Linear Algebra - Cheat Sheet (HS22)

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1 Complex Numbers

General

$z = \underbrace{x}_{\text{Re}} + i \underbrace{y}_{\text{Im}} = \underbrace{r \cdot (\cos(\varphi) + i \cdot \sin(\varphi)) = r \cdot e^{i\varphi}}_{\text{Polarform}}$

$\overline{z} = x - iy = r \cdot e^{i(2\pi - \varphi)}$

$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$

$$\varphi = \begin{cases} arctan(\frac{y}{x}), & \text{I Q.} \\ arctan(\frac{y}{x}) + \pi, & \text{II/III Q.} \\ arctan(\frac{y}{x}) + 2\pi, & \text{IV Q.} \end{cases}$$

Operations

$+/-: \; (x_1 + x_2) + (y_1 + y_2)i$

$z_1 \cdot z_2: \; (x_1 + y_1 i)(x_2 + y_2 i) = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$

$\frac{z_1}{z_2}: \; \frac{z_1 \cdot \overline{z_2}}{|z_2|^2} = \frac{r_1}{r_2} \cdot e^{i(\varphi_1 - \varphi_2)}$

$z^n: \; r^n \cdot e^{i\varphi n}$

$$\sqrt{a}: \; a = z^n \Leftrightarrow |a| \cdot e^{i\alpha} = r^n \cdot e^{i\varphi n} \begin{cases} r = \sqrt[n]{|a|} \\ \varphi = \frac{\alpha + 2k\pi}{n} \\ k=0,...,n-1 \end{cases}$$

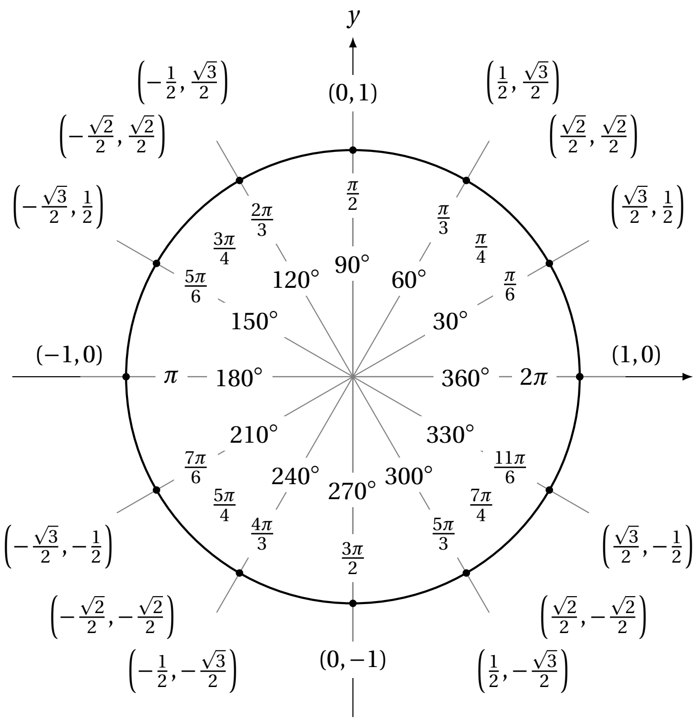
Polynomials

degree 2: $z = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

special case: $az^n + c = 0 \Leftrightarrow z = \sqrt[n]{-\frac{c}{a}}$

With polynomials with complex roots, the roots occur as a complex-conjugate pair.

Polynomials over $\mathbb{C}$ with an odd degree have at least one root in $\mathbb{R}$.



2 LSE

Gauss-Algorithm

runtime: $O(n^3)$

elementary row operations:

switch, multiply and add/subtract rows

goal: row echelon form

$$\begin{array}{cccc|c} x_1 & x_2 & ... & x_n & \\ \hline a_{11} & a_{12} & ... & a_{1n} & b_1 \\ 0 & a_{22} & ... & a_{2n} & b_2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & a_{rn} & b_r \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & b_m \end{array}$$

rank (rows 1 to r); consistency conditions (rows below r); LHS (coefficient columns); RHS (b column)

if RHS always 0: homogeneous LSE

consistency conditions (VB): $b_{r+1} = ... = b_m = 0$

Number of solutions

T 1.1 $Ax = b$ min. 1 solution $\Leftrightarrow$ $(r = m)$ or $(r < m + VB)$ if this is the case:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions

$\Rightarrow r = m$:

- $r = n = m$: only 1 solution, non-singular LSE
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

$\Rightarrow r < m$:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

K 1.5 In a homogeneous LSE non-trivial solutions exist only if $r < n$.

T 2.5 $Ax = b$ has a solution $\Leftrightarrow$ b is a linear combination of columnvectors of A

3 Matrices & Vectors

General

mxn Matrices have m rows and n columns.

The element (i, j) can be denoted as $a_{i,j}$ or $(A)_{i,j}$

T 2.1

$(\alpha\beta)A = \alpha(\beta A)$	$(A + B) + C = A + (B + C)$
$(\alpha A)B = \alpha(AB)$	$(AB) \cdot C = A \cdot (BC)$
$(\alpha + \beta) \cdot A = \alpha A + \beta A$	$(A + B) \cdot C = AC + BC$
$\alpha(A + B) = \alpha A + \alpha B$	$A \cdot (B + C) = AB + AC$
$A + B = B + A$	

⚠ in general $AB \neq BA$

If $AB = BA$ we say «A and B commute»

Def. If $AB = O$, we call A and B divisors of zero.

Def. A linear combination of vectors $a_1 ... a_n$ is an expression of the following type: $\alpha_n \cdot a_n + ... + \alpha_1 \cdot a_1$

Def. A matrix is symmetric when $A^T = A$ and Hermitian when $A^H = A$ (real diagonal).

Def. A matrix is skew-symmetric when $A^T = -A$. (zeros on diagonal)

T 2.6 $(A^T)^T = A$ $\quad$ $(\alpha A)^T = \alpha(A^T)$

$(AB)^T = B^T A^T$ $\quad$ $(A + B)^T = A^T + B^T$

T 2.7 If A, B symmetric: $AB = BA \Leftrightarrow AB$ symmetric

For any A: $A^T A = AA^T$ (symmetric)

Scalar Product and Norm

Def. Eucl. scalar product (SP): $\langle x, y\rangle := x^H y$
(inner product)

T 2.9 linearity in second factor: $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
$\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$
symmetric / hermitian: $\langle x, y\rangle = \overline{\langle y, x\rangle}$
positive definite: $\langle x, x\rangle \geq 0$; if $'='\Rightarrow\ x = 0$

C 2.10 bilinearity in $\mathbb{R}^n$: $\langle w+x, y\rangle = \langle w, y\rangle + \langle x, y\rangle$
$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$
sesquilinearity in $\mathbb{C}^n$: $\langle w+x, y\rangle = \langle w, y\rangle + \langle x, y\rangle$
$\langle \alpha x, y\rangle = \overline{\alpha}\langle x, y\rangle$

Def. Eucl. norm / 2-norm: $||x|| :\equiv \sqrt{\langle x, x\rangle}$

Cauchy-Schwarz inequality (CBS inequality)
$|\langle x, y\rangle| \leq ||x|| \cdot ||y||$
The equality holds iff y is a multiple of x or vice versa

T 2.12 The following holds for the 2-norm:
(N1) positive definite: $||x|| \geq 0$, if $'='\Rightarrow\ x = 0$
(N2) $||\alpha x|| = |\alpha|\ ||x||$
(N3) triangle inequality: $||x \pm y|| \leq ||x|| + ||y||$

Def. angle φ between x and y: $\varphi = arc\ cos\left(\frac{\langle x, y\rangle}{||x||\cdot||y||}\right)$

Def. x and y are orthogonal, if $\langle x, y\rangle = 0$; $x \perp y$

T 2.13 Pythagoras: $||x \pm y||^2 = ||x||^2 + ||y||^2$, if $x \perp y$

Def. p-Norm: $||x||_p :\equiv (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}$

Outer Product and Projection

Def. The outer product is the matrix that is returned, when multiplying the vectors x and y: $x \cdot y^H$
($rank = 1$)

T 2.15 The orthogonal projection $P_y x$ of x on y is given by: $P_y x :\equiv \frac{1}{||y||^2} y y^H x$

Def. The projection matrix $P_y = \frac{1}{||y||^2} \cdot y y^H$
$P_y^H = P_y$ (Hermitian), $P_y^2 = P_y$ (Idempotent)

Linear Transformations

For all $x, \tilde{x} \in \mathbb{E}^n$ and $\gamma \in \mathbb{E}$:
$A(\gamma x + \tilde{x}) = \gamma(Ax) + (A\tilde{x})$

Def. image of A: $imA :\equiv \{Ax \in \mathbb{E}^m; x \in \mathbb{E}^n\}$

Inverse

Def. A nxn matrix A is invertible, if there exists a matrix A^{-1}, such that $A \cdot A^{-1} = I_n = A^{-1}A$.

T 2.17 4 equivalent statements:
i) A is invertible
ii) $\exists X$ such that $AX = I_n$
iii) X is definitive
iv) A is non-singular, i.e. rank A = n

T 2.18 With two non-singular nxn matrices A and B:
i) A^{-1} is non-singular and $(A^{-1})^{-1} = A$
ii) AB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$
iii) A^H is non-singular and $(A^H)^{-1} = (A^{-1})^H$

T 2.19 If A is non-singular, $Ax = b$ has exactly one solution for every b: $x = A^{-1}b$

Find inverse $O(n^3)$: $[\ A \mid I\ ] \longrightarrow [\ I \mid A^{-1}\ ]$
-> using elementary row operations

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } detA \neq 0 \Leftrightarrow A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Orthogonal and Unitary Matrices

Def. We call a matrix unitary or orthogonal, if $A^H A = I_n$, $A^T A = I_n$ respectively.

T 2.20 Let A and B be unitary:
i) A is non-singular and $A^{-1} = A^H$
ii) $AA^H = I_n$
iii) A^{-1} is unitary (/orthogonal)
iv) AB is unitary (/orthogonal)

T 2.21 A linear transformation defined by an orthogonal or unitary nxn matrix A is length preserving (/isometric) and angle preserving:
$||Ax|| = ||x||$, $\langle Ax, Ay\rangle = \langle x, y\rangle$

Examples of Important Matrices

Rotationmatrices (orthogonal)

$$\begin{bmatrix} cos(\varphi) & sin(\varphi) \\ -sin(\varphi) & cos(\varphi) \end{bmatrix} \text{ or } \begin{bmatrix} cos(\varphi) & 0 & sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\varphi) & 0 & cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Permutationmatrices (orthogonal)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Blockmatrices

$$A = \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right), \text{ if invertible } A^{-1} = \left(\begin{array}{c|c} a_{11}^{-1} & a_{12}^{-1} \\ \hline a_{21}^{-1} & a_{22}^{-1} \end{array}\right)$$

4 LU-Decomposition

LU-Decomposition

The LU-Decomposition is a tool to solve SLE. It does this by factorizing a matrix, making it easy to solve the same matrix vor different RHS.

1. Find $PA = LR$
2. Solve $Lc = Pb$ (forward subst.)
3. Solve $Rx = c$ (backward subst.)

If rows are swapped P gets permutated.

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 2 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 2 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{4}{3} & 1 & \frac{2}{3} & 1 \end{array}\right]$$
$$\underbrace{\qquad}_{P} \quad \underbrace{\qquad}_{R} \quad \underbrace{\qquad}_{L}$$

partial pivoting as a pivot strategy to minimize rounding errors

5 Vector Spaces

General

Def. A vector space V over $\mathbb{E}$ is a non-empty set, on which addition and scalar multiplication are defined.

Axioms
- $(V1)$ $x+y=y+x$ $(\forall x,y\in V)$
- $(V2)$ $(x+y)+z=x+(y+z)$ $(\forall x,y,z\in V)$
- $(V3)$ $\exists o\in V: x+o=x$ $(\forall x\in V)$ zero vector
- $(V4)$ $\forall x\,\exists(-x): x+(-x)=o$ $(\forall x\in V)$
- $(V5)$ $\alpha(x+y)=\alpha x+\alpha y$ $(\forall\alpha\in\mathbb{E},\ \forall x,y\in V)$
- $(V6)$ $(\alpha+\beta)x=\alpha x+\beta x$ $(\forall\alpha,\beta\in\mathbb{E},\ \forall x\in V)$
- $(V7)$ $(\alpha\beta)x=\alpha(\beta x)$ $(\forall\alpha,\beta\in\mathbb{E},\ \forall x\in V)$
- $(V8)$ $1x=x$ $(\forall x\in V)$

T 4.1 $\forall\alpha\in\mathbb{E},\forall x,y\in V$
- i) $0\cdot x=o$ ii) $\alpha o=o$
- iii) $\alpha\cdot x=o\Rightarrow x=o\ or\ \alpha=0$
- iv) $(-\alpha)\cdot x=\alpha\cdot(-x)=-(\alpha x)$

T 4.2 $\forall x,y\in V,\ \exists z\in V$
$x+z=y$, where z is definite and $z=y+(-x)$

Fields

Def. A Field is a non-empty set $\mathbb{K}$, on which addition and multiplication are defined.

Axioms
- $(K1)$ $\alpha+\beta=\beta+\alpha$ $(\forall\alpha,\beta\in\mathbb{K})$
- $(K2)$ $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ $(\forall\alpha,\beta,\gamma\in\mathbb{K})$
- $(K3)$ $\exists o\in V:\alpha+o=\alpha$ $(\forall\alpha\in\mathbb{K})$ zero element
- $(K4)$ $\forall\alpha\,\exists(-\alpha):\alpha+(-\alpha)=o$ $(\forall\alpha\in\mathbb{K})$
- $(K5)$ $\alpha\cdot\beta=\beta\cdot\alpha$ $(\forall\alpha,\beta\in\mathbb{K})$
- $(K6)$ $(\alpha\cdot\beta)\cdot\gamma=\alpha\cdot(\beta\cdot\gamma)$ $(\forall\alpha,\beta,\gamma\in\mathbb{K})$
- $(K7)$ $\exists 1\in\mathbb{K}:\alpha\cdot 1=\alpha$ $(\forall\alpha\in\mathbb{K})$ identity element
- $(K8)$ $\forall\alpha\in\mathbb{K},\ \alpha\neq 0,\ \exists\alpha^{-1}\in\mathbb{K}:$ $\alpha\cdot(\alpha^{-1})=1$ inverse
- $(K9)$ $\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma$ $(\forall\alpha,\beta,\gamma\in\mathbb{K})$
- $(K10)$ $(\alpha+\beta)\cdot\gamma)=\alpha\cdot\gamma+\beta\cdot\gamma$ $(\forall\alpha,\beta,\gamma\in\mathbb{K})$

Subspaces

Def. A subspace U is a non-empty subset of a vector space V, which is closed under sums and scalar multiples.

T 4.3 A subspace is a vector space itself.

T 4.4 With $A\in\mathbb{R}^{mxn}$ and L_0 containing solutions of $Ax=o$, L_0 is a subspace of $\mathbb{R}^n$.

Def. The set of all linear combinations of $v_1,...,v_n$ is a subspace spanned by these vectors.
$span\{v_1,...,v_n\}$ / linear hull of $v_1,...,v_n$

Def. The vectors $v_1,...,v_n$ are a spanning set of V, if $\forall w\in V\ \Rightarrow\ w\in span\{v_1,...,v_n\}$.

Linear Dependencies, Bases, Dimensions

Def. Vectors $v_1,...,v_n$ are linearly independent, if no vector is a linear combination of the others.
$\sum_{k=0}^{n}\alpha_k v_k=0$, only if $\alpha_1=...=\alpha_k=0$

Def. A $span\{v_1,...,v_n\}=V$ is a basis of V, if $v_1,...,v_n$ are linearly independent.
$\Rightarrow$ standard basis consists of unit vectors

Def. The dimension of V is denoted as,
$dimV=|spanV|$. $dim\{0\}=0$

L 4.8 Any set $\{v_1,...,v_m\}\subset V$ with $|B_V|<m$ is linearly dependent.

T 4.9 Any set of linearly independent vectors of V can be extended to a basis of V. (as long as V has a finite spanning set)

C 4.10 The set of n linearly independent vectors is a basis of V in any finite vector space, if $dimV=n$.

Def. The coefficients ξ_k are coordinates of x in basis B. $\xi=(\xi_1...\xi_n)^T$ is the coordinate vector and $x=\sum_{i=1}^{n}\xi_i b_i$ is the representation of x in coordinates of B.

Def. Two subspaces $U,\ U'\subset V$ are complementary, if every $v\in V$ has a specific representation $v=u+u'$, with $u\in U,\ u'\in U'$. In that case V is the direct sum of U and U':
$V=U\bigoplus U'$.

Change Of Basis, Coordinate Transformation

Def. To change from an old basis B to a new basis B' one can get the new basis vectors b'_k as follows:
$$b'_k=\sum_{i=1}^{n}\tau_{ik}b_i$$
$T=(\tau_{ik})$ is called transformation matrix.

T 4.13 $\xi=T\xi'$ and $\xi'=T^{-1}\xi$. T is invertible (non-singular)

Def. $B'=B\cdot T$ to get the new basis.

6 Linear Maps

General

Def. A transformation $F:X\rightarrow Y,\ x\mapsto F(x)$ is linear, if the following holds:
$F(x+\tilde{x})=F(x)+F(\tilde{x})$
$F(\gamma x)=\gamma F(x)$

Functions

Def.
- Injective: $\forall x,x'\in X: F(x)=F(x')\Rightarrow x=x'$
- Surjective: $F(X)=Y$
- Bijective: injective and surjective $\Rightarrow$ f^{-1} existiert

Transformation matrix

Let F be a linear transformation $X\rightarrow Y$. $F(b_l)\in Y$ can be written using a linear combination of the basis of Y:
$$F(b_l)=\sum_{k=1}^{m}a_{kl}c_k$$

Def. The mxn matrix $A=(a_{kl})$ is the matrix for F relative to the given bases in X and Y.

$F(x)=y\Leftrightarrow A\xi=\eta$
$$\begin{tikzcd} x\in X \arrow[r, "F"] \arrow[d, shift left, "k_x"] & y\in Y \arrow[d, shift left, "k_y"] \\ \xi\in\mathbb{E}^n \arrow[r, "A"] \arrow[u, shift left, "k_x^{-1}"] & \eta\in\mathbb{E}^m \arrow[u, shift left, "k_y^{-1}"] \end{tikzcd}$$

Def. If F is bijective, we call it an isomorphism, if $X=Y$, we call it an automorphism.

L 5.1 If F is an isomorphism, there exists F^{-1} and F^{-1} is also an isomorphism.

Kernel, Image and Rank

Def. The kernel of F $ker\ F :\equiv \{x \in X | F(x) = o\}$
is a subspace of X. F injective $\Leftrightarrow ker\ F = \{o\}$ (T 5.6)

Def. The image of F $im\ F :\equiv \{F(x) | x \in X\}$
is a subspace of Y. F surjective $\Leftrightarrow im\ F = Y$

T 5.7 Rank-nullity theorem:
$dim\ X - dim(ker\ F) = dim(im\ F) = rank F$

Def. The rank F is defined as: $rank F = dim(im\ F)$.

K 5.8
- $F: X \to Y$ injective $\Leftrightarrow rank F = dim\ X$
- $F: X \to Y$ surjective $\Leftrightarrow rank F = dim\ Y$
- $F: X \to Y$ bijective $\Leftrightarrow rank F = dim\ X = dim\ Y$
  Isomorphism
- $F: X \to X$ bijective $\Leftrightarrow rank F = dim\ X$
  Automorphism $\Leftrightarrow ker\ F = o$

Def. Two vector spaces X, Y are isomorphic, if there exists an isomorphism $F: X \to Y$.

T 5.9 Two finite vector spaces X, Y are isomorphic $\Leftrightarrow dim\ X = dim\ Y$

C 5.10
i) $rank(G \circ F) \le min\{rank F, rank G\}$
ii) G injective $\Rightarrow rank(G \circ F) = rank F$
iii) G surjective $\Rightarrow rank(G \circ F) = rank G$

Matrices as Linear Mappings

Def. The column space / range of A is the subspace
$R(A) = span\{a_1, ..., a_n\} = im\ A$

Def. The null space of A is the subspace
$N(A) = L_0(Ax = o) = ker\ A$
#free_parameters $= dim(N(A))$

T 5.12 With $rank A = r$ and solution set L_0 of $Ax = o$:
$dim\ L_0 \equiv dim\ N(A) \equiv dim(ker\ A) = n - r$

T 5.13 for A^{mxn}: $rank A = ...$
i) #pivotelements in reduced form of A
ii) $dim(im\ A)$
iii) $dim(column\ space)$
iv) $dim(row\ space)$

C 5.14 $rank A^T = rank A^H = rank A$

T 5.16 With $A \in \mathbb{E}^{mxn}$, $B \in \mathbb{E}^{pxm}$:
i) $rank(BA) \le min\{rank B, rank A\}$
ii) $rank B = m (\le p) \Rightarrow rank(BA) = rank A$
iii) $rank A = m (\le n) \Rightarrow rank(BA) = rank B$

T 5.18 For square matrices the following statements are equivalent:

i) A is invertible
ii) A is non-singular
iii) $rank A = n$
iv) columns are lin. independent
v) rows are linearly independent
vi) $im\ A \equiv R(A) = \mathbb{E}^n$
vii) $ker\ A \equiv N(A) = \{o\}$
viii) $A: \mathbb{E}^n \to \mathbb{E}^n$ is an automorphism
ix) A is the transformation matrix for a coordinate transformation

Affine Spaces and solutions to inhomogeneous LSE

Def. Let U be a subspace of V and $u_0 \in V$:
$u_0 + U :\equiv \{u_0 + u | u \in U\}$ is an affine (sub)space.

Def. Let $F: X \to Y$ be a linear mapping and $y_0 \in Y$:
$H: X \to y_0 + Y$, $x \mapsto y_0 + F(x)$ is an affine mapping.

T 5.19 Let x_0 be any solution of $Ax = b$ and
let L_0 be the solution set for $Ax = o$:
The solution set L_b to $Ax = b$ is an affine subspace:
$L_b = x_0 + L_0$

The Transformation matrix for coord. transformation

Let X, Y be vector spaces with $dim\ X = n$, $dim\ Y = m$:

F: $X \to Y$, $x \mapsto y$ a linear map,
A: $\mathbb{E}^n \to \mathbb{E}^m$, $\xi \mapsto \eta$ a (mapping) matrix,
B: $\mathbb{E}^n \to \mathbb{E}^m$, $\xi' \mapsto \eta'$ a (mapping) matrix,
T: $\mathbb{E}^n \to \mathbb{E}^n$, $\xi' \mapsto \xi$ a transformation matrix in $\mathbb{E}^n$,
S: $\mathbb{E}^m \to \mathbb{E}^m$, $\eta' \mapsto \eta$ a transformation matrix in $\mathbb{E}^m$,

$$\begin{tikzcd}
x \in X \arrow[r, "F"] \arrow[d, shift left, "k_x"] & y \in Y \arrow[d, shift left, "k_y"] \\
\xi \in \mathbb{E}^n \arrow[r, "A"] \arrow[u, shift left, "k_x^{-1}"] \arrow[d, shift left, "T^{-1}"] & \eta \in \mathbb{E}^m \arrow[u, shift left, "k_y^{-1}"] \arrow[d, shift left, "S^{-1}"] \\
\xi' \in \mathbb{E}^n \arrow[r, "B"] \arrow[u, shift left, "T"] & \eta' \in \mathbb{E}^m \arrow[u, shift left, "S"]
\end{tikzcd}$$

$A = S \cdot B \cdot T^{-1}$
$B = S^{-1} \cdot A \cdot T$

If $F: X \to X$:
$A = T \cdot B \cdot T^{-1}$
$B = T^{-1} \cdot A \cdot T$

Def. Two nxn matrices A and B are similar,
if there exists a non-singular matrix T, such that
either $B = T^{-1} \cdot A \cdot T$ or $A = T \cdot B \cdot T^{-1}$.
We call $A \mapsto B = T^{-1} \cdot A \cdot T$ a similarity transformation.

7 Vector Spaces with Scalar Product

General

Def. A norm in a vector space is a function
$\|\cdot\|: V \to \mathbb{R}$, $x \mapsto \|x\|$, which is:
(N1) positive definite: $\|x\| \ge 0$, $\|x\| = 0 \Leftrightarrow x = 0$
(N2) homogeneous: $\|\alpha x\| = |\alpha|\ \|x\|$
(N3) triangle inequality: $\|x + y\| \le \|x\| + \|y\|$
A vector space with norm is a normed vector space.

Def. A inner product in a vector space is a function
$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{E}$, $x, y \mapsto \langle x, y \rangle$, which is:
(S1) linearity (2nd factor): $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
$\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$
(S2) hermitian (sym. in $\mathbb{R}$): $\langle x, y \rangle = \overline{\langle y, x \rangle}$
(S3) positive definite: $\langle x, x \rangle \ge 0$
$\langle x, x \rangle = 0 \Rightarrow x = o$

Def. The induced norm / length of a vector is
defined as: $\|\cdot\|: V \to \mathbb{R}$, $\|x\| \mapsto \sqrt{\langle x, x \rangle}$

T 6.1 Cauchy-Schwarz inequality: $|\langle x, y \rangle| \le \|x\|\ \|y\|$

Def. The angle $\varphi = \angle(x, y)$, $0 \le \varphi \le \pi$, is defined as:
$\varphi :\equiv arc\ cos\left(\frac{\langle x,y \rangle}{\|x\|\ \|y\|}\right)$ or $\varphi :\equiv arc\ cos\left(\frac{\mathrm{Re}\langle x,y \rangle}{\|x\|\ \|y\|}\right)$

Def. Two vectors are orthogonal, if $\langle x, y \rangle = 0$, $x \perp y$.
Two subsets are orthogonal, if:
$\forall x \in M$, $\forall y \in N \quad \langle x, y \rangle = 0$, $M \perp N$

T 6.2 Pythagoras: $x \perp y \Rightarrow \|x \pm y\|^2 = \|x\|^2 + \|y\|^2$

Orthonormal Bases

Def. A basis is orthogonal, if the basis vectors are
pairwise orthogonal: $\langle b_k, b_l \rangle = 0$, for $k \ne l$.
We call the basis orthonormal, if all basis vectors
are of length 1: $\langle b_k, b_k \rangle = 1$.

T 6.4 With an orthonormal basis $\{b_1, ..., b_n\}$ and $x \in V$:
$$x = \sum_{k=1}^{n} \langle b_k, x \rangle b_k \quad \Rightarrow \quad \xi_k = \langle b_k, x \rangle$$

T 6.5 Parseval's Identity: with $\xi_k :\equiv \langle b_k, x \rangle_V$, $\eta_k :\equiv \langle b_k, y \rangle_V$
$$\langle x, y \rangle_V = \sum_{k=1}^{n} \overline{\xi_k} \eta_k = \xi^H \eta = \langle \xi, \eta \rangle_{\mathbb{E}^n}$$
Therefore, the inner product of two vectors in V
is equal to the scalar product of the respective
coordinate vectors in $\mathbb{E}^n$:
$\|x\|_V = \|\xi\|_{\mathbb{E}^n} \quad \angle(x, y)_V = \angle(\xi, \eta)_{\mathbb{E}^n} \quad x \perp y \Leftrightarrow \xi \perp \eta$

Gram-Schmidt Process

$$b_1 :\equiv \frac{a_1}{||a_1||_V}$$
$$\tilde{b_k} :\equiv a_k - \sum_{j=1}^{k-1} \langle b_j, a_k \rangle_V b_j$$
$$b_k :\equiv \frac{\tilde{b_k}}{||\tilde{b_k}||_V}$$

T 6.6 After k steps $\{b_1, ..., b_k\}$ are pairwise orthonormal. If $\{a_1, ..., a_n\}$ is a basis of V, so is $\{b_1, ..., b_k\}$.

Orthogonal Complements

Def. $\mathrm{U}^\perp$ is the orthogonal complement of the subspace U: $\mathrm{U}^\perp \bigoplus \mathrm{U} = \mathrm{V}$

T 6.9 For a complex mxn matrix with rank r:

- $N(\mathrm{A}) = (R(\mathrm{A}^\mathrm{H}))^\perp \subset \mathbb{E}^n$
- $N(\mathrm{A}^\mathrm{H}) = (R(\mathrm{A}))^\perp) \subset \mathbb{E}^n$
- $N(\mathrm{A}) \bigoplus R(\mathrm{A}^\mathrm{H}) = \mathbb{E}^n$
- $N(\mathrm{A}^\mathrm{H}) \bigoplus R(\mathrm{A}) = \mathbb{E}^m$
- $dim\ R(\mathrm{A}) = r$
- $dim\ N(\mathrm{A}) = n - r$
- $dim\ R(\mathrm{A}^\mathrm{H}) = r$
- $dim\ N(\mathrm{A}^\mathrm{H}) = m - r$

We call these subspaces fundamental subspaces of A.

Orthogonal and Unitary Change Of Basis

To get from B to B', where both are orthonormal bases, we can write $b'_k = \sum_{i=1}^{n} \tau_{ik} b_i$ and get the transformation matrix T, with $\mathrm{T}^{-1} = \mathrm{T}^\mathrm{H}$.

T 6.11 Therefore: $\xi = \mathrm{T}\xi'$, $\xi' = \mathrm{T}^\mathrm{H}\xi$

Additionally, $\mathrm{B} = \mathrm{B}'\mathrm{T}$ and $\mathrm{B}' = \mathrm{B}\mathrm{T}^\mathrm{H}$, where all matrices are unitary/orthogonal.

T 6.10 The transformation matrix for a change of basis between orthonormal bases is unitary/orthogonal.

C 6.12 $\langle x, z \rangle_V = \xi^\mathrm{H}\eta = (\xi')^\mathrm{H}\eta' = \langle \xi', \eta' \rangle$
$\Rightarrow$ T is length and angle preserving

$$\begin{tikzcd}
x \in X \arrow[r, "F"] \arrow[d, shift left, "k_x"] & z \in Y \arrow[d, shift left, "k_y"] \\
\xi \in \mathbb{E}^n \arrow[r, "\mathrm{A}"] \arrow[u, shift left, "k_x^{-1}"] \arrow[d, shift left, "\mathrm{T}^{-1} = \mathrm{T}^\mathrm{H}"] & \eta \in \mathbb{E}^m \arrow[u, shift left, "k_y^{-1}"] \arrow[d, shift left, "\mathrm{S}^{-1} = \mathrm{S}^\mathrm{H}"] \\
\xi' \in \mathbb{E}^n \arrow[r, "\mathrm{B}"] \arrow[u, shift left, "\mathrm{T}"] & \eta' \in \mathbb{E}^m \arrow[u, shift left, "\mathrm{S}"]
\end{tikzcd}$$

$\mathrm{A} = \mathrm{S} \cdot \mathrm{B} \cdot \mathrm{T}^\mathrm{H}$
$\mathrm{B} = \mathrm{S}^\mathrm{H} \cdot \mathrm{A} \cdot \mathrm{T}$

If $Y = X, k_y = k_x$ and $\mathrm{S} = \mathrm{T}$:
$\mathrm{A} = \mathrm{T} \cdot \mathrm{B} \cdot \mathrm{T}^\mathrm{H}$
$\mathrm{B} = \mathrm{T}^\mathrm{H} \cdot \mathrm{A} \cdot \mathrm{T}$
$\Rightarrow$ A, B are Hermitian

Orthogonal and Unitary Transformations

Def. A linear transformation $F : X \to Y$ is unitary/symmetric, if: $\langle F(x), F(y) \rangle_Y = \langle x, y \rangle_X$

T 6.13
- i) F is length preserving: $||F(x)||_Y = ||x||_X$
- ii) F is angle preserving: $x \perp y \Leftrightarrow F(x) \perp F(y)$
- iii) $ker\ F = o$, F is injective

If additionally $dim\ X = dim\ Y < \infty$:

- iv) F is an isomorphism
- v) $\{b_1, ..., b_n\}$ is an orthonormal basis for X $\Leftrightarrow \{F(b_1), ..., F(b_n)\}$ is an orthonormal basis for Y
- vi) F^{-1} is unitary/orthogonal
- vii) The transformation matrix A is unitary/orthogonal.

8 Least Squares Method

General

Let Ax = y be an overdetermined linear system (more equations than unknowns). As an exact solution is not guaranteed we try to minimize the length of the residual vector (orthogonal to $R(\mathrm{A})$): $\mathrm{r} :\equiv \mathrm{y} - \mathrm{Ax}$

Therefore, to find the least squares solution we solve: $x^* = \arg\min_{x \in \mathbb{E}^n} ||\mathrm{Ax} - \mathrm{b}||_2^2$

Def. The normal equations, $\mathrm{A}^\mathrm{H}\mathrm{Ax} = \mathrm{A}^\mathrm{H}\mathrm{y}$ can be used to solve the given LSE.

Lemma: $\mathrm{A}^\mathrm{H}\mathrm{A}$ non-singular $\Leftrightarrow rank A = n$

Def. Moore-Penrose Pseudo-Inverse: $\mathrm{A}^+ :\equiv (\mathrm{A}^\mathrm{H}\mathrm{A})^{-1}\mathrm{A}^\mathrm{H}$ (for when $rank A = n$)

9 QR Factorization & Decomposition

General

A = QR Q: orthogonal matrix; R: upper triangular matrix

This decomposition is definite, if $m \geq n$ and $rank A = n$.

Calculate QR Factorization

I: Gram-Schmidt process with columns of A $\Rightarrow$ Q
II: Solve $\mathrm{R} = \mathrm{Q}^\mathrm{T}\mathrm{A}$
Alternatively:

- $r_{11} :\equiv ||a_1||$
- $r_{jk} :\equiv \langle q_j, a_k \rangle$
- $r_{kk} :\equiv ||\tilde{q}_k||$

Least-Squares with QR Factorization

Normal equations:
$$\begin{aligned}
\mathrm{A}^\mathrm{T}\mathrm{A}x &= \mathrm{A}^\mathrm{T}b \\
(\mathrm{QR})^\mathrm{T}(\mathrm{QR})x &= (\mathrm{QR})^\mathrm{T}b \\
\mathrm{R}^\mathrm{T}\mathrm{Q}^\mathrm{T}\mathrm{QR}x &= \mathrm{R}^\mathrm{T}\mathrm{Q}^\mathrm{T}b \\
\mathrm{R}^\mathrm{T}\mathrm{R}x &= \mathrm{R}^\mathrm{T}\mathrm{Q}^\mathrm{T}b \\
\mathrm{R}x &= \mathrm{Q}^\mathrm{T}b
\end{aligned}$$
$^\mathrm{T}$ can be replaced with $^\mathrm{H}$

10 Determinant

General

T 8.1 There are $n!$ permutations in S_n.

Def. The sign of a permutation is defined as follows:
$$\text{sign } p = \begin{cases} +1, \text{ if \#permutations even} \\ -1, \text{ if \#permutations odd} \end{cases}$$

Def. The determinant of an nxn matrix A is:
$$det(\mathrm{A}) = \sum_{p \in S_n} \text{sign } p \cdot a_{1,p(1)} a_{2,p(2)} \ldots a_{n,p(n)}$$

T 8.3 Properties of $\det : \mathbb{E}^{nxn} \to \mathbb{E}, \mathrm{A} \mapsto \det(\mathrm{A})$:

- i) det(+) is linear in every row
- ii) when swapping two rows, det(A) switches sign
- iii) $\det(I) = 1$

T 8.4
- iv) if A has a row of zeros, det(A)= 0
- v) $\det(\gamma\mathrm{A}) = \gamma^n \det(\mathrm{A})$
- vi) if A has two identical rows, det(A)= 0
- vii) if we add multiples of two rows to each other, det(A) doesn't change
- viii) if A is triangular or diagonal, $\det(\mathrm{A}) = \prod_{i=1}^{n} a_{ii}$

det(A) $\Leftrightarrow$ A is singular
det(A) $\Leftrightarrow$ A is non-singular

T 8.5 After applying the gauss algorithm to A:
$$\det(\mathrm{A}) = (-1)^\nu \prod_{k=1}^{n} r_{kk}$$

T 8.7 Multiplicativity: $\det(\mathrm{AB}) = \det(\mathrm{A}) \cdot \det(\mathrm{B})$

C 8.8 A non-singular $\Rightarrow \det(\mathrm{A}^{-1}) = (\det(\mathrm{A}))^{-1}$

T 8.9 $\det(\mathrm{A})^\mathrm{T} = \det(\mathrm{A})$ and $\det(\mathrm{A}^\mathrm{H}) = \overline{\det(\mathrm{A})}$

Determinant for block matrices

C 8.14 $\begin{vmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{O} & \mathrm{D} \end{vmatrix} = \det(\mathrm{A}) \cdot \det(\mathrm{D})$

11 Eigenvalues and Eigenvectors

General

Def. We call the number $\lambda \in \mathbb{E}$ eigenvalue (EVal) of the linear transformation $F: V \to V$, if there exists an eigenvector (EVec) $v \in V$, $v \neq o$, such that: $F(v) = \lambda v$

Def. We denote the eigenspace E_λ, containing all eigenvectors for λ: $E_\lambda :\equiv \{v \in V | F(v) = \lambda v\}$
(the eigenspace is a subspace of V)

Def. The set of all eigenvalues of F is called spectrum, denoted $\sigma(F)$.

Def. $\xi \in \mathbb{E}^n$ is an eigenvalue of of $\lambda \Leftrightarrow A\xi = \xi\lambda$

L 9.1 A linear transformation F and it's matrix representation have the same eigenvalues and the eigenvalues are related respecting the coordinate transformation.

L 9.2 λ is an eigenvalue, iff ker $(A - \lambda I)$ doesn't contain just the zero vector (singular). $E_\lambda = ker$ $(A - \lambda I)$

Def. The geometric multiplicity of λ is $dim\, E_\lambda$.

Def. The characteristic polynomial of A$\in \mathbb{E}^{nxn}$ is defined as $x_A(\lambda) :\equiv \det(A - \lambda I)$. We call $x_A(\lambda) = 0$ the characteristic equation.

L 9.4 $x_A(\lambda) = (-\lambda)^n + Tr(A)(-\lambda)^{n-1} + \ldots + \det(A)$

L 9.5 $\lambda \in \mathbb{E}$ is an eigenvalue of $A \in \mathbb{E}^{nxn}$
$\Leftrightarrow \lambda$ is a root of x_A
$\Leftrightarrow \lambda$ is a solution to the characteristic equation

Def. The algebraic multiplicity of an eigenvalue λ is the multiplicity of λ as a root for the characteristic equation.

L 9.6 A singular $\Leftrightarrow 0 \in \sigma(A)$

T 9.7 Similar matrices have the same characteristic equation, determinant, trace and the same eigenvalues.

Def. A basis made of eigenvectors is a eigen basis of F:

$$x = \sum_{k=1}^{n} \xi_k v_k \quad \mapsto \quad F(x) = \sum_{k=1}^{n} \lambda_k \xi_k v_k$$

T 9.9 There is a similar diagonal matrix Λ to $A \in \mathbb{E}^{nxn}$
$\Leftrightarrow$ A has an eigen basis $\quad$ $AV = V\Lambda$

Spectral/Eigenvalue Decomposition

Def. We call a matrix A to which a spectral decomposition $A = V\Lambda V^{-1}$ exists diagonalizable.

T 9.13 geom. mult. $\leq$ alg. mult.

T 9.14 A matrix is diagonalizable, if for every eigenvalue holds: geom. mult. = alg. mult.

T 9.15 Spectral Theorem: Let A$\in \mathbb{C}^{nxn}$ be hermitian:
i) all eigenvalues are real
ii) the eigenvectors are pairwise orthogonal
iii) there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A
iv) for this unitary matrix U holds:
$U^H A U = \Lambda :\equiv \mathrm{diag}(\lambda_1, ..., \lambda_n)$
(also holds for real-symmetric A$\in \mathbb{R}^{nxn}$)

12 Singular Value Decomposition

General

- exists for every Matrix
- $A^H A$ is always hermitian and positive semi-definite

T 11.1 For a complex matrix A^{mxn} of rank r, there exist unitary matrices U and V, and a mxn matrix Σ:
$\Sigma = \begin{pmatrix} \Sigma_r & O \\ O & O \end{pmatrix}$, with $\Sigma_r :\equiv \mathrm{diag}\,\{\sigma_1, \ldots, \sigma_r\}$
where the singular values σ_i are positive and ordered, such that $A = U\Sigma V^H = \sum_{k=1}^{r} u_k \sigma_k v_k^H$

$AA^H = U\Sigma_m^2 U^H, \qquad A^H A = V\Sigma_n^2 V^H$

$\{u_1, \ldots, u_r\}$ is a basis of im A$\equiv R(A)$
$\{u_{r+1}, \ldots, u_m\}$ is a basis of ker $A^H \equiv N(A^H)$
$\Rightarrow \{u_1, \ldots, u_m\}$ are the left singular vectors

$\{v_1, \ldots, v_r\}$ is a basis of im $A^H \equiv R(A^H)$
$\{v_{r+1}, \ldots, v_n\}$ is a basis of ker $A \equiv N(A)$
$\Rightarrow \{v_1, \ldots, v_n\}$ are the right singular vectors

Least Squares using SVD

$$||Ax - b||_2^2 = ||\Sigma \underbrace{V^H x}_{y} - \underbrace{U^H b}_{c}||_2^2 = ||\Sigma y - c||_2^2$$

$x^* = V\Sigma^+ U^H b \quad \Rightarrow \infty$ solutions, here smallest 2-norm