.

$$f(t)=u(t-2)-u(t-4)$$

Discrete version of the unit step

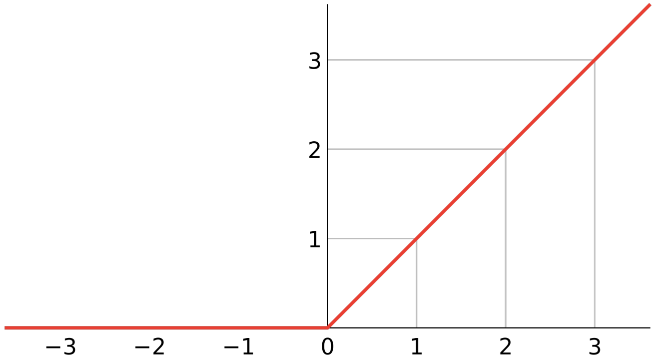
- Continuous and discrete time unit step functions





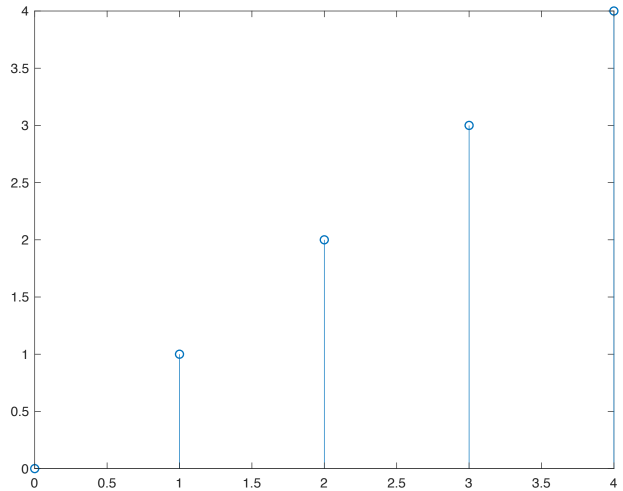
Ramp signal in an interval

Continuous time



$$R(x) := \begin{cases} x, & x \geq 0; \\ 0, & x < 0 \end{cases}$$

Discrete time approximation

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Matlab code
n=0:4;
stem(n,n)
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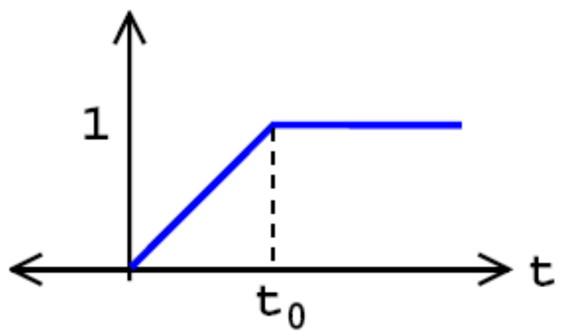
Other signals

- The ramp function followed by a plateau (continuous time)

$$r(t) = \begin{cases} 0 \text{ if } t < 0 \\ \frac{t}{t_0} \text{ if } 0 \leq t \leq t_0 \\ 1 \text{ if } t > t_0 \end{cases}$$

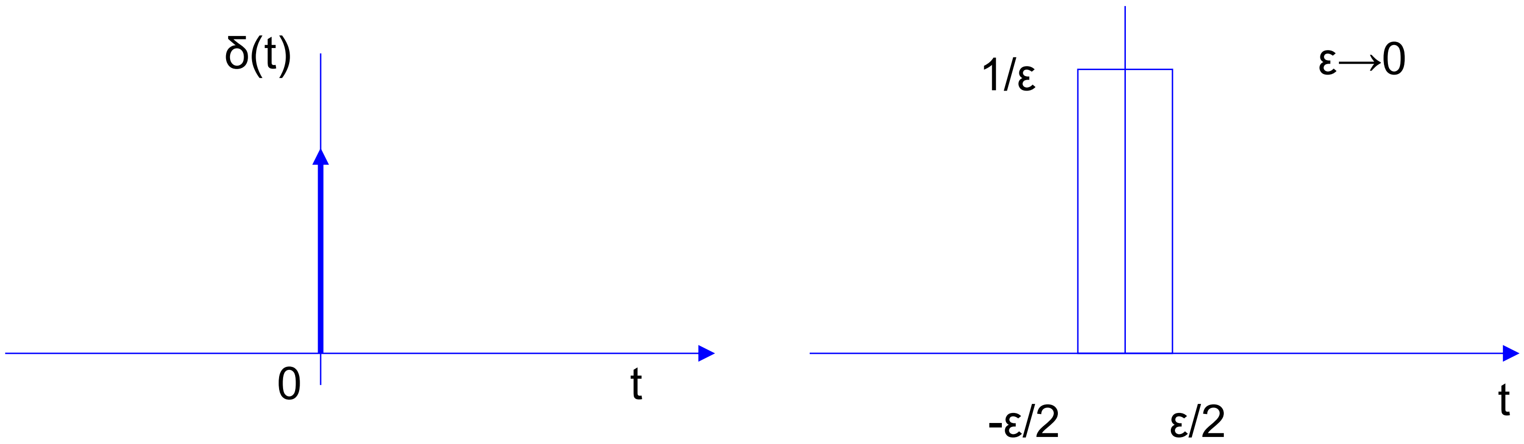


Impulse (delta) function

- The unit impulse function

$$\delta(t) = 0, t \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$



The unit delta function is zero for all t except t=0 and its integral is equal to one

Properties of the unit impulse function

- Multiplication of a function by impulse

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$
$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

- **Sampling property of the unit function**

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t)\,dt = \int_{-\infty}^{+\infty} \phi(0)\delta(t)\,dt = \phi(0)\int_{-\infty}^{+\infty} \delta(t)\,dt = \phi(0)$$

$$\boxed{\int_{-\infty}^{+\infty} \phi(t)\delta(t-T)\,dt = \phi(T)}$$

- The area under the curve given by the product of the unit impulse function shifted by T and $\varphi(t)$ is the value of the function $\varphi(t)$ for t=T

Properties of the unit impulse function

- The unit step function is the integral of the unit impulse function

$$\frac{du}{dt} = \delta(t)$$

$$\int_{-\infty}^{t} \delta(t)\,dt = u(t)$$

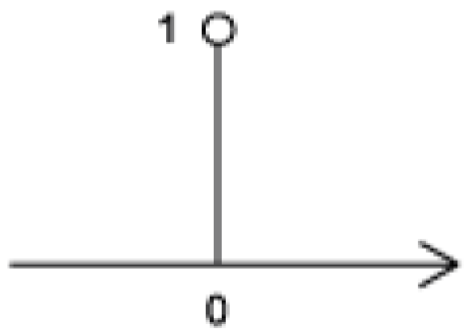
  - Thus

$$\int_{-\infty}^{t} \delta(t)\,dt = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Properties of the unit impulse function

- Discrete time impulse function

$$\delta[n] = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ otherwise} \end{cases}$$



Useful functions

- The continuous time complex exponential

$$f(t) = Ae^{j\omega t}$$

- Euler's relations

$$Ae^{jwt} = Acos(\omega t) + j(Asin(\omega t))$$

$$cos(\omega t) = \frac{e^{jwt} + e^{-(jwt)}}{2}$$

$$sin(\omega t) = \frac{e^{jwt} - e^{-(jwt)}}{2j}$$

$$e^{jwt} = cos(\omega t) + jsin(\omega t)$$

- Discrete time complex exponential
  - n: time index
  - k: frequency index

$$f[n] = \exp j\frac{2\pi k}{N}n$$

Useful functions

- Exponential function e^{st}
  - Generalization of the function $e^{j\omega t}$

$$s = \sigma + j\omega$$

Therefore

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(\cos \omega t + j \sin \omega t) \tag{1.30a}$$

If $s^* = \sigma - j\omega$ (the conjugate of s), then

$$e^{s^*t} = e^{\sigma - j\omega} = e^{\sigma t}e^{-j\omega t} = e^{\sigma t}(\cos \omega t - j \sin \omega t) \tag{1.30b}$$

and

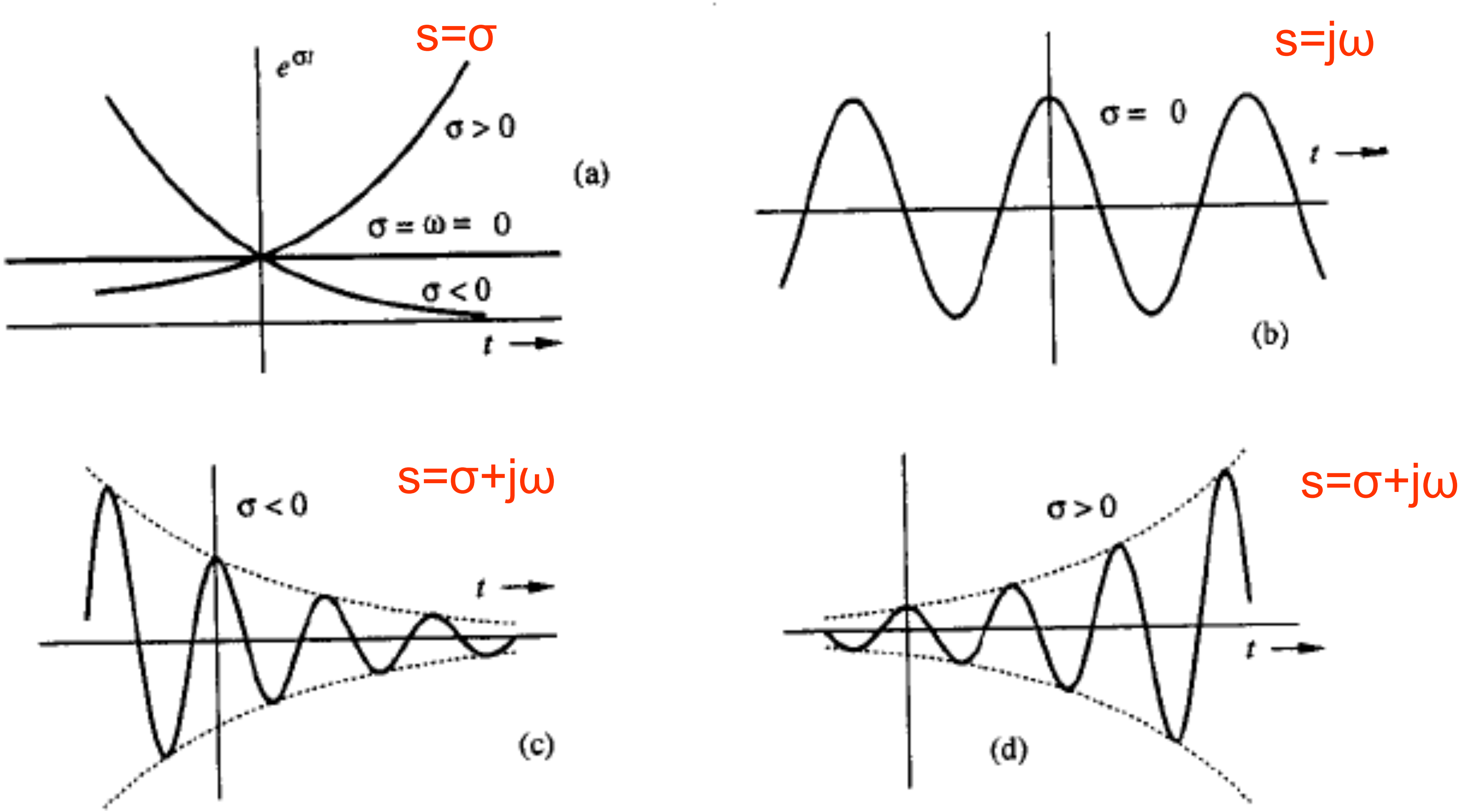
$$e^{\sigma t} \cos \omega t = \frac{1}{2}(e^{st} + e^{s^*t}) \tag{1.30c}$$

The exponential function

Special cases

- A constant $K = K \times \exp 0$
- A monotonic exponential $\exp^{\sigma t}(\omega = 0, s = \sigma)$
- A trigonometric function $\sigma = 0, s = j\omega$
- A trigonometric function with exponential envelop $s = \sigma + j\omega$

The exponential function



Complex frequency plan