Linear Algebra - Cheat Sheet (HS22)

Mattia Taiana

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1 Complex Numbers

General

$z = \underbrace{x}_{\text{Re}} + i \underbrace{y}_{\text{Im}} = \underbrace{r \cdot (\cos(\varphi) + i \cdot \sin(\varphi)) = r \cdot e^{i\varphi}}_{\text{Polarform}}$

$\overline{z} = x - iy = r \cdot e^{i(2\pi - \varphi)}$

$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$

$$\varphi = \begin{cases} arctan(\frac{y}{x}), & \text{I Q.} \\ arctan(\frac{y}{x}) + \pi, & \text{II/III Q.} \\ arctan(\frac{y}{x}) + 2\pi, & \text{IV Q.} \end{cases}$$

Operations

$+/-:\quad (x_1 + x_2) + (y_1 + y_2)i$

$z_1 \cdot z_2:\quad (x_1 + y_1 i)(x_2 + y_2 i) = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$

$\frac{z_1}{z_2}:\quad \frac{z_1 \cdot \overline{z_2}}{|z_2|^2} = \frac{r_1}{r_2} \cdot e^{i(\varphi_1 - \varphi_2)}$

$z^n:\quad r^n \cdot e^{i\varphi n}$

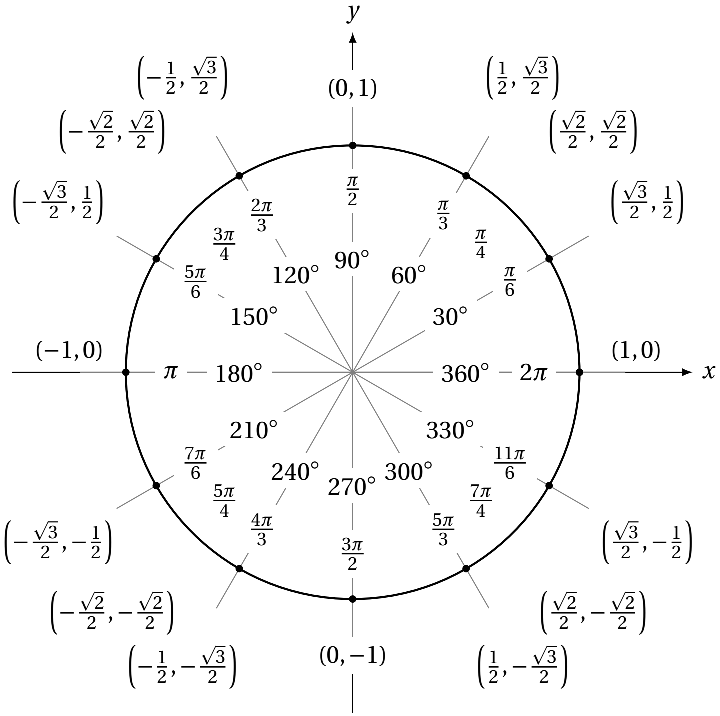
$$\sqrt{a}:\quad a = z^n \Leftrightarrow |a| \cdot e^{i\alpha} = r^n \cdot e^{i\varphi n} \begin{cases} r = \sqrt[n]{|a|} \\ \varphi = \frac{\alpha + 2k\pi}{n} \end{cases}_{k=0,\dots,n-1}$$

Polynomials

Grad 2: $\quad z = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

Sonderfall: $\quad az^n + c = 0 \Leftrightarrow z = \sqrt[n]{-\frac{c}{a}}$

With polynomials with complex roots, the roots occur as a complex-conjugate pair.
Polynomials over $\mathbb{C}$ with an odd degree have at least one root in $\mathbb{R}$.



2 LSE

Gauss-Algorithm

runtime: $O(n^3)$
elementary row operations:
switch, multiply and add/subtract rows
goal: row echelon form

$$\begin{array}{cccc|c} x_1 & x_2 & \dots & x_n & \\ \hline a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_{rn} & b_r \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & b_m \end{array}$$

rank (rows 1 to r); consistency conditions (rows $r+1$ to m); LHS (columns $x_1 \dots x_n$); RHS (b column)

if RHS always 0: homogeneous LSE
consistency conditions (VB): $b_{r+1} = \dots = b_m = 0$

Number of solutions

S 1.1
$Ax = b$ min. 1 solution $\Leftrightarrow$ $(r = m)$ or $(r < m + VB)$
if this is the case:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions

$\Rightarrow r = m$:

- $r = n = m$: only 1 solution, non-singular LSE
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

$\Rightarrow r < m$:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

K 1.5 In a homogeneous LSE non-trivial solutions exist only if $r < n$.

S 2.5 $\quad Ax = b$ has a solution $\Leftrightarrow$
b is a linear combination of columnvectors of A

3 Matrices & Vectors

General

mxn Matrices have m rows and n columns.
The element (i, j) can be denoted as $a_{i,j}$ or $(A)_{i,j}$

S 2.1

$(\alpha\beta)A = \alpha(\beta A)$	$(A + B) + C = A + (B + C)$
$(\alpha A)B = \alpha(AB)$	$(AB) \cdot C = A \cdot (BC)$
$(\alpha + \beta) \cdot A = \alpha A + \beta A$	$(A + B) \cdot C = AC + BC$
$\alpha(A + B) = \alpha A + \alpha B$	$A \cdot (B + C) = AB + AC$
$A + B = B + A$	

⚠ in general $AB \neq BA$
If $AB = BA$ we say «A and B commute»

Def. If $AB = O$, we call A and B divisors of zero.

Def. A linear combination of vectors $a_1 \dots a_n$ is an expression of the following type:
$\alpha_n \cdot a_n + \dots + \alpha_1 \cdot a_1$

Def. A matrix is symmetric when $A^T = A$ and Hermitian when $A^H = A$ (real diagonal). S

Def. A matrix is skew-symmetric when $A^T = -A$.
(zeros on diagonal)

2.6

$(A^T)^T = A$	$(\alpha A)^T = \alpha(A^T)$
$(AB)^T = B^T A^T$	$(A + B)^T = A^T + B^T$

S 2.7
If A, B symmetric: $AB = BA \Leftrightarrow AB$ symmetric
For any A: $A^T A = AA^T$ (symmetric)