Linear Algebra - Cheat Sheet (HS22)

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1 Complex Numbers

General

$$z = \underbrace{x}_{\text{Re}} + i \underbrace{y}_{\text{Im}} = \underbrace{r \cdot (\cos(\varphi) + i \cdot \sin(\varphi)) = r \cdot e^{i\varphi}}_{\text{Polarform}}$$

$$\overline{z} = x - iy = r \cdot e^{i(2\pi - \varphi)}$$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$$

$$\varphi = \begin{cases} arctan(\frac{y}{x}), & \text{I}\,Q. \\ arctan(\frac{y}{x}) + \pi, & \text{II/III}\,Q. \\ arctan(\frac{y}{x}) + 2\pi, & \text{IV}\,Q. \end{cases}$$

Operations

$+/-:$ $(x_1 + x_2) + (y_1 + y_2)i$

$z_1 \cdot z_2:$ $(x_1 + y_1 i)(x_2 + y_2 i) = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$

$\frac{z_1}{z_2}:$ $\frac{z_1 \cdot \overline{z}_2}{|z_2|^2} = \frac{r_1}{r_2} \cdot e^{i(\varphi_1 - \varphi_2)}$

$z^n:$ $r^n \cdot e^{i\varphi n}$

$\sqrt{a}:$ $a = z^n \Leftrightarrow |a| \cdot e^{i\alpha} = r^n \cdot e^{i\varphi n} \begin{cases} r = \sqrt[n]{|a|} \\ \varphi = \frac{\alpha + 2k\pi}{n} \end{cases}_{k=0,\dots,n-1}$

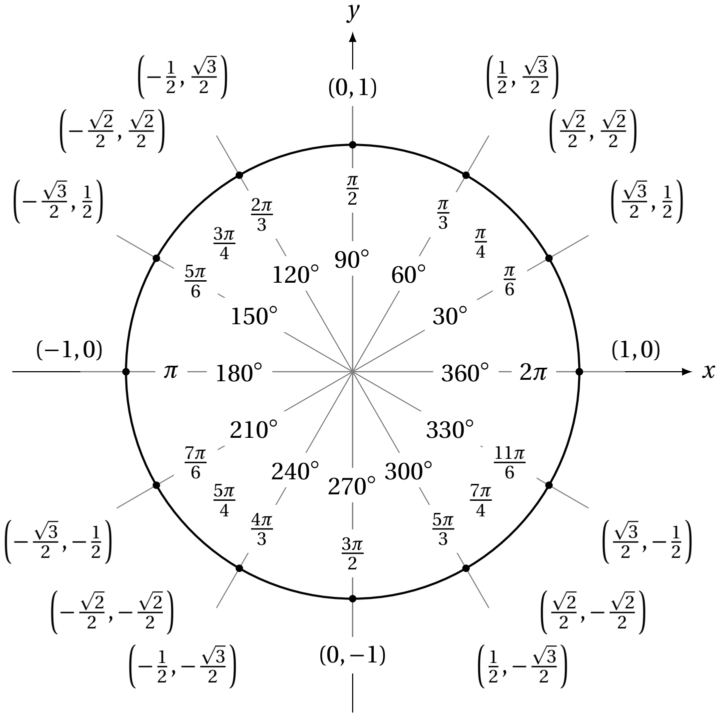
Polynomials

degree 2: $z = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

special case: $az^n + c = 0 \Leftrightarrow z = \sqrt[n]{-\frac{c}{a}}$

With polynomials with complex roots, the roots occur as a complex-conjugate pair.

Polynomials over $\mathbb{C}$ with an odd degree have at least one root in $\mathbb{R}$.



2 LSE

Gauss-Algorithm

runtime: $O(n^3)$

elementary row operations:

switch, multiply and add/subtract rows

goal: row echelon form

$$\begin{array}{cccc|cl} x_1 & x_2 & \dots & x_n & & \\ \hline \textcircled{a_{11}} & a_{12} & \dots & a_{1n} & b_1 & \\ 0 & \textcircled{a_{22}} & \dots & a_{2n} & b_2 & \Big\}\ \text{rank} \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \textcircled{a_{rn}} & b_r & \\ \dots & \dots & \dots & \dots & \dots & \Big\}\ \text{consistency conditions} \\ 0 & 0 & \dots & 0 & b_m & \\ \multicolumn{4}{c}{\underbrace{\hphantom{a_{11}\quad a_{12}\quad \dots \quad a_{1n}}}_{\text{LHS}}} & \underbrace{\hphantom{b_m}}_{\text{RHS}} & \end{array}$$

if RHS always 0: homogeneous LSE

consistency conditions (VB): $b_{r+1} = \dots = b_m = 0$

Number of solutions

T 1.1

$Ax = b$ min. 1 solution $\Leftrightarrow$ $(r = m)$ or $(r < m + VB)$

if this is the case:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions

$\Rightarrow r = m$:

- $r = n = m$: only 1 solution, non-singular LSE
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

$\Rightarrow r < m$:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

K 1.5 In a homogeneous LSE non-trivial solutions exist only if $r < n$.

T 2.5

$Ax = b$ has a solution $\Leftrightarrow$

b is a linear combination of columnvectors of A

3 Matrices & Vectors

General

mxn Matrices have m rows and n columns.

The element (i, j) can be denoted as $a_{i,j}$ or $(A)_{i,j}$

T 2.1

$(\alpha\beta)A = \alpha(\beta A)$ $\quad$ $(A + B) + C = A + (B + C)$

$(\alpha A)B = \alpha(AB)$ $\quad$ $(AB) \cdot C = A \cdot (BC)$

$(\alpha + \beta) \cdot A = \alpha A + \beta A$ $\quad$ $(A + B) \cdot C = AC + BC$

$\alpha(A + B) = \alpha A + \alpha B$ $\quad$ $A \cdot (B + C) = AB + AC$

$A + B = B + A$

⚠in general $AB \neq BA$

If $AB = BA$ we say «A and B commute»

Def. If $AB = O$, we call A and B divisors of zero.

Def. A linear combination of vectors $a_1 \dots a_n$ is an expression of the following type:
$\alpha_n \cdot a_n + \dots + \alpha_1 \cdot a_1$

Def. A matrix is symmetric when $A^T = A$ and Hermitian when $A^H = A$ (real diagonal).

Def. A matrix is skew-symmetric when $A^T = -A$. (zeros on diagonal)

T 2.6

$(A^T)^T = A$ $\quad$ $(\alpha A)^T = \alpha(A^T)$

$(AB)^T = B^T A^T$ $\quad$ $(A + B)^T = A^T + B^T$

T 2.7

If A, B symmetric: $AB = BA \Leftrightarrow AB$ symmetric

For any A: $A^T A = AA^T$ (symmetric)

Scalar Product and Norm

Def. Eucl. scalar product (SP): $\langle x, y\rangle :\equiv x^H y$ (inner product)

T 2.9

linearity in second factor: $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
$\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$

symmetric / hermitian: $\langle x, y\rangle = \overline{\langle y, x\rangle}$

positive definite: $\langle x, x\rangle \geq 0$; if $' =' \Rightarrow\ x = 0$

C 2.10

bilinearity in $\mathbb{R}^n$: $\langle w+x, y\rangle = \langle w, y\rangle + \langle x, y\rangle$
$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

sesquilinearity in $\mathbb{C}^n$: $\langle w+x, y\rangle = \langle w, y\rangle + \langle x, y\rangle$
$\langle \alpha x, y\rangle = \overline{\alpha}\langle x, y\rangle$

Def. Eucl. norm / 2-norm: $||x|| :\equiv \sqrt{\langle x, x\rangle}$

(Cauchy-)Schwarz inequality (CBS inequality)

$|\langle x, y\rangle| \leq ||x|| \cdot ||y||$

The equality holds iff y is a multiple of x or vice versa

T 2.12

The following holds for the 2-norm:

(N1) positive definite: $||x|| \geq 0$, if $' =' \Rightarrow\ x = 0$

(N2) $||\alpha x|| = |\alpha|\ ||x||$

(N3) triangle inequality: $||x \pm y|| \leq ||x|| + ||y||$

Def. angle φ between x and y: $\varphi = arc\ cos\left(\frac{\langle x,y\rangle}{||x||\cdot||y||}\right)$

Def. x and y are orthogonal, if $\langle x, y\rangle = 0$; $x \perp y$

T 2.13

Pythagoras $||x \pm y||^2 = ||x||^2 + ||y||^2$, if $x \perp y$

Def. p-Norm: $||x||_p :\equiv (|x_1|^p + ... + |x_n|^p)^{\frac{1}{p}}$

Outer Product and Projection

Def. The outer product is the matrix that is returned, when multiplying the vectors x and y: $x \cdot y^H$ (rank = 1)

T 2.15

The orthogonal projection $P_y x$ of x on y is given by:
$P_y x :\equiv \frac{1}{||y||^2} y y^H x$

Def. the projection matrix $P_y = \frac{1}{||y||^2} \cdot y y^H$

$P_y^H = P_y$ (Hermitian), $P_y^2 = P_y$ (Idempotent)

Linear Transformations

For all $x, \tilde{x} \in \mathbb{E}^n$ and $\gamma \in \mathbb{E}$:
$A(\gamma x + \tilde{x}) = \gamma(Ax) + (A\tilde{x})$

Def. image of A: $imA :\equiv \{Ax \in \mathbb{E}^m; x \in \mathbb{E}^n\}$

Inverse

Def. A nxn matrix A is invertible, if there exists a matrix A^{-1}, such that $A \cdot A^{-1} = I_n = A^{-1}A$.

T 2.17 4 equivalent statements:

i) A is invertible
ii) $\exists X$ such that $AX = I_n$
iii) X is definitive
iv) A is non-singular, i.e. rank A = n

T 2.18 With two non-singular nxn matrices A and B:

i) A^{-1} is non-singular and $(A^{-1})^{-1} = A$
ii) AB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$
iii) A^H is non-singular and $(A^H)^{-1} = (A^{-1})^H$

T 2.19

If A is non-singular, $Ax = b$ has exactly one solution for every b: $x = A^{-1}b$

Find inverse $O(n^3)$: $[\ A\ |\ I\ ] \longrightarrow [\ I\ |\ A^{-1}\ ]$
-> using elementary row operations

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } detA \neq 0 \Leftrightarrow A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Orthogonal and Unitary Matrices

Def. We call a matrix unitary or orthogonal, if $A^H A = I_n$, $A^T A = I_n$ respectively.

T 2.20 Let A and B be unitary:

i) A is non-singular and $A^{-1} = A^H$
ii) $AA^H = I_n$
iii) A^{-1} is unitary (/orthogonal)
iv) AB is unitary (/orthogonal)

T 2.21

A linear transformation defined by an orthogonal or unitary nxn matrix A is length preserving (/isometric) and angle preserving:
$||Ax|| = ||x||$, $\langle Ax, Ay\rangle = \langle x, y\rangle$

Examples of Important Matrices

Rotationmatrices (orthogonal)

$$\begin{bmatrix} cos(\varphi) & sin(\varphi) \\ -sin(\varphi) & cos(\varphi) \end{bmatrix} \text{ or } \begin{bmatrix} cos(\varphi) & 0 & sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\varphi) & 0 & cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Permutationmatrices (orthogonal)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Blockmatrices

$A = \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right)$, if invertible $A^{-1} = \left(\begin{array}{c|c} a_{11}^{-1} & a_{12}^{-1} \\ \hline a_{21}^{-1} & a_{22}^{-1} \end{array}\right)$

4 LU-Decomposition

LU-Decomposition

The LU-Decomposition is a tool to solve SLE. It does this by factorizing a matrix, making it easy to solve the same matrix vor different RHS.

1. Find $PA = LR$
2. solve $Lc = Pb$
   (forward substitution)
3. solve $Rx = c$
   (backward substitution)

If rows are swapped
P gets permutated.

partial pivoting as a pivot strategy to minimize rounding errors

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$