

Linear Algebra - Cheat Sheet (HS22)

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1 Complex Numbers

General

$$z = \underbrace{x}_{\text{Re}} + i \underbrace{y}_{\text{Im}} = r \cdot (\cos(\varphi) + i \cdot \sin(\varphi)) = r \cdot e^{i\varphi} \quad \text{Polarform}$$

$$\bar{z} = x - iy = r \cdot e^{i(2\pi - \varphi)}$$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}}$$

$$\varphi = \begin{cases} \arctan(\frac{y}{x}), & \text{I Q.} \\ \arctan(\frac{y}{x}) + \pi, & \text{II/III Q.} \\ \arctan(\frac{y}{x}) + 2\pi, & \text{IV Q.} \end{cases}$$

Operations

$$+/- : (x_1 + x_2) + (y_1 + y_2)i$$

$$z_1 \cdot z_2 : (x_1 + y_1 i)(x_2 + y_2 i) = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$$

$$\frac{z_1}{z_2} : \frac{z_1 \cdot \bar{z}_2}{|z_2|^2} = \frac{r_1}{r_2} \cdot e^{i(\varphi_1 - \varphi_2)}$$

$$z^n : r^n \cdot e^{i\varphi n}$$

$$\sqrt[n]{a} : a = z^n \Leftrightarrow |a| \cdot e^{i\alpha} = r^n \cdot e^{i\varphi n} \begin{cases} r = \sqrt[n]{|a|} \\ \varphi = \frac{\alpha + 2k\pi}{n} \end{cases} \quad k=0, \dots, n-1$$

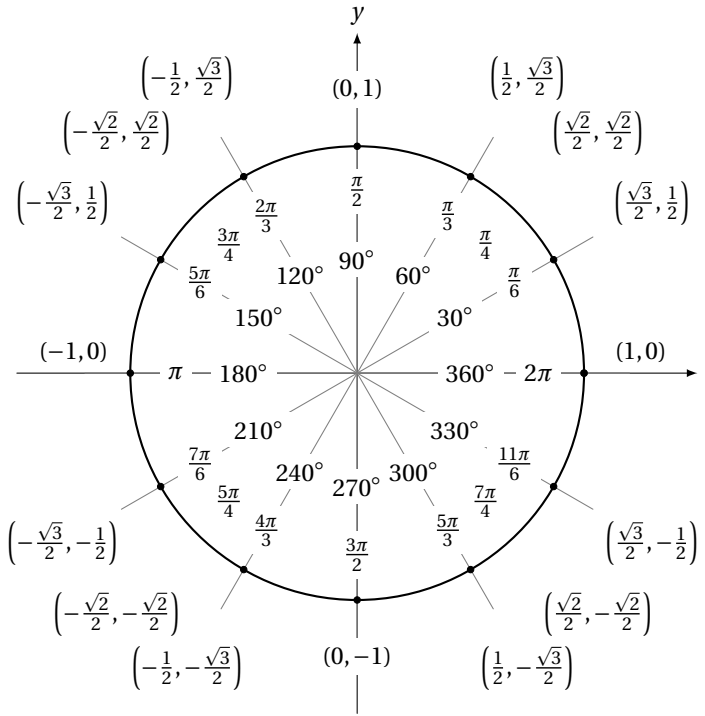
Polynomials

$$\text{degree 2: } z = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{special case: } az^n + c = 0 \Leftrightarrow z = \sqrt[n]{-\frac{c}{a}}$$

With polynomials with complex roots, the roots occur as a complex-conjugate pair.

Polynomials over \mathbb{C} with an odd degree have at least one root in \mathbb{R} .



2 LSE

Gauss-Algorithm

runtime: $O(n^3)$
 elementary row operations:
 switch, multiply and add/subtract rows
 goal: row echelon form

x_1	x_2	...	x_n	
a_{11}	a_{12}	...	a_{1n}	b_1
0	a_{22}	...	a_{2n}	b_2
...
0	0	0	a_{rn}	b_r
...
0	0	...	0	b_m

LHS RHS

} rank

} consistency conditions

if RHS always 0: homogeneous LSE
 consistency conditions (VB): $b_{r+1} = \dots = b_m = 0$

Number of solutions

T 1.1 $Ax = b$ min. 1 solution $\Leftrightarrow (r = m)$ or $(r < m + VB)$
 if this is the case:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions

$\Rightarrow r = m$:

- $r = n = m$: only 1 solution, **non-singular** LSE
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

$\Rightarrow r < m$:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

K 1.5 In a homogeneous LSE non-trivial solutions exist only if $r < n$.

T 2.5 $Ax = b$ has a solution \Leftrightarrow
 b is a linear combination of columnvectors of A

3 Matrices & Vectors

General

$m \times n$ Matrices have m rows and n columns.

The element (i, j) can be denoted as $a_{i,j}$ or $(A)_{i,j}$

T 2.1	$(\alpha\beta)A = \alpha(\beta A)$	$(A+B) + C = A + (B+C)$
	$(\alpha A)B = \alpha(AB)$	$(AB) \cdot C = A \cdot (BC)$
	$(\alpha + \beta) \cdot A = \alpha A + \beta A$	$(A+B) \cdot C = AC + BC$
	$\alpha(A+B) = \alpha A + \alpha B$	$A \cdot (B+C) = AB + AC$
	$A+B = B+A$	

\triangleq in general $AB \neq BA$

If $AB = BA$ we say «A and B commute»

Def. If $AB = O$, we call A and B **divisors of zero**.

Def. A **linear combination** of vectors $a_1 \dots a_n$ is an expression of the following type:

$$\alpha_n \cdot a_n + \dots + \alpha_1 \cdot a_1$$

Def. A matrix is **symmetric** when $A^T = A$ and **Hermitian** when $A^H = A$ (real diagonal).

Def. A matrix is **skew-symmetric** when $A^T = -A$.
 (zeros on diagonal)

T 2.6	$(A^T)^T = A$	$(\alpha A)^T = \alpha(A^T)$
	$(AB)^T = B^T A^T$	$(A+B)^T = A^T + B^T$

T 2.7 If A, B symmetric: $AB = BA \Leftrightarrow AB$ symmetric
 For any A : $A^T A = A A^T$ (symmetric)

Scalar Product and Norm

Def. **Eucl. scalar product** (SP): $\langle x, y \rangle := x^H y$
(inner product)

T 2.9 linearity in second factor: $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
 $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$
 symmetric / hermitian: $\langle x, y \rangle = \langle y, x \rangle$
 positive definite: $\langle x, x \rangle \geq 0$; if $' \Rightarrow x = 0$

C 2.10 bilinearity in \mathbb{R}^n : $\langle w + x, y \rangle = \langle w, y \rangle + \langle x, y \rangle$
 $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
 sesquilinearity in \mathbb{C}^n : $\langle w + x, y \rangle = \langle w, y \rangle + \langle x, y \rangle$
 $\langle \alpha x, y \rangle = \bar{\alpha} \langle x, y \rangle$

Def. **Eucl. norm / 2-norm**: $\|x\| := \sqrt{\langle x, x \rangle}$

Cauchy-Schwarz inequality (CBS inequality)

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

The equality holds iff y is a multiple of x or vice versa

T 2.12 The following holds for the 2-norm:
 (N1) positive definite: $\|x\| \geq 0$, if $' \Rightarrow x = 0$
 (N2) $\|\alpha x\| = |\alpha| \|x\|$
 (N3) **triangle inequality**: $\|x \pm y\| \leq \|x\| + \|y\|$

Def. angle φ between x and y: $\varphi = \arccos \left(\frac{\langle x, y \rangle}{\|x\| \cdot \|y\|} \right)$

Def. x and y are **orthogonal**, if $\langle x, y \rangle = 0$; $x \perp y$

T 2.13 Pythagoras: $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$, if $x \perp y$

Def. **p-Norm**: $\|x\|_p := (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$

Outer Product and Projection

Def. The **outer product** is the matrix that is returned, when multiplying the vectors x and y: $x \cdot y^H$
(rank = 1)

T 2.15 The **orthogonal projection** $P_y x$ of x on y is given by: $P_y x := \frac{1}{\|y\|^2} y y^H x$

Def. The **projection matrix** $P_y = \frac{1}{\|y\|^2} \cdot y y^H$
 $P_y^H = P_y$ (Hermitian), $P_y^2 = P_y$ (Idempotent)

Linear Transformations

For all $x, \tilde{x} \in \mathbb{E}^n$ and $\gamma \in \mathbb{E}$:

$$A(\gamma x + \tilde{x}) = \gamma(Ax) + (A\tilde{x})$$

Def. **image of A**: $imA := \{Ax \in \mathbb{E}^m; x \in \mathbb{E}^n\}$

Inverse

Def. A nxn matrix A is **invertible**, if there exists a matrix A^{-1} , such that $A \cdot A^{-1} = I_n = A^{-1} A$.

T 2.17 4 equivalent statements:

- A is invertible
- $\exists X$ such that $AX = I_n$
- X is definitive
- A is non-singular, i.e. rank A = n

T 2.18 With two non-singular nxn matrices A and B:

- A^{-1} is non-singular and $(A^{-1})^{-1} = A$
- AB is non-singular and $(AB)^{-1} = B^{-1} A^{-1}$
- A^H is non-singular and $(A^H)^{-1} = (A^{-1})^H$

T 2.19 If A is non-singular, $Ax = b$ has exactly one solution for every b: $x = A^{-1} b$

Find inverse $O(n^3)$: $[A | I] \longrightarrow [I | A^{-1}]$

-> using elementary row operations

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \det A \neq 0 \Leftrightarrow A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Orthogonal and Unitary Matrices

Def. We call a matrix unitary or orthogonal, if $A^H A = I_n$, $A^T A = I_n$ respectively.

T 2.20 Let A and B be unitary:

- A is non-singular and $A^{-1} = A^H$
- $AA^H = I_n$
- A^{-1} is unitary (/orthogonal)
- AB is unitary (/orthogonal)

T 2.21 A linear transformation defined by an orthogonal or unitary nxn matrix A is **length preserving** (/isometric) and **angle preserving**:
 $\|Ax\| = \|x\|$, $\langle Ax, Ay \rangle = \langle x, y \rangle$

Examples of Important Matrices

Rotation matrices (orthogonal)

$$\begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix} \text{ or } \begin{bmatrix} \cos(\varphi) & 0 & \sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Permutation matrices (orthogonal)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Block matrices

$$A = \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array} \right), \text{ if invertible } A^{-1} = \left(\begin{array}{c|c} a_{11}^{-1} & a_{12}^{-1} \\ \hline a_{21}^{-1} & a_{22}^{-1} \end{array} \right)$$

4 LU-Decomposition

LU-Decomposition

The **LU-Decomposition** is a tool to solve SLE. It does this by factorizing a matrix, making it easy to solve the same matrix for different RHS.

1. Find $PA = LR$

2. Solve $Lc = Pb$
(forward subst.)

3. Solve $Rx = c$
(backward subst.)

If rows are swapped
P gets permuted.

partial pivoting as a **pivot strategy** to minimize rounding errors

$$\begin{array}{l} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 & 0 & 0 & 1 \end{array} \right] \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 2 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 2 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array} \right] \\ \underbrace{\hspace{1.5cm}}_P \quad \underbrace{\hspace{1.5cm}}_R \quad \underbrace{\hspace{1.5cm}}_L \end{array}$$

5 Vector Spaces

General

Def. A **vector space** V over \mathbb{E} is a non-empty set, on which addition and scalar multiplication are defined.

- Axioms**
- (V1) $x + y = y + x$ ($\forall x, y \in V$)
 - (V2) $(x + y) + z = x + (y + z)$ ($\forall x, y, z \in V$)
 - (V3) $\exists o \in V : x + o = x$ ($\forall x \in V$) **zero vector**
 - (V4) $\forall x \exists (-x) : x + (-x) = o$ ($\forall x \in V$)
 - (V5) $\alpha(x + y) = \alpha x + \alpha y$ ($\forall \alpha \in \mathbb{E}, \forall x, y \in V$)
 - (V6) $(\alpha + \beta)x = \alpha x + \beta x$ ($\forall \alpha, \beta \in \mathbb{E}, \forall x \in V$)
 - (V7) $(\alpha\beta)x = \alpha(\beta x)$ ($\forall \alpha, \beta \in \mathbb{E}, \forall x \in V$)
 - (V8) $1x = x$ ($\forall x \in V$)

- T 4.1** $\forall \alpha \in \mathbb{E}, \forall x, y \in V$
- i) $0 \cdot x = o$
 - ii) $\alpha o = o$
 - iii) $\alpha \cdot x = o \Rightarrow x = o \text{ or } \alpha = 0$
 - iv) $(-\alpha) \cdot x = \alpha \cdot (-x) = -(\alpha x)$

T 4.2 $\forall x, y \in V, \exists z \in V$
 $x + z = y$, where z is definite and $z = y + (-x)$

Fields

Def. A **Field** is a non-empty set \mathbb{K} , on which addition and multiplication are defined.

- Axioms**
- (K1) $\alpha + \beta = \beta + \alpha$ ($\forall \alpha, \beta \in \mathbb{K}$)
 - (K2) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ ($\forall \alpha, \beta, \gamma \in \mathbb{K}$)
 - (K3) $\exists o \in V : \alpha + o = \alpha$ ($\forall \alpha \in \mathbb{K}$) **zero element**
 - (K4) $\forall \alpha \exists (-\alpha) : \alpha + (-\alpha) = o$ ($\forall \alpha \in \mathbb{K}$)
 - (K5) $\alpha \cdot \beta = \beta \cdot \alpha$ ($\forall \alpha, \beta \in \mathbb{K}$)
 - (K6) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ ($\forall \alpha, \beta, \gamma \in \mathbb{K}$)
 - (K7) $\exists 1 \in \mathbb{K} : \alpha \cdot 1 = \alpha$ ($\forall \alpha \in \mathbb{K}$) **identity element**
 - (K8) $\forall \alpha \in \mathbb{K}, \alpha \neq 0, \exists \alpha^{-1} \in \mathbb{K} : \alpha \cdot (\alpha^{-1}) = 1$ **inverse**
 - (K9) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ ($\forall \alpha, \beta, \gamma \in \mathbb{K}$)
 - (K10) $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ ($\forall \alpha, \beta, \gamma \in \mathbb{K}$)

Subspaces

Def. A **subspace** U is a non-empty subset of a vector space V , which is closed under sums and scalar multiples.

T 4.3 A subspace is a vector space itself.

T 4.4 With $A \in \mathbb{R}^{m \times n}$ and L_0 containing solutions of $Ax = o$, L_0 is a subspace of \mathbb{R}^n .

Def. The set of all linear combinations of v_1, \dots, v_n is a subspace spanned by these vectors.
 $span\{v_1, \dots, v_n\}$ / linear hull of v_1, \dots, v_n

Def. The vectors v_1, \dots, v_n are a **spanning set** of V , if $\forall w \in V \Rightarrow w \in span\{v_1, \dots, v_n\}$.

Linear Dependencies, Bases, Dimensions

Def. Vectors v_1, \dots, v_n are **linearly independent**, if no vector is a linear combination of the others.

$$\sum_{k=0}^n \alpha_k v_k = 0, \text{ only if } \alpha_1 = \dots = \alpha_n = 0$$

Def. A $span\{v_1, \dots, v_n\} = V$ is a **basis** of V , if v_1, \dots, v_n are linearly independent.
 \Rightarrow standard basis consists of unit vectors

Def. The **dimension** of V is denoted as,
 $dim V = |span V|$. $dim\{0\} = 0$

L 4.8 Any set $\{v_1, \dots, v_m\} \subset V$ with $|B_V| < m$ is linearly dependent.

T 4.9 Any set of linearly independent vectors of V can be extended to a basis of V .
 (as long as V has a finite spanning set)

C 4.10 The set of n linearly independent vectors is a basis of V in any finite vector space, if $dim V = n$.

Def. The coefficients ξ_k are **coordinates** of x in basis B . $\xi = (\xi_1 \dots \xi_n)^T$ is the **coordinate vector** and $x = \sum_{i=1}^n \xi_i b_i$ is the representation of x in coordinates of B .

Def. Two subspaces $U, U' \subset V$ are complementary, if every $v \in V$ has a specific representation $v = u + u'$, with $u \in U, u' \in U'$. In that case V is the **direct sum** of U and U' :
 $V = U \oplus U'$.

Change Of Basis, Coordinate Transformation

Def. To change from an old basis B to a new basis B' one can get the new basis vectors b'_k as follows:

$$b'_k = \sum_{i=1}^n \tau_{ik} b_i$$

$T = (\tau_{ik})$ is called **transformation matrix**.

T 4.13 $\xi = T\xi'$ and $\xi' = T^{-1}\xi$. T is invertible (non-singular)

Def. $B' = B \cdot T$ to get the new basis.

6 Linear Maps

General

Def. A transformation $F : X \rightarrow Y, x \mapsto F(x)$ is **linear**, if the following holds:

$$F(x + \tilde{x}) = F(x) + F(\tilde{x})$$

$$F(\gamma x) = \gamma F(x)$$

Functions

Def. **Injective**: $\forall x, x' \in X : F(x) = F(x') \Rightarrow x = x'$

Surjective: $F(X) = Y$

Bijjective: injective and surjective $\Rightarrow f^{-1}$ existiert

Transformation matrix

Let F be a linear transformation $X \rightarrow Y$. $F(b_l) \in Y$ can be written using a linear combination of the basis of Y :

$$F(b_l) = \sum_{k=1}^m a_{kl} c_k$$

Def. The $m \times n$ matrix $A = (a_{kl})$ is the **matrix for F relative to the given bases** in X and Y .

$$\begin{array}{ccc} x \in X & \xrightarrow{F} & y \in Y \\ \uparrow k_x^{-1} & & \uparrow k_y^{-1} \\ F(x) = y \Leftrightarrow A\xi = \eta & & \\ \downarrow k_x & & \downarrow k_y \\ \xi \in \mathbb{E}^n & \xrightarrow{A} & \eta \in \mathbb{E}^m \end{array}$$

Def. If F is bijective, we call it an **isomorphism**, if $X = Y$, we call it an **automorphism**.

L 5.1 If F is an isomorphism, there exists F^{-1} and F^{-1} is also an isomorphism.

Kernel, Image and Rank

Def. The **kernel** of F $\ker F := \{x \in X | F(x) = o\}$
is a subspace of X . **F injective** $\Leftrightarrow \ker F = \{o\}$ (T 5.6)

Def. The **image** of F $\operatorname{im} F := \{F(x) | x \in X\}$
is a subspace of Y . **F surjective** $\Leftrightarrow \operatorname{im} F = Y$

T 5.7 Rank-nullity theorem:

$$\dim X - \dim(\ker F) = \dim(\operatorname{im} F) = \operatorname{rank} F$$

Def. The **rank** F is defined as: $\operatorname{rank} F = \dim(\operatorname{im} F)$.

K 5.8 • $F: X \rightarrow Y$ injective $\Leftrightarrow \operatorname{rank} F = \dim X$
• $F: X \rightarrow Y$ surjective $\Leftrightarrow \operatorname{rank} F = \dim Y$
• $F: X \rightarrow Y$ bijective $\Leftrightarrow \operatorname{rank} F = \dim X = \dim Y$
Isomorphism
• $F: X \rightarrow X$ bijective $\Leftrightarrow \operatorname{rank} F = \dim X$
Automorphism $\Leftrightarrow \ker F = o$

Def. Two vector spaces X, Y are isomorphic, if there exists an isomorphism $F: X \rightarrow Y$.

T 5.9 Two finite vector spaces X, Y are isomorphic
 $\Leftrightarrow \dim X = \dim Y$

C 5.10 i) $\operatorname{rank}(G \circ F) \leq \min\{\operatorname{rank} F, \operatorname{rank} G\}$
ii) G injective $\Rightarrow \operatorname{rank}(G \circ F) = \operatorname{rank} F$
iii) G surjective $\Rightarrow \operatorname{rank}(G \circ F) = \operatorname{rank} F$

Matrices as Linear Mappings

Def. The **column space / range** of A is the subspace
 $R(A) = \operatorname{span}\{a_1, \dots, a_n\} = \operatorname{im} A$

Def. The **null space** of A is the subspace
 $N(A) = L_0(Ax = o) = \ker A$
 $\# \text{free_parameters} = \dim(N(A))$

T 5.12 With $\operatorname{rank} A = r$ and solution set L_0 of $Ax = o$:
 $\dim L_0 \equiv \dim N(A) \equiv \dim(\ker A) = n - r$

T 5.13 for $A^{m \times n}$: $\operatorname{rank} A = \dots$
i) $\# \text{pivotelements in reduced form of } A$
ii) $\dim(\operatorname{im} A)$
iii) $\dim(\text{column space})$
iv) $\dim(\text{row space})$

C 5.14 $\operatorname{rank} A^T = \operatorname{rank} A^H = \operatorname{rank} A$

T 5.16 With $A \in \mathbb{E}^{m \times n}, B \in \mathbb{E}^{p \times m}$:
i) $\operatorname{rank}(BA) \leq \min\{\operatorname{rank} B, \operatorname{rank} A\}$
ii) $\operatorname{rank} B = m(\leq p) \Rightarrow \operatorname{rank}(BA) = \operatorname{rank} A$
iii) $\operatorname{rank} A = m(\leq n) \Rightarrow \operatorname{rank}(BA) = \operatorname{rank} B$

T 5.18 For square matrices the following statements are equivalent:

- i) A is invertible
- ii) A is non-singular
- iii) $\operatorname{rank} A = n$
- iv) columns are lin. independent
- v) rows are linearly independent
- vi) $\operatorname{im} A \equiv R(A) = \mathbb{E}^n$
- vii) $\ker A \equiv N(A) = \{o\}$
- viii) $A: \mathbb{E}^n \rightarrow \mathbb{E}^n$ is an automorphism
- ix) A is the transformation matrix for a coordinate transformation

Affine Spaces and solutions to inhomogeneous LSE

Def. Let U be a subspace of V and $u_0 \in V$:
 $u_0 + U := \{u_0 + u | u \in U\}$ is an **affine (sub)space**.

Def. Let $F: X \rightarrow Y$ be a linear mapping and $y_0 \in Y$:
 $H: X \rightarrow y_0 + Y, x \mapsto y_0 + F(x)$ is an **affine mapping**.

T 5.19 Let x_0 be any solution of $Ax = b$ and let L_0 be the solution set for $Ax = o$:
The solution set L_b to $Ax = b$ is an affine subspace:
 $L_b = x_0 + L_0$

The Transformation matrix for coord. transformation

Let X, Y be vector spaces with $\dim X = n, \dim Y = m$:

$F: X \rightarrow Y, x \mapsto y$ a linear map,
 $A: \mathbb{E}^n \rightarrow \mathbb{E}^m, \xi \mapsto \eta$ a (mapping) matrix,
 $B: \mathbb{E}^n \rightarrow \mathbb{E}^m, \xi' \mapsto \eta'$ a (mapping) matrix,
 $T: \mathbb{E}^n \rightarrow \mathbb{E}^n, \xi' \mapsto \xi$ a transformation matrix in \mathbb{E}^n ,
 $S: \mathbb{E}^m \rightarrow \mathbb{E}^m, \eta' \mapsto \eta$ a transformation matrix in \mathbb{E}^m ,

$$\begin{array}{ccc} x \in X & \xrightarrow{F} & y \in Y \\ \uparrow k_x^{-1} & & \uparrow k_y^{-1} \\ \xi \in \mathbb{E}^n & \xrightarrow{A} & \eta \in \mathbb{E}^m \\ \uparrow T & & \uparrow S \\ \xi' \in \mathbb{E}^n & \xrightarrow{B} & \eta' \in \mathbb{E}^m \end{array} \quad \begin{array}{l} A = S \cdot B \cdot T^{-1} \\ B = S^{-1} \cdot A \cdot T \\ \text{If } F: X \rightarrow Y: \\ A = T \cdot B \cdot T^{-1} \\ B = T^{-1} \cdot A \cdot T \end{array}$$

Def. Two $n \times n$ matrices A and B are **similar**, if there exists a non-singular matrix T , such that either $B = T^{-1} \cdot A \cdot T$ or $A = T \cdot B \cdot T^{-1}$.

7 Vector Spaces with Scalar Product

General

Def. A **norm** in a vector space is a function $\|\cdot\|: V \rightarrow \mathbb{R}, x \mapsto \|x\|$, which is:
(N1) **positive definite**: $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = o$
(N2) **homogeneous**: $\|\alpha x\| = |\alpha| \|x\|$
(N3) **triangle inequality**: $\|x + y\| \leq \|x\| + \|y\|$
A vector space with norm is a **normed vector space**.

Def. A **inner product** in a vector space is a function $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{E}, x, y \mapsto \langle x, y \rangle$, which is:
(S1) **linearity (2nd factor)**: $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
 $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$
(S2) **hermitian (sym. in \mathbb{R})**: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
(S3) **positive definite**: $\langle x, x \rangle \geq 0$
 $\langle x, x \rangle = 0 \Rightarrow x = o$

Def. The **induced norm / length** of a vector is defined as: $\|\cdot\|: V \rightarrow \mathbb{R}, \|x\| \mapsto \sqrt{\langle x, x \rangle}$

T 6.1 **Cauchy-Schwarz inequality**: $|\langle x, y \rangle| \leq \|x\| \|y\|$

Def. The **angle** $\varphi = \angle(x, y), 0 \leq \varphi \leq \pi$, is defined as:
 $\varphi := \arccos\left(\frac{\langle x, y \rangle}{\|x\| \|y\|}\right)$ or $\varphi := \arccos\left(\frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|}\right)$

Def. Two vectors are **orthogonal**, if $\langle x, y \rangle = 0, x \perp y$.
Two subsets are orthogonal, if:
 $\forall x \in M, \forall y \in N \quad \langle x, y \rangle = 0, M \perp N$

T 6.2 **Pythagoras**: $x \perp y \Rightarrow \|x \pm y\|^2 = \|x\|^2 + \|y\|^2$

Orthonormal Bases

Def. A basis is **orthogonal**, if the basis vectors are pairwise orthogonal: $\langle b_k, b_l \rangle = 0$, for $k \neq l$.
We call the basis **orthonormal**, if all basis vectors are of length 1: $\langle b_k, b_k \rangle = 1$.

T 6.4 With an orthonormal basis $\{b_1, \dots, b_n\}$ and $x \in V$:

$$x = \sum_{k=1}^n \langle b_k, x \rangle b_k \Rightarrow \xi_k = \langle b_k, x \rangle$$

T 6.5 **Parseval's Identity**: with $\xi_k := \langle b_k, x \rangle_V, \eta_k := \langle b_k, y \rangle_V$
 $\langle x, y \rangle_V = \sum_{k=1}^n \overline{\xi_k} \eta_k = \xi^H \eta = \langle \xi, \eta \rangle_{\mathbb{E}^n}$

Therefore, the inner product of two vectors in V is equal to the scalar product of the respective coordinate vectors in \mathbb{E}^n :

$$\|x\|_V = \|\xi\|_{\mathbb{E}^n} \quad \angle(x, y)_V = \angle(\xi, \eta)_{\mathbb{E}^n} \quad x \perp y \Leftrightarrow \xi \perp \eta$$

Gram-Schmidt Process

$$b_1 := \frac{a_1}{\|a_1\|_V}$$

$$\tilde{b}_k := a_k - \sum_{j=1}^{k-1} \langle b_j, a_k \rangle_V b_j$$

$$b_k := \frac{\tilde{b}_k}{\|\tilde{b}_k\|_V}$$

T 6.6 After k steps $\{b_1, \dots, b_k\}$ are pairwise orthonormal.
If $\{a_1, \dots, a_n\}$ is a basis of V , so is $\{b_1, \dots, b_k\}$.

Orthogonal Complements

Def. U^\perp is the orthogonal complement of the subspace U : $U^\perp \oplus U = V$

T 6.9 For a complex $m \times n$ matrix with rank r :

$$\begin{aligned} \bullet N(A) &= (R(A^H))^\perp \subset \mathbb{E}^n & \bullet N(A^H) &= (R(A))^\perp \subset \mathbb{E}^m \\ \bullet N(A) \oplus R(A^H) &= \mathbb{E}^n & \bullet N(A^H) \oplus R(A) &= \mathbb{E}^m \\ \bullet \dim R(A) &= r & \bullet \dim N(A) &= n - r \\ \bullet \dim R(A^H) &= r & \bullet \dim N(A^H) &= m - r \end{aligned}$$

We call these subspaces **fundamental subspaces** of A .

Orthogonal and Unitary Change Of Basis

To get from B to B' , where both are orthonormal

bases, we can write $b'_k = \sum_{i=1}^n \tau_{ik} b_i$ and get

the transformation matrix T , with $T^{-1} = T^H$.

T 6.11 Therefore: $\xi = T\xi'$, $\xi' = T^H\xi$

Additionally, $B = B'T$ and $B' = BT^H$, where all matrices are unitary/orthogonal.

T 6.10 The transformation matrix for a change of basis between orthonormal bases is unitary/orthogonal.

C 6.12 $\langle x, z \rangle_V = \xi^H \eta = (\xi')^H \eta' = \langle \xi', \eta' \rangle$
 $\Rightarrow T$ is length and angle preserving

$$\begin{array}{ccc} x \in X & \xrightarrow{F} & z \in Y \\ \uparrow k_x^{-1} & & \uparrow k_y^{-1} \\ \xi \in \mathbb{E}^n & \xrightarrow{A} & \eta \in \mathbb{E}^m \\ \uparrow T & & \uparrow S \\ \xi' \in \mathbb{E}^n & \xrightarrow{B} & \eta' \in \mathbb{E}^m \end{array}$$

$$\begin{aligned} A &= S \cdot B \cdot T^H \\ B &= S^H \cdot A \cdot T \\ \text{If } Y = X, k_y &= k_x \\ \text{and } S &= T: \\ A &= T \cdot B \cdot T^H \\ B &= T^H \cdot A \cdot T \\ \Rightarrow A, B &\text{ are Hermitian} \end{aligned}$$

Orthogonal and Unitary Transformations

Def. A linear transformation $F: X \rightarrow Y$ is **unitary** / **symmetric**, if: $\langle F(x), F(y) \rangle_Y = \langle x, y \rangle_X$

- T 6.13**
- i) F is length preserving: $\|F(x)\|_Y = \|x\|_X$
 - ii) F is angle preserving: $x \perp y \Leftrightarrow F(x) \perp F(y)$
 - iii) $\ker F = 0$, F is injective

If additionally $\dim X = \dim Y < \infty$:

- iv) F is an isomorphism
- v) $\{b_1, \dots, b_n\}$ is an orthonormal basis for X
 $\Leftrightarrow \{F(b_1), \dots, F(b_n)\}$ is an orthonormal basis for Y
- vi) F^{-1} is unitary/orthogonal
- vii) The transformation matrix A is unitary/orthogonal.

8 Least Squares Method

General

Let $Ax = y$ be an **overdetermined linear system** (more equations than unknowns). As an exact solution is not guaranteed we try to minimize the length of the **residual vector** (orthogonal to $R(A)$): $r := y - Ax$

Therefore, to find the **least squares solution** we solve: $x^* = \arg \min_{x \in \mathbb{E}^n} \|Ax - b\|_2^2$

Def. The **normal equations**, $A^H Ax = A^H y$ can be used to solve the given LSE.

Lemma: $A^H A$ non-singular $\Leftrightarrow \text{rank } A = n$

Def. **Moore-Penrose Pseudo-Inverse**: $A^+ := (A^H A)^{-1} A^H$
(for when $\text{rank } A = n$)

9 QR Factorization & Decomposition

General

$A = QR$ Q : orthogonal matrix; R : upper triangular matrix

This decomposition is definite, if $m \geq n$ and $\text{rank } A = n$.

Calculate QR Factorization

I: Gram-Schmidt process with columns of $A \Rightarrow Q$

II: Solve $R = Q^T A$

Alternatively:

$$\bullet r_{11} := \|a_1\| \quad \bullet r_{jk} := \langle q_j, a_k \rangle \quad \bullet r_{kk} := \|\tilde{q}_k\|$$

Least-Squares with QR Factorization

Normal equations: $A^T A x = A^T b$

$$\begin{aligned} (QR)^T (QR)x &= (QR)^T b \\ R^T Q^T Q R x &= R^T Q^T b \\ R^T R x &= R^T Q^T b \\ R x &= Q^T b \end{aligned}$$

T can be replaced with H

10 Determinant

General

T 8.1 There are $n!$ permutations in S_n .

Def. The **sign** of a permutation is defined as follows:

$$\text{sign } p = \begin{cases} +1, & \text{if \#permutations even} \\ -1, & \text{if \#permutations odd} \end{cases}$$

Def. The **determinant** of an $n \times n$ matrix A is:

$$\det(A) = \sum_{p \in S_n} \text{sign } p \cdot a_{1,p(1)} a_{2,p(2)} \dots a_{n,p(n)}$$

T 8.3 Properties of $\det: \mathbb{E}^{n \times n} \rightarrow \mathbb{E}, A \mapsto \det(A)$:

- i) $\det(+)$ is linear in every row
- ii) when swapping two rows, $\det(A)$ switches sign
- iii) $\det(I) = 1$

T 8.4 iv) if A has a **row of zeros**, $\det(A) = 0$

v) $\det(\gamma A) = \gamma^n \det(A)$

vi) if A has **two identical rows**, $\det(A) = 0$

vii) if we add multiples of two rows to each other, $\det(A)$ doesn't change

viii) if A is triangular or diagonal, $\det(A) = \prod_{i=1}^n a_{ii}$

$\det(A) \Leftrightarrow A$ is singular

$\det(A) \Leftrightarrow A$ is non-singular

T 8.5 After applying the gauss algorithm to A :

$$\det(A) = (-1)^v \prod_{k=1}^n r_{kk}$$

T 8.7 Multiplicativity: $\det(AB) = \det(A) \cdot \det(B)$

C 8.8 A non-singular $\Rightarrow \det(A^{-1}) = (\det(A))^{-1}$

T 8.9 $\det(A)^T = \det(A)$ and $\det(A^H) = \overline{\det(A)}$

Determinant for block matrices

C 8.14 $\begin{vmatrix} A & B \\ O & D \end{vmatrix} = \det(A) \cdot \det(D)$

11 Eigenvalues and Eigenvectors

General

Def. We call the number $\lambda \in \mathbb{E}$ **eigenvalue (EVal)** of the linear transformation $F: V \rightarrow V$, if there exists an **eigenvector (EVec)** $v \in V$, $v \neq 0$, such that: $F(v) = \lambda v$

Def. We denote the **eigenspace** E_λ , containing all eigenvectors for λ : $E_\lambda := \{v \in V | F(v) = \lambda v\}$ (the eigenspace is a subspace of V)

Def. The set of all eigenvalues of F is called **spectrum**, denoted $\sigma(F)$.

Def. $\xi \in \mathbb{E}^n$ is an eigenvector of A if $A\xi = \xi\lambda$

L 9.1 A linear transformation F and its matrix representation have the same eigenvalues and the eigenvalues are related respecting the coordinate transformation.

L 9.2 λ is an eigenvalue, iff $\ker(A - \lambda I)$ doesn't contain just the zero vector (singular). $E_\lambda = \ker(A - \lambda I)$

Def. The **geometric multiplicity** of λ is $\dim E_\lambda$.