Linear Algebra - Cheat Sheet (HS22)

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1 Complex Numbers

General

$z = \underbrace{x}_{\text{Re}} + i \underbrace{y}_{\text{Im}} = \underbrace{r \cdot (\cos(\varphi) + i \cdot \sin(\varphi)) = r \cdot e^{i\varphi}}_{\text{Polarform}}$

$\overline{z} = x - iy = r \cdot e^{i(2\pi - \varphi)}$

$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$

$$\varphi = \begin{cases} arctan(\frac{y}{x}), & \text{I Q.} \\ arctan(\frac{y}{x}) + \pi, & \text{II/III Q.} \\ arctan(\frac{y}{x}) + 2\pi, & \text{IV Q.} \end{cases}$$

Operations

$+/-: \ (x_1 + x_2) + (y_1 + y_2)i$

$z_1 \cdot z_2: \ (x_1 + y_1 i)(x_2 + y_2 i) = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$

$\frac{z_1}{z_2}: \ \frac{z_1 \cdot \overline{z}_2}{|z_2|^2} = \frac{r_1}{r_2} \cdot e^{i(\varphi_1 - \varphi_2)}$

$z^n: \ r^n \cdot e^{i\varphi n}$

$\sqrt{a}: \ a = z^n \Leftrightarrow |a| \cdot e^{i\alpha} = r^n \cdot e^{i\varphi n} \begin{cases} r = \sqrt[n]{|a|} \\ \varphi = \frac{\alpha + 2k\pi}{n} \end{cases}_{k=0,\ldots,n-1}$

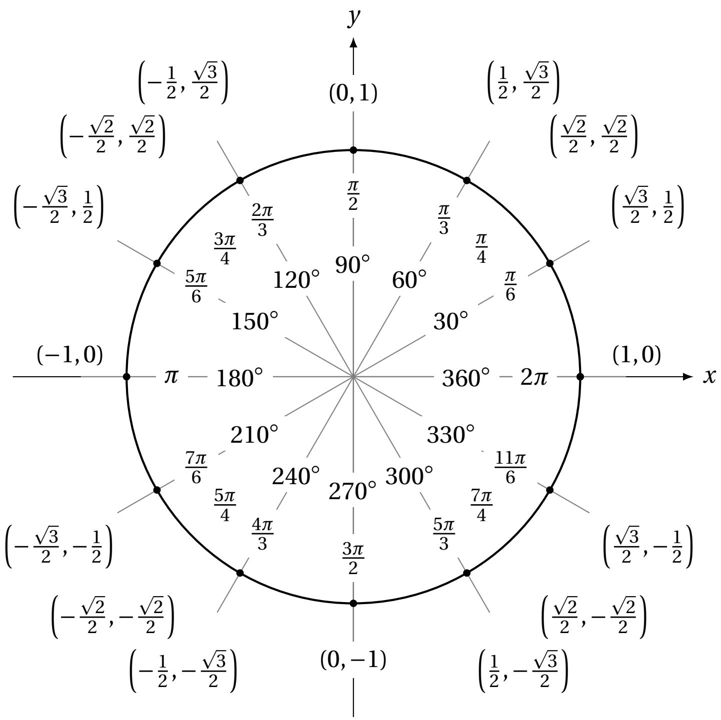
Polynomials

degree 2: $z = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

special case: $az^n + c = 0 \Leftrightarrow z = \sqrt[n]{-\frac{c}{a}}$

With polynomials with complex roots, the roots occur as a complex-conjugate pair.

Polynomials over $\mathbb{C}$ with an odd degree have at least one root in $\mathbb{R}$.



2 LSE

Gauss-Algorithm

runtime: $O(n^3)$

elementary row operations:

switch, multiply and add/subtract rows

goal: row echelon form

x_1	x_2	...	x_n		
a_{11}	a_{12}	...	a_{1n}	b_1	rank
0	a_{22}	...	a_{2n}	b_2	
...	...	...	...	...	
0	0	0	a_{rn}	b_r	
...	...	...	...	...	consistency conditions
0	0	...	0	b_m	
LHS				RHS	

if RHS always 0: homogeneous LSE

consistency conditions (VB): $b_{r+1} = \ldots = b_m = 0$

Number of solutions

T 1.1 $Ax = b$ min. 1 solution $\Leftrightarrow$ $(r = m)$ or $(r < m + VB)$ if this is the case:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions

$\Rightarrow r = m$:

- $r = n = m$: only 1 solution, non-singular LSE
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

$\Rightarrow r < m$:

- $r = n$: only 1 solution
- $r < n$: ∞ many solutions, $(n - r)$ free parameters

K 1.5 In a homogeneous LSE non-trivial solutions exist only if $r < n$.

T 2.5 $Ax = b$ has a solution $\Leftrightarrow$ b is a linear combination of columnvectors of A

3 Matrices & Vectors

General

mxn Matrices have m rows and n columns.
The element (i, j) can be denoted as $a_{i,j}$ or $(A)_{i,j}$

T 2.1

$(\alpha\beta)A = \alpha(\beta A)$	$(A + B) + C = A + (B + C)$
$(\alpha A)B = \alpha(AB)$	$(AB) \cdot C = A \cdot (BC)$
$(\alpha + \beta) \cdot A = \alpha A + \beta A$	$(A + B) \cdot C = AC + BC$
$\alpha(A + B) = \alpha A + \alpha B$	$A \cdot (B + C) = AB + AC$
$A + B = B + A$	

⚠ in general $AB \neq BA$

If $AB = BA$ we say «A and B commute»

Def. If $AB = O$, we call A and B divisors of zero.

Def. A linear combination of vectors $a_1 \ldots a_n$ is an expression of the following type: $\alpha_n \cdot a_n + \ldots + \alpha_1 \cdot a_1$

Def. A matrix is symmetric when $A^T = A$ and Hermitian when $A^H = A$ (real diagonal).

Def. A matrix is skew-symmetric when $A^T = -A$. (zeros on diagonal)

T 2.6 $(A^T)^T = A$ $\quad (\alpha A)^T = \alpha(A^T)$
$(AB)^T = B^T A^T$ $\quad (A + B)^T = A^T + B^T$

T 2.7 If A, B symmetric: $AB = BA \Leftrightarrow AB$ symmetric
For any A: $A^T A = AA^T$ (symmetric)

Scalar Product and Norm

Def. Eucl. scalar product (SP): $\langle x, y\rangle :\equiv x^H y$
(inner product)

T 2.9 linearity in second factor: $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
$\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$
symmetric / hermitian: $\langle x, y\rangle = \overline{\langle y, x\rangle}$
positive definite: $\langle x, x\rangle \geq 0$; if $'='\Rightarrow\ x = 0$

C 2.10 bilinearity in $\mathbb{R}^n$: $\langle w+x, y\rangle = \langle w, y\rangle + \langle x, y\rangle$
$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$
sesquilinearity in $\mathbb{C}^n$: $\langle w+x, y\rangle = \langle w, y\rangle + \langle x, y\rangle$
$\langle \alpha x, y\rangle = \overline{\alpha}\langle x, y\rangle$

Def. Eucl. norm / 2-norm: $||x|| :\equiv \sqrt{\langle x, x\rangle}$

(Cauchy-)Schwarz inequality (CBS inequality)
$|\langle x, y\rangle| \leq ||x|| \cdot ||y||$
The equality holds iff y is a multiple of x or vice versa

T 2.12 The following holds for the 2-norm:
(N1) positive definite: $||x|| \geq 0$, if $'='\Rightarrow\ x = 0$
(N2) $||\alpha x|| = |\alpha|\ ||x||$
(N3) triangle inequality: $||x \pm y|| \leq ||x|| + ||y||$

Def. angle φ between x and y: $\varphi = arc\ cos\left(\frac{\langle x, y\rangle}{||x|| \cdot ||y||}\right)$

Def. x and y are orthogonal, if $\langle x, y\rangle = 0$; $x \perp y$

T 2.13 Pythagoras: $||x \pm y||^2 = ||x||^2 + ||y||^2$, if $x \perp y$

Def. p-Norm: $||x||_p :\equiv (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}$

Outer Product and Projection

Def. The outer product is the matrix that is returned, when multiplying the vectors x and y: $x \cdot y^H$
$(rank = 1)$

T 2.15 The orthogonal projection $P_y x$ of x on y is given by: $P_y x :\equiv \frac{1}{||y||^2} y y^H x$

Def. The projection matrix $P_y = \frac{1}{||y||^2} \cdot y y^H$
$P_y^H = P_y$ (Hermitian), $P_y^2 = P_y$ (Idempotent)

Linear Transformations

For all $x, \tilde{x} \in \mathbb{E}^n$ and $\gamma \in \mathbb{E}$:
$A(\gamma x + \tilde{x}) = \gamma(Ax) + (A\tilde{x})$
Def. image of A: $imA :\equiv \{Ax \in \mathbb{E}^m; x \in \mathbb{E}^n\}$

Inverse

Def. A nxn matrix A is invertible, if there exists a matrix A^{-1}, such that $A \cdot A^{-1} = I_n = A^{-1}A$.

T 2.17 4 equivalent statements:
i) A is invertible
ii) $\exists X$ such that $AX = I_n$
iii) X is definitive
iv) A is non-singular, i.e. rank A = n

T 2.18 With two non-singular nxn matrices A and B:
i) A^{-1} is non-singular and $(A^{-1})^{-1} = A$
ii) AB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$
iii) A^H is non-singular and $(A^H)^{-1} = (A^{-1})^H$

T 2.19 If A is non-singular, $Ax = b$ has exactly one solution for every b: $x = A^{-1}b$

Find inverse $O(n^3)$: $[\ A\ |\ I\ ] \longrightarrow [\ I\ |\ A^{-1}\ ]$
-> using elementary row operations

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } detA \neq 0 \Leftrightarrow A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Orthogonal and Unitary Matrices

Def. We call a matrix unitary or orthogonal, if $A^H A = I_n$, $A^T A = I_n$ respectively.

T 2.20 Let A and B be unitary:
i) A is non-singular and $A^{-1} = A^H$
ii) $AA^H = I_n$
iii) A^{-1} is unitary (/orthogonal)
iv) AB is unitary (/orthogonal)

T 2.21 A linear transformation defined by an orthogonal or unitary nxn matrix A is length preserving (/isometric) and angle preserving:
$||Ax|| = ||x||$, $\langle Ax, Ay\rangle = \langle x, y\rangle$

Examples of Important Matrices

Rotationmatrices (orthogonal)

$$\begin{bmatrix} cos(\varphi) & sin(\varphi) \\ -sin(\varphi) & cos(\varphi) \end{bmatrix} \text{ or } \begin{bmatrix} cos(\varphi) & 0 & sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\varphi) & 0 & cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Permutationmatrices (orthogonal)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Blockmatrices

$$A = \left(\begin{array}{c|c} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right), \text{ if invertible } A^{-1} = \left(\begin{array}{c|c} a_{11}^{-1} & a_{12}^{-1} \\ a_{21}^{-1} & a_{22}^{-1} \end{array}\right)$$

4 LU-Decomposition

LU-Decomposition

The LU-Decomposition is a tool to solve SLE. It does this by factorizing a matrix, making it easy to solve the same matrix vor different RHS.

1. Find $PA = LR$

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

2. Solve $Lc = Pb$
(forward subst.)

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 2 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

3. Solve $Rx = c$
(backward subst.)

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 2 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{4}{3} & 1 & \frac{2}{3} & 1 \end{array}\right]$$

$\underbrace{\qquad}_{P}\quad \underbrace{\qquad}_{R}\quad \underbrace{\qquad}_{L}$

If rows are swapped
P gets permutated.

partial pivoting as a pivot strategy to minimize rounding errors

5 Vector Spaces

General

Def. A vector space V over $\mathbb{E}$ is a non-empty set, on which addition and scalar multiplication are defined.

Axioms
- $(V1)$ $x+y=y+x$ $(\forall x,y\in V)$
- $(V2)$ $(x+y)+z=x+(y+z)$ $(\forall x,y,z\in V)$
- $(V3)$ $\exists o\in V: x+o=x$ $(\forall x\in V)$ zero vector
- $(V4)$ $\forall x\,\exists(-x): x+(-x)=o$ $(\forall x\in V)$
- $(V5)$ $\alpha(x+y)=\alpha x+\alpha y$ $(\forall\alpha\in\mathbb{E},\ \forall x,y\in V)$
- $(V6)$ $(\alpha+\beta)x=\alpha x+\beta x$ $(\forall\alpha,\beta\in\mathbb{E},\ \forall x\in V)$
- $(V7)$ $(\alpha\beta)x=\alpha(\beta x)$ $(\forall\alpha,\beta\in\mathbb{E},\ \forall x\in V)$
- $(V8)$ $1x=x$ $(\forall x\in V)$

T 4.1 $\forall\alpha\in\mathbb{E},\forall x,y\in V$
- i) $0\cdot x=o$
- ii) $\alpha o=o$
- iii) $\alpha\cdot x=o\Rightarrow x=o\ or\ \alpha=0$
- iv) $(-\alpha)\cdot x=\alpha\cdot(-x)=-(\alpha x)$

T 4.2 $\forall x,y\in V,\ \exists z\in V$
$x+z=y$, where z is definite and $z=y+(-x)$

Fields

Def. A Field is a non-empty set $\mathbb{K}$, on which addition and multiplication are defined.

Axioms
- $(K1)$ $\alpha+\beta=\beta+\alpha$ $(\forall\alpha,\beta\in\mathbb{K})$
- $(K2)$ $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ $(\forall\alpha,\beta,\gamma\in\mathbb{K})$
- $(K3)$ $\exists o\in V:\alpha+o=\alpha$ $(\forall\alpha\in\mathbb{K})$ zero element
- $(K4)$ $\forall\alpha\,\exists(-\alpha):\alpha+(-\alpha)=o$ $(\forall\alpha\in\mathbb{K})$
- $(K5)$ $\alpha\cdot\beta=\beta\cdot\alpha$ $(\forall\alpha,\beta\in\mathbb{K})$
- $(K6)$ $(\alpha\cdot\beta)\cdot\gamma=\alpha\cdot(\beta\cdot\gamma)$ $(\forall\alpha,\beta,\gamma\in\mathbb{K})$
- $(K7)$ $\exists 1\in\mathbb{K}:\alpha\cdot 1=\alpha$ $(\forall\alpha\in\mathbb{K})$ identity element
- $(K8)$ $\forall\alpha\in\mathbb{K},\ \alpha\neq 0,\ \exists\alpha^{-1}\in\mathbb{K}:$ $\alpha\cdot(\alpha^{-1})=1$ inverse
- $(K9)$ $\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma$ $(\forall\alpha,\beta,\gamma\in\mathbb{K})$
- $(K10)$ $(\alpha+\beta)\cdot\gamma)=\alpha\cdot\gamma+\beta\cdot\gamma$ $(\forall\alpha,\beta,\gamma\in\mathbb{K})$

Subspaces

Def. A subspace U is a non-empty subset of a vector space V, which is closed under sums and scalar multiples.

T 4.3 A subspace is a vector space itself.

T 4.4 With $A\in\mathbb{R}^{mxn}$ and L_0 containing solutions of $Ax=o$, L_0 is a subspace of $\mathbb{R}^n$.

Def. The set of all linear combinations of $v_1,...,v_n$ is a subspace spanned by these vectors.
$span\{v_1,...,v_n\}$ / linear hull of $v_1,...,v_n$

Def. The vectors $v_1,...,v_n$ are a spanning set of V, if $\forall w\in V\Rightarrow w\in span\{v_1,...,v_n\}$.

Linear Dependencies, Bases, Dimensions

Def. Vectors $v_1,...,v_n$ are linearly independent, if no vector is a linear combination of the others.
$\Sigma_{k=0}^n\alpha_k v_k=0$, only if $\alpha_1=...=\alpha_k=0$

Def. A $span\{v_1,...,v_n\}=V$ is a basis of V, if $v_1,...,v_n$ are linearly independent.
$\Rightarrow$ standard basis consists of unit vectors

Def. The dimension of V is denoted as,
$dimV=|spanV|$. $dim\{0\}=0$

L 4.8 Any set $\{v_1,...,v_m\}\subset V$ with $|B_V|<m$ is linearly dependent.

T 4.9 Any set of linearly independent vectors of V can be extended to a basis of V. (as long as V has a finite spanning set)

C 4.10 The set of n linearly independent vectors is a basis of V in any finite vector space, if $dimV=n$.

Def. The coefficients ξ_k are coordinates of x in basis B. $\xi=(\xi_1...\xi_n)^T$ is the coordinate vector and $x=\Sigma_{i=1}^n\xi_i b_i$ is the representation of x in coordinates of B.

Def. Two subspaces $U,\ U'\subset V$ are complementary, if every $v\in V$ has a specific representation $v=u+u'$, with $u\in U,\ u'\in U'$. In that case V is the direct sum of U and U':
$V=U\bigoplus U'$.