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## Final Dissertation

Complex-Valued Deep Learning for Condition Monitoring

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# Abstract

At present, the vast majority of deep learning architectures are based on real-valued operations and representations. However, recent works and fundamental theoretical analyses suggest that complex numbers can have richer expressiveness: many deterministic wave signals, such as seismic, electrical, or vibrational, contain information in their phase, which risks being lost when studied using a real-valued model. However, despite their attractive properties and potential, only recently complex-valued algorithms have started to be introduced in the deep neural networks frameworks.

In this work, we move forward in this direction developing and implementing a coherent and working structure to train complex-valued models, remaining rigorous from a mathematical perspective but, at the same time, seeking for stability and accuracy of the training process.

As a first application of this solution, we show the results obtained applying complex-valued deep neural networks for condition monitoring in industrial applications. Different Deep Network architectures have been trained on vibrational signals extracted from sensors attached to gearmotors to detect failures. Finally, we compare the performances obtained with real and complex-valued neural networks.



# Chapter 1

## Introduction

### 1.1 Intro

- At present, the vast majority of deep learning architectures are based on real-valued operations and representations. However, recent works and fundamental theoretical analyses suggest that complex numbers can have richer expressiveness: many deterministic wave signals, such as seismic, electrical, or vibrational, contain information in their phase, which risks being lost when studied using a real-valued model.
- most methods nowadays work like: drop phase info, take modulus or 2 independent channels, ignoring inherent relationships among them
- complex-valued deep learning grew very slowly during the years, due to a long list of factors that we will list during the thesis. there are inherently complex data (e.g. MRI signals) and also real data can be "moved" to the complex space (e.g. via Fourier transform)

### 1.2 Previous work

- [4]: Modern networks apply real-valued transformations on the data. Particularly, convolutions in convolutional neural networks discard phase information entirely. Many deterministic signals, such as seismic data or electrical signals, contain significant information in the phase of the signal. We explore complex-valued deep convolutional networks to leverage non-linear feature maps. While it has been shown that phase content can be restored in deep neural networks, we show how including phase information in feature maps improves both training and inference from deterministic physical data. Furthermore, we show that the reduction of parameters in a complex network outperforms larger real-valued networks.

### 1.3 Neuronal Synchrony

### 1.4 Thesis Organization



## Part I

# Complex-Valued Deep Learning



# Chapter 2

# Complex Analysis

## Introduction

At present, the vast majority of deep learning architectures are based on real-valued operations and representations. However people knew that there was a numerical domain even larger, and more general, than the real one: the complex space  $\mathbb{C}$ . We cannot forget also about the fact that Quantum Mechanics told us that the "true nature" of reality is inherently complex (since quantum states and operators live in a complex Hilbert space).

All of this is implicitly suggesting that actual deep learning techniques can be further extended to an even more general, and hopefully more efficient, representation. This possibility was already under the eyes of researchers from the dawn of deep learning, since the first publications covering this hypotheses are almost thirty years old.

In this chapter we are going to recall the main concepts, from complex analysis, that will be helpful during the practical extent of real-valued deep learning techniques in the complex world. We will address the main issues encountered in this development, mainly due to characteristics of the complex domain without an effective real counterpart (e.g. Liouville's and Identity theorems), but we will also focus on the property known as *circularity*, that in principle could be the key to prove the effective advantages brought by complex-valued models. In the end, we will examine the core of modern deep learning approach in presence of a complex dataset, focused on the existing mapping between numbers in  $\mathbb{C}$  and points in  $\mathbb{R}^2$ , proving that in many cases it is inadequate, and should be replaced with a new and proper methodology living in the complex domain.

## 2.1 Complex Numbers

A natural way to build complex-valued neural networks is to extend real-valued models to handle complex-valued neurons. Obviously, this extension requires also the parameters (weights, biases) and the activations to be complex-valued. In contrast, the loss function should be real-valued, in order to allow for an empirical risk minimization during the training process. Despite all the operations that need to be redefined for this adaptation, the most challenging part of the problem consists into constructing a coherent and stable algorithm to train such networks.

The benefits of retaining a complex representation remains an open question; however, several works and implementations, have recently proved that complex networks could leverage mathematical properties of  $\mathbb{C}$  to learn more efficient transform functions than with real-valued networks.

But before getting into the motivation and background for complex-valued deep learning, we should first recall a few concepts of *complex analysis*<sup>1</sup>.

**Definition 2.1.1.** A **complex number**  $z \in \mathbb{C}$  takes the form

$$z = x + iy$$

---

<sup>1</sup>A nice book to recall all the concepts explained is [5].

where  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$  are its real and imaginary components, respectively, and  $i = \sqrt{-1}$ .

The same complex number may be written in exponential, or polar, form in terms of its magnitude  $m \in \mathbb{R}^+$ , and its phase  $\theta \in \mathbb{R}$ , as

$$z = me^{i\theta}$$

Magnitude and phase can then be extracted as follow:

$$m = \|z\| = (z\bar{z})^{1/2} \quad \theta = -i \log \frac{z}{\|z\|}$$

Each complex number in  $\mathbb{C}$  can be viewed as a point  $(x, y)$  in the Euclidean space  $\mathbb{R}^2$ ; the two, however, are not isomorphic, especially because the first is a field, while the second a vector space, and also because they have distinct notions of differentiability.

**Observation 2.1.1.** Given two complex numbers in polar form,  $z = re^{i\theta}$  and  $w = se^{i\varphi}$ , then

$$zw = rse^{i(\theta+\varphi)}$$

So, multiplication by a complex number corresponds to an *homothety* (rotation composed with a dilation) in  $\mathbb{R}^2$ .

**Definition 2.1.2.** The **complex conjugate** of  $z = x + iy$  is defined by  $\bar{z} = x - iy$  (or, equivalently,  $\bar{z} = me^{-i\theta}$ ), and it is obtained by a *reflection* across the real axis in the plane.

The functions defined in the complex plane can be decomposed as well:

$$f : D \subseteq \mathbb{C} \rightarrow \mathbb{C} \quad f(z) = u(x, y) + iv(x, y)$$

with  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$  begin real-valued functions. Also the notion of continuity is the same that holds for functions in  $\mathbb{R}$ .

Now, since the notion of *convergence* is the same for  $\mathbb{C}$  and  $\mathbb{R}^2$  (because absolute values in the first and euclidean distances in the second coincide), we can recover as well the triangle inequality,

$$\|z + w\| \leq \|z\| + \|w\| \quad \forall z, w \in \mathbb{C}$$

that will hold also in the complex plane. Because of the latter relation, it is immediate that if  $f : D \rightarrow \mathbb{C}$  is continuous, then the real-valued function defined by  $z \mapsto \|f(z)\|$  is continuous too. We say that  $f$  attains a **maximum** at the point  $z_0 \in D$  if

$$\|f(z)\| \leq \|f(z_0)\| \quad \forall z \in D$$

with the inequality reversed for the definition of a **minimum**.

This definition will be relevant when reformulating the learning process as an optimization problem, since  $\mathbb{C}$  is not an ordered field with respect to the canonical addition and multiplication. In order to compare two complex numbers, in fact, one needs first to establish a *total ordering* on the set, like the lexicographical order (which prioritizes real parts), or the polar order (which prioritizes magnitudes). The next notions are central in complex analysis, effectively without any real counterpart.

**Definition 2.1.3.** Let  $D$  be and open set in  $\mathbb{C}$ , and  $f$  a complex-valued function on  $D$ . The function  $f$  is **holomorphic** at the point  $z_0 \in D$  if the limit

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

that corresponds to the *derivative* of  $f$ , exists and is finite.

Equivalently,  $f$  is holomorphic on a domain  $D \subseteq \mathbb{C}$  if it is for every  $z_0 \in D$ .

It should be emphasized that in the above limit,  $h \in \mathbb{C}$  is a complex number that may approach 0 from any direction. The concept of differentiability is, in fact, much stronger in  $\mathbb{C}$  than with functions of real variables: a holomorphic function will actually be infinitely many times complex differentiable, i.e., the existence of the first derivative will guarantee the existence of derivatives of any order. In the real case, instead, derivability does not necessary implies the continuity of the derivative. Furthermore, the slope of  $f$  needs to be identical for every trajectory in the complex plane through  $z_0$  [2]. Even more is true: every holomorphic function is also *analytic*, in the sense that it has a power series expansion near every point. Again, the same does not hold in  $\mathbb{R}$ .

## 2.2 Complex Differentiability

We have already given a brief overview on the concept of complex differentiation 2.1.3. The fact is that, unfortunately, even seemingly simple functions are not holomorphic. The square modulus itself,  $f(z) = \|z\|^2 = z\bar{z}$ , for example.

But, under this perspective, how can we provide an efficient and coherent way to *optimize* any loss function associated to a machine learning problem? How can we exploit gradient descent if most of the times losses and neurons' activations are non-holomorphic?

A possible alternative exists: just as subgradients enable us to optimize functions that are not differentiable across their domain (e.g. ReLU), we can leverage **Wirtinger calculus** (or **CR-calculus**) to optimize functions that are not holomorphic but still differentiable with respect to their real and imaginary components. The core idea behind this approach is basically a "change of basis" for the complex number  $z = x + iy$ , from the traditional representation as a vector in  $\mathbb{R}^2$ , i.e.  $r = (x, y)^T$ , to the so called **conjugate coordinates** [6]:

$$c \equiv (z, \bar{z})^T \in \mathbb{C} \times \mathbb{C}, \quad z = x + iy \quad \text{and} \quad \bar{z} = x - iy \quad (2.1)$$

Wirtinger operators permit the construction of a differential calculus for complex-valued functions, that is entirely analogous to its ordinary real-valued counterpart. Furthermore, it allows us to avoid the expansion of the input into its real and imaginary components and the successive derivation (that would be just a simple real derivative). Those operators we were talking about are the **R-derivative**,  $\partial f / \partial z$  (computed treating  $z$  as a real variable and holding instances of  $\bar{z}$  constant) and the **conjugate R-derivative**,  $\partial f / \partial \bar{z}$  (same but this time treating  $z$  like a constant). More formally, we can construct the following operators:

$$\frac{\partial}{\partial z} \equiv \partial_z = \frac{1}{2} (\partial_x - i\partial_y) \quad \frac{\partial}{\partial \bar{z}} \equiv \partial_{\bar{z}} = \frac{1}{2} (\partial_x + i\partial_y) \quad (2.2)$$

In backpropagation we need to compute the derivatives of the final loss function with respect to the parameters of the network across the various layers. The task can be achieved applying the *chain rule* multiplying the upstream derivatives times the local derivatives for a specific layer function. For this reason it may be useful to write down also the chain rule for the CR-derivatives, even if, in the end, it is just an extension of the real case (in which we consider  $z$  and  $\bar{z}$  as independent random variables):

$$\frac{\partial(f \circ g)}{\partial z} = \frac{\partial f}{\partial g} \frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{g}} \left( \overline{\frac{\partial g}{\partial \bar{z}}} \right) \quad \frac{\partial(f \circ g)}{\partial \bar{z}} = \frac{\partial f}{\partial g} \frac{\partial g}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{g}} \left( \overline{\frac{\partial g}{\partial z}} \right) \quad (2.3)$$

where  $g$  is a complex-valued function of  $z$ .

Rearranging the previous functions one can derive also the following identities:

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\left( \frac{\partial f}{\partial z} \right)} \quad \frac{\partial \bar{f}}{\partial z} = \overline{\left( \frac{\partial f}{\partial \bar{z}} \right)} \quad (2.4)$$

Furthermore, in the case of a real-valued output, with  $f(z) : \mathbb{C} \rightarrow \mathbb{R}$ , there is an additional pair of identities that hold:

$$\frac{\partial f}{\partial z} = \overline{\left( \frac{\partial f}{\partial \bar{z}} \right)} \quad \frac{\partial f}{\partial \bar{z}} = \overline{\left( \frac{\partial f}{\partial z} \right)} \quad (2.5)$$

We will see, in the next chapter, when implementing a complex backpropagation algorithm, that these last identities will greatly simplify the computations, since the final loss function of any network must be real-valued.

## 2.3 Other theorems

There is a couple of further theorems that is worth to recall, that we will need to take into account in order to define a learning algorithm for complex neural networks.

**Theorem 2.3.1. (Liouville's theorem)** If  $f$  is entire (holomorphic on all  $\mathbb{C}$ ) and bounded (i.e.  $\exists N \in \mathbb{R}$  such that  $\|f(z)\| \leq N \forall z \in \mathbb{C}$ ), then  $f$  is constant.

Equivalently, non-constant holomorphic functions on  $\mathbb{C}$  are unbounded.

This theorem is particularly relevant when trying to define a set of activation functions suitable for the complex layers of our network. Because of this, in fact, we need to choose functions that are unbounded, otherwise they would return always a constant output, that is not desirable behavior. Moreover, a direct consequence of the Liouville's theorem is the following corollary:

**Corollary 2.3.1.1.** If  $f$  is smaller or equal to a scalar times its input (i.e.  $\exists M \in \mathbb{R}$ ,  $M > 0$ , such that  $\|f(z)\| \leq M\|z\|$ ), then  $f$  is linear.

This is another undesirable behavior, since non-linearity is an important requirement in the setup of a backpropagation algorithm.

**Theorem 2.3.2. (Identity theorem)** Let  $D \in \mathbb{C}$  be a domain, and  $f$ ,  $g$  holomorphic functions on  $D$ . If the set  $E = \{z \in D : f(z) = g(z)\}$  contains a non-isolated (i.e. accumulation) point, then  $f(z) = g(z)$  for all  $z \in D$ .

We reported the identity theorem because, according to [7], there are clues that a complex-valued network is able to learn any complex function just training over part of its domain.

## 2.4 Circularity

An important characteristic of a complex random variable is the so-called circularity property, or lack of it. Circular random variables have, in fact, vanishing pseudo-variance, index that its real and imaginary parts are statistically uncorrelated. Under this perspective, a recent work [8] have shown that, the circularity property of a dataset can significantly impact on the different performances obtained using a complex-valued model with respect to its real counterpart. At least in principle, in fact, complex-valued networks seems to benefit more of dataset presenting inherent correlations.

Let us denote the vector  $\mathbf{u} \triangleq [X, Y]^T$  as the real vector built by stacking the real and imaginary parts of a complex random variable  $Z = X + iY$ . The probability density function of  $Z$  can be identified with the pdf of  $\mathbf{u}$ . The *variance* of  $Z$  is defined by:

$$\sigma_Z^2 \triangleq \mathbb{E} [|Z - \mathbb{E}[Z]|^2] = \mathbb{E} [|Z|^2] - |\mathbb{E}[Z]|^2 = \sigma_X^2 + \sigma_Y^2$$

where  $\sigma_X^2$  and  $\sigma_Y^2$  are respectively the variance of  $X$  and  $Y$ . However, this parameters does not bring any information about the *covariance* of  $Z$ ,

$$\sigma_{XY} \triangleq \mathbb{E} [(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

for which we need to rely on another statistical quantity defined for complex random variables, i.e. the *pseudo-variance*:

$$\tau_Z \triangleq \mathbb{E} [(Z - \mathbb{E}[Z])^2] = \sigma_X^2 - \sigma_Y^2 + 2i\sigma_{XY}$$

Unlike the variance of  $Z$ , which is always real and positive, the pseudo-variance is in general complex. We define the **circular quotient**  $\rho_Z$  as:

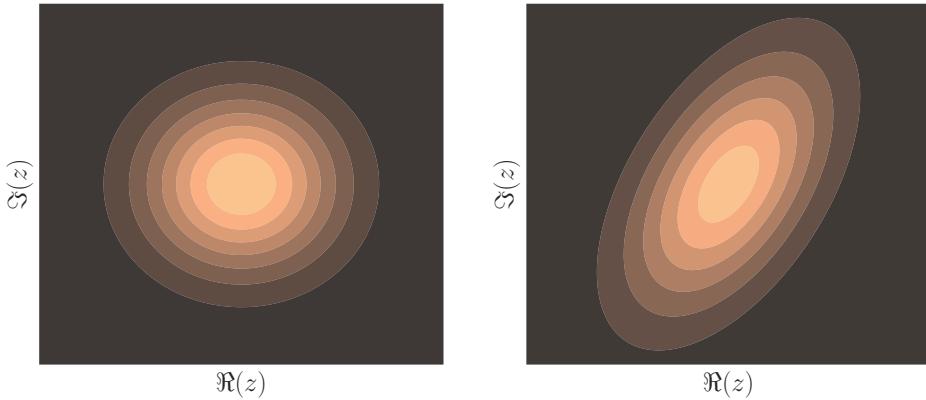
$$\rho_Z = \frac{\tau_Z}{\sigma_Z^2}$$

Additionally, we can define also a *correlation coefficient* among real and imaginary parts of  $Z$ :

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

In some papers, the circular quotient is defined as a covariance measure between  $Z$  and  $\bar{Z}$ , so among a random variable and its complex conjugate, rather than considering real and imaginary parts. We believe that those formulations in the end are equivalent from a practical point of view and so they can be used interchangeably.

Circular vs non-circular distributions


 Figure 2.1: Example of a perfectly circular distribution ( $\rho_z = 0$ ) and a non-circular one (right).

From another perspective, a complex random vector  $\mathbf{Z}$  is called circularly symmetric if, for every  $\varphi \in [-\pi, \pi)$ , the distribution of  $e^{i\varphi}\mathbf{Z}$  is the same of  $\mathbf{Z}$ . The vector's PDF then satisfies  $f(\mathbf{Z}) = cg(|\mathbf{Z}|^2)$  for some non-negative function  $g$  and normalizing constant  $c$ . Hence, the regions of constant contours, for this distribution, are circles in the complex plane (as can be observed from figure 2.1).

Another interesting fact is that  $\rho_Z$  possess an intuitive geometrical interpretation since the modulus and phase of its principal square-root are equal to the eccentricity and angle of orientation of the ellipse defined by the covariance matrix of the real and imaginary parts of  $Z$ .

## 2.5 Why not to prefer real-valued network with two channels?

As we explained in the previous section, complex numbers can also be represented as points in  $\mathbb{R}^2$ , considering the "isomorphism"  $\mathbb{C} \ni x + iy \leftrightarrow (x, y)^T \in \mathbb{R}^2$ . Consequently, one of the main approach that are followed when a complex input is fed to a real-valued model is exactly to split such input into two independent channels, one for the real components and one for the imaginary. That's similar to what happens for the three channels of an RGB image, that are sent, one at a time independently, through a convolutional (or fully-connected) layer.

Unfortunately, even though this approach is efficient and coherent with the way complex numbers are managed in modern calculators (real and imaginary parts stored as separated floating point numbers), it turns out to be non-proper, even "dangerous", from an analytic point of view, since the real-valued inner product implemented in linear layers dismisses the mathematical correlation between the real and imaginary parts.

### Complex Multiplication

Complex multiplication, in fact, has a specific protocol that combines the real and imaginary components of the input to the corresponding ones of the output (figure 2.2, top left). In the case of a complex input that has been split into its components, trying to reproduce the multiplication using the naive real-valued implementation of the "inner product" breaks this protocol. The reason is simply that real and imaginary parts of the output depend on the constituents of both factors.

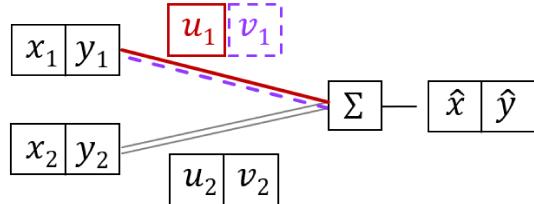
Nevertheless, it is possible to connect the (real-valued) network's components in order to "mimic" exactly the behavior of complex multiplication: you just need to exploit *shared weights*, as in the example in figure 2.2 (on the right). However, this does not come without any cost, since  $6x$  floating point operations are required (4 multiplications + 2 sums), with respect to a standard product, making this approach quite inefficient.

For a better understanding of the differences between complex and real multiplication it may be worth to give a look at figure 2.3, in which are represented the various degrees of freedom of those operations. Complex multiplication naturally has two degrees of freedom, *scaling* and *rotation* (check 2.1.1), com-

### Complex multiplication

$$\begin{aligned}\hat{z} &= zw \\ &= (x + iy)(u + iv) \\ &= xu - yv + i(xv + yu) \\ &= \hat{x} + i(\hat{y})\end{aligned}$$

### Complex inner product



### 2-ch inner product w/ shared weights

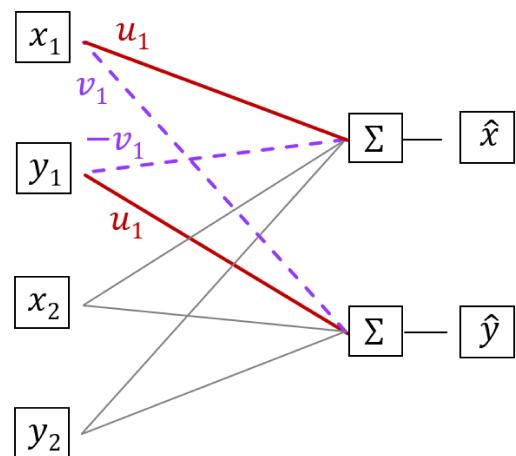


Figure 2.2: Visual representation of the multiplication and the corresponding implementation in a complex-valued network (bottom left). In order to replicate the same behavior on a real-valued network, the weights' components must be replicated, or shared, in the 2-channels layer (on the right). (source: [9])

pared to the four that characterize a product among vectors in  $\mathbb{R}^2$ . Let's consider the black letter "R" in the image, as the input  $z_{in}$  of a fully-connected layer in our network. This letter may change in different way depending on the multiplication protocol implemented:

- with single-channel real weights (2.3, left), the only possible effects is a re-scaling of the input;
- in the complex plane (2.3, center), as repeated several times, the multiplication depends on two parameters (magnitude and phase of the complex-valued weights) and represents and homothety in the plane;
- if the multiplication is instead implemented with a real-valued layer with two inputs and two outputs (2.3, right), there are four weights to be learned; that's will be for sure more expensive, but can guarantees two additional transformation like *reflect* and *shear* (bad or good thing depending on the dataset used).

In many applications the relative change in phase between neighboring pixels may be of importance and the absolute phase value irrelevant. Two-channel real networks that treat real and imaginary parts independently can be problematic if a certain application needs to be invariant with respect to this arbitrary global complex scaling across an image.

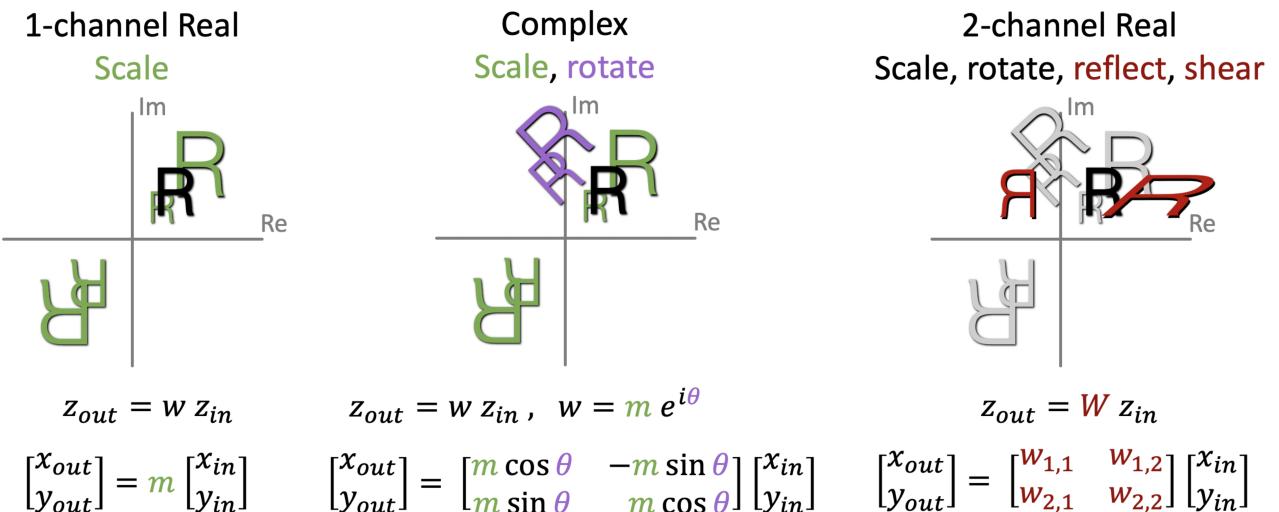


Figure 2.3: Visual representation of the complex multiplication (center) compared to its real-valued implementations (1-ch left, 2-ch right). (source: [9])

## Non-Linear Activations

Not only linear operations would be affected by this complex decomposition, but also non-linear ones. In generic neural networks the main non-linear contribute is due to activation functions. But while complex multiplication can be in some way reproduced also by real-valued layers, for those non-linear activations the situation is not so straightforward. Working with a 2-channels input implies that the components will pass as independent variables through the activation, which again dismisses eventual natural correlations between real and imaginary parts. Furthermore, when working, for example, with the canonical ReLU activation, one can fall into undesirable scenarios like a number with a large magnitude not allowed to pass (because of a negative component) or even unexpected rotations (because either real or imaginary parts got strongly altered). We cannot even renounce to them, since they are basically the core of a backpropagation algorithm. The only possibility, then, is to define a whole set of new activation functions, more suitable for working in  $\mathbb{C}$ .

Later, we will see how the choice of the activation was one of the most discussed obstacles in developing a definitive complex-valued deep learning framework.



# Chapter 3

## Extent

From Wirtinger calculus backpropagation to specific software implementation challenges, in this chapter are described the fundamental details of complex-valued neural network components and how they are related to existing real-valued network implementations. We will show how existing layers and functionalities can be extended to work also with a complex-valued input and which of them needs to be completely redefined.

We address the problem of re-adapting the training process building a complex backpropagation algorithm on top of many prior works, that allows for an optimization when the loss function is real-valued, thanks to Wirtinger calculus.

Furthermore, we will discuss in details the problem of building complex-valued activation functions, that was one of the main obstacles in the development of deep learning in this direction.

In the end, we will provide a brief presentation of the high level library, built on top of JAX, that we have realized in order simplify the setup and train of those kind of networks. Nowadays, in fact, the internet is full of deep learning libraries implementing basically every kind of known model, with different optimizations, parallelizations, etc. However, for some reason, many of them still does not provide support to complex data types: a huge obstacle in the growth of complex-valued deep learning.

### 3.1 Problems in the extent

Considering just their fundamental structure, complex-valued neural networks work exactly like their real counterpart, and are again constituted by neurons connected among each other (figure 3.1): the only difference is that now those neurons are complex-valued.

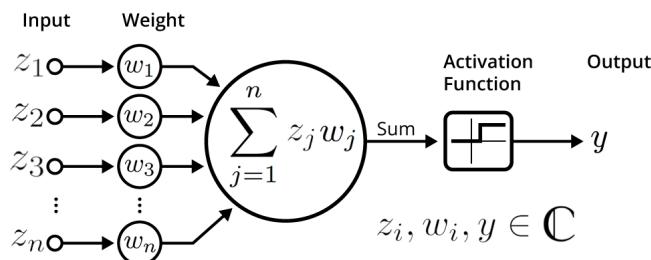


Figure 3.1: Fundamental unit (neuron) of a complex-valued neural network.

Each neuron receives a weighted input signal  $\mathbf{z}$ , that this time is complex valued (as the weights  $\mathbf{w}_i$ ); this signal is summed up and added to a bias  $\mathbf{b}$ , and then passed through an activation function  $f : \mathbb{C} \rightarrow \mathbb{C}$ , that most of the times is non-linear. If we denote with the subscript  $\ell$  the forward pass of

a neuron in the  $\ell$ -th layer, then the output can be expressed with the following formula:

$$\mathbf{y}_\ell = f_\ell \left( \sum_{i=1}^N \mathbf{w}_i z_i + \mathbf{b}_\ell \right)$$

where  $N$  are the neurons in layer  $\ell$ ,  $M$  the neurons in layer  $(\ell - 1)$ ,  $\mathbf{z}_{\ell-1} \in \mathbb{C}^M$  was the output of the previous layer,  $\mathbf{w}_\ell \in \mathbb{C}^{N \times M}$  and  $\mathbf{b}_\ell \in \mathbb{C}^N$  are the learnable parameters of this level,  $f_\ell$  the activation function and  $\mathbf{y}_\ell \in \mathbb{C}^N$  the effective output.

However, when considering a possible extension from  $\mathbb{R}$  to  $\mathbb{C}$ , we need to take into account a few inconveniences, since we look for a coherent and rigorous framework.

### Max operator is undefined

As explained also in the introductory mathematical section,  $\mathbb{C}$  is not an ordered field, in the sense that we cannot define a comparison relation among complex numbers that makes everybody agree. In principle you can define one, like the lexicographical ordering, that compares first the real part and only after the imaginary, or relying on establishing this relation among the magnitudes of those numbers. The latter is actually the preferred approach. This brief overview is important, since many non-linear functions in deep learning, like ReLU and Max-Pooling necessitate of a *maximum* operation in order to fulfill their purpose of increasing numerical stability and dimensionality reduction, respectively.

### Unstable Activations

As we will see in a dedicated section, the problem of defining stable and coherent activation functions is one of the main issues that limited the development of complex-valued deep learning during the years. Complex functions, in fact, necessitate of further limitations to be suitable as activations: because of the Liouville's theorem 2.3.1, for example, they can't be limited, and neither grow too slow, otherwise their derivative would always vanish during the backpropagation. So, simply re-adapting existing activations to support complex-valued inputs, maybe redefining ambiguous operations like `max`, is not enough, especially because you need to care about the eventual loss of complex correlations if the activation is applied independently on the real and imaginary components.

### Lost Probabilistic Interpretation

One nice property of real-valued neural network classifiers is the probabilistic interpretation that we can associate to its final layer, mainly due to the normalization in the range  $[0, 1]$  provided by sigmoid/softmax activation functions. But now, the final output of the network will be a set of complex numbers, that we cannot interpret anymore as a probability distribution over a set of probabilistic outcomes. This nice property can be partially recovered if we add a *magnitude* layer just before the last activation: in this way we drop all the phase information but we move back to a real-valued problem. Anyway, it always depends on the final objective of the model.

### Optimization of a Complex-Valued Output

Another problem related to having a complex-valued output, beside its interpretation, is the loss associated. If you have a complex final loss, how can you effectively minimize it? In the first chapter we have defined the minimum of a function defined in  $\mathbb{C}$  as the point  $z_0 \in \mathbb{C}$  in which its modulus is minimized. But this definition, provided by the author of that complex analysis book, is referred to a total ordering in which we refer first to the magnitudes, and so also this is just a convention. Notice that, at the end, minimizing the modulus of a complex loss is equivalent to defining a real-valued loss, i.e.  $\mathcal{L} : \mathbb{C} \rightarrow \mathbb{R}$ , and minimizing it. And that's exactly what we are going to do when setup a backpropagation.

After all this steps, it is clear that we cannot simply reuse deep learning architectures designed for real values without first understanding the complex mathematics in neural networks forward and backward.

## 3.2 Complex Backpropagation

As anticipated in the introductory section, the interest of researchers in this complex-valued deep learning area started arising many years ago; in fact, the first trial to setup a complex backpropagation algorithm even dates back to 1991 [7]. According to the author, not only its algorithm turns out to have an higher convergence speed (and the same generalization performances) with respect to its real counterpart, but also that is able to learn an entire class of transformations (e.g. rotations, similarities, parallel displacements, etc.) that the real method cannot. However, having read the work of Nitta [7], I feel it is better to remark a couple of things, mainly because many years have passed from its publication. First of all, the author used a suboptimal setup:

- the network followed one of the "conventional" approaches that we are proving to be unefficient, i.e. treating real and imaginary parts of the data as independent random variables;
- he computed the derivatives  $\partial f/\partial x$  and  $\partial f/\partial y$ , instead of relying on Wirtinger calculus 2.2; even if this is a working alternative to ours, we will see that it is suboptimal;
- he relied on "bad" activation functions, since, as told by he himself, many times the algorithm failed to converge.

I decided to report his work because it was still one of the first and working attempts to develop a complex backpropagation algorithm, but also because of the purely theoretical analysis realized on the transformations that a complex network can learn. Nitta, managed to teach its networks several transformations in  $\mathbb{R}^2$ , like rotations, reductions and parallel displacements, that the corresponding real-valued model didn't make. He understood first that this was possible thanks to the higher degrees of freedom offered by complex multiplication (discussed in section 2.5). But what I believe it is even more interesting, is the relation that Nitta have found among complex-valued networks and the **Identity theorem 2.3.2**:

*"We believe that Complex-BP networks satisfy the Identity Theorem, that is, Complex-BP networks can approximate complex functions just by training them only over a part of the domain of the complex functions."*

This means that exploiting holomorphic functions when building a complex-valued network can sometimes impact on its generalization capabilities (since its shape will be rigidly determined by its characteristics on a small local region of its domain) [10]. Unfortunately, no additional work have been realized on this statement during the years, but I think it is an aspect deserving further attention.

In section 2.2 we have discussed about complex differentiability, and we also said that holomorphicity is not a property assured for most functions, and even simple ones, like the square modulus, can be not differentiable in the complex sense. In our architectures we have mainly two sources of *nonholomorphicity*: the loss and the activations. For reasons that will be clearer later on, boundedness and analiticity cannot be achieved simultaneously in the complex domain, and the first feature is often preferred [11].

An elegant approach that can save computational labor is the usage of Wirtinger calculus to setup optimization problems, solvable via gradient descent, for functions that are not holomorphic but at least differentiable with respect to their real and imaginary components.

Also for neural network functions we have a similar problem: requiring them to have complex differentiable properties can be quite limiting, and we should again rely on CR-calculus. To complicate matters, the chain rule requires now to compute two terms rather than one (both the R-derivative and the conjugate R-derivative), causing more than  $4x$  calculations during the backward pass.

"Fortunately", we can significantly reduce both memory and computation time during complex backpropagation by assuming that our final network loss function is *real-valued*. This is a strong but valid assumption since, as already discussed in section 2.1, minimizing a complex-valued function is an **ambiguous** operation. Also, minimizing the modulus of a complex loss, as suggested by the magnitude ordering introduced, in the end is exactly like optimizing a real-valued function.

But why does it turn out to be more efficient?

### 3.2.1 Steepest Complex Gradient Descent

In the previous sections we have widely discussed about the fact that complex-valued functions cannot be minimized, and so we ended up with the conclusion that we must use a real-valued loss in order to formulate the training process of a complex model as an optimization problem, in continuation to what happens inside real models.

Let's then consider a function  $f(z) : \mathbb{C} \rightarrow \mathbb{R}$ : if we decide to proceed with a gradient descent algorithm, which direction are we supposed to take since there are two different derivatives that one can compute ( $\partial/\partial z$  and  $\partial/\partial \bar{z}$ )?

It can be proved ([3, 11] and appendix A) that the direction of the *steepest gradient descent* is the **complex cogradient**,  $\nabla_{\bar{z}} f$ . So, given a function  $f$  that depends on a complex random variable  $z \in \mathbb{C}$ , the update rule that minimizes it is:

$$\mathbf{z} \leftarrow \mathbf{z} - \alpha \nabla_{\bar{z}} f \quad (3.1)$$

where  $\alpha \in \mathbb{R}$  is the learning rate.

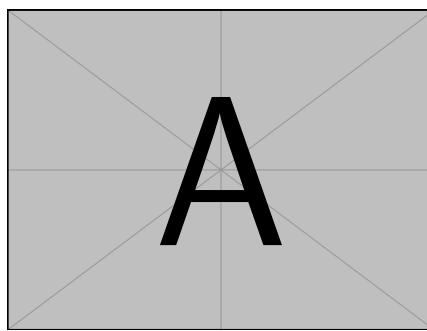


Figure 3.2: Complex Gradient Descent.

In order to provide also a visual representation, in figure 3.2 we have considered a simple, non holomorphic, real-valued function like  $f(z) = z\bar{z} = \|z\|^2$ , that has a unique global minimum at  $z = 0 + 0j$ . We have then applied the gradient descent and ascent rules in both directions of the gradient,  $\nabla_z f$ , and the cogradient  $\nabla_{\bar{z}} f$ , in order to verify what said above. In the plot we clearly see that the only direction that approaches the true minimum (starting from a random point in the dominium of  $f$ ) is exactly the one determined by the complex cogradient, while the complex gradient moves in a completely wrong direction. Also considering the ascent rules we observe that the steepest direction maximizing  $f$  is again the one determined by the cogradient.

### 3.2.2 Backpropagation with a Real-valued Loss

With the real-valued loss assumption (proposed in 3.2) and the update rule 3.1, complex backpropagation turns out to be quite efficient and not so computationally expensive as we thought in section 3.2. The situation can be summarized with table 3.1 and figure 3.3.

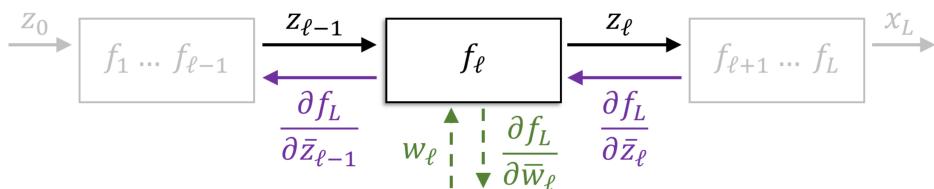


Figure 3.3: Forward and backward pass through a complex layer. (source [9])

Now the algorithm needs to pass only one of the two CR derivatives back to the earlier layers, even tho, because of the chain rule 2.3, it must compute both of them, at least for the final layer. In figure

Standard Real Calculus	Complex Calculus	Complex Calculus, assuming real-valued loss
Input to layer $\ell + 1$ :		
$\frac{\partial f_L}{\partial x_\ell}$	$\frac{\partial f_L}{\partial z_\ell}$ and $\frac{\partial f_L}{\partial \bar{z}_\ell}$	$\frac{\partial f_L}{\partial \bar{z}_\ell}$
Output from layer $\ell$ :		
$\frac{\partial f_L}{\partial x_{\ell-1}} = \frac{\partial f_L}{\partial x_\ell} \frac{\partial f_\ell}{\partial x_{\ell-1}}$	$\frac{\partial f_L}{\partial z_{\ell-1}} = \frac{\partial f_L}{\partial z_\ell} \frac{\partial f_\ell}{\partial z_{\ell-1}} + \frac{\partial f_L}{\partial \bar{z}_\ell} \left( \overline{\frac{\partial f_\ell}{\partial z_{\ell-1}}} \right)$	
	$\frac{\partial f_L}{\partial \bar{z}_{\ell-1}} = \frac{\partial f_L}{\partial z_\ell} \frac{\partial f_\ell}{\partial \bar{z}_{\ell-1}} + \frac{\partial f_L}{\partial \bar{z}_\ell} \left( \overline{\frac{\partial f_\ell}{\partial z_{\ell-1}}} \right)$	$\frac{\partial f_L}{\partial \bar{z}_{\ell-1}} = \left( \overline{\frac{\partial f_L}{\partial \bar{z}_\ell}} \right) \frac{\partial f_\ell}{\partial \bar{z}_{\ell-1}} + \frac{\partial f_L}{\partial \bar{z}_\ell} \frac{\overline{\partial f_\ell}}{\partial z_{\ell-1}}$

Table 3.1: Comparison of backpropagation calculus. (source: [9])

3.3 it is effectively illustrated how input and derivatives flow through the layers during the forward and backward pass, respectively.

Given the steepest complex gradient descent rule 3.1, we can then write down the equivalent expression for our network function depending on a set of parameters  $\mathbf{w}$ :

$$\mathbf{w}_n \leftarrow \mathbf{w}_n - \alpha \frac{\partial f}{\partial \bar{\mathbf{w}}_n}$$

### 3.3 Re-definition of the main neural network layers

We have started writing down this chapter with the purpose of building a working and coherent framework for complex-valued machine learning. And we also In section 3.1 we have analyzed the main obstacles that we encounter during this extension, while in 3.2 we

#### Fully-Connected Layers

For the fundamental layers of a neural network the extension is quite trivial. Consider a layer with  $K$  complex-valued units, a weight vector  $\mathbf{w} \in \mathbb{C}^K$  and a bias term  $\beta \in \mathbb{C}$ ; then, the response to a certain input vector  $\mathbf{z} \in \mathbb{C}^N$  (given also the activation function  $f$ ) is:

$$\mathbf{y} = f(\mathbf{z}, \{\mathbf{w}, \beta\}) = f \left( \sum_k z_k w_k + \beta \right) = f(\mathbf{z}^T \mathbf{w} + \beta)$$

So it is just like the real case, with the only difference that now the multiplication is in  $\mathbb{C}$  and not in  $\mathbb{R}$ , with all the consequences already discussed in 2.5.

What we should care about is, instead, the initial values of the networks' parameters. Proper initialization can, in principle, help reducing the risk of vanishing or exploding gradients. Conventionally, researchers follow the approaches proposed by Glorot [12] or by He [13] (the latter designed specifically for ReLU activations), with the final objective of ensuring that the variance of input, output and their gradients are the same. According to these two methods, weights should be initialized with a normal distribution (or truncated-normal) with zero mean and standard deviation that depends on the number of units in that specific layer. Thanks to Trabelsi [1] (derivation in appendix A.1) we could provide two equivalent procedures, but in the complex domain.

To put in place them, we need first to consider the weights in polar form, i.e.  $\|\mathbf{w}\|e^{i\theta}$ , and then:

- random sampling the magnitude according to a **Rayleigh distribution** with parameter  $\sigma = 1/\sqrt{n_{in} + n_{out}}$  (**Complex Xavier** initialization) or  $\sigma = 1/\sqrt{n_{in}}$  (**Complex He** initialization);
- random sampling the phases according to a **uniform distribution** in  $[-\pi, \pi]$ .

The biases  $\beta$ , instead, can all be initialized simply to 0, or uniformly in a small interval  $[-\varepsilon, \varepsilon]$ .

## Convolutional Layers

In order to perform the equivalent of a traditional real-valued 2D convolution in the complex domain, we convolve a complex filter matrix  $\mathbf{W} = \mathbf{A} + i\mathbf{B}$  by a complex vector  $\mathbf{h} = \mathbf{x} + i\mathbf{y}$ , where  $\mathbf{A}, \mathbf{B}$  are real matrices and  $\mathbf{x}, \mathbf{y}$  are real vectors, since we are simulating complex arithmetic using real-valued entries. As the convolution operator is distributive, we have:

$$\mathbf{W} * \mathbf{h} = (\mathbf{A} * \mathbf{x} - \mathbf{B} * \mathbf{y}) + i(\mathbf{B} * \mathbf{x} + \mathbf{A} * \mathbf{y}) \quad (3.2)$$

This is a very nice behavior, since the a complex convolution can be decomposed into two real-valued independent operations. And this means that we can exploit already existing algorithms (like this one, from Gauss<sup>1</sup>) to perform efficiently this (already expensive) computation.

Notice also that complex convolutional layers' weights basically, can learn to rotate the phase of desirable data toward the positive real axis; in fact, if we rewrite 3.2 in a matrix form:

$$\begin{bmatrix} \Re(\mathbf{W} * \mathbf{h}) \\ \Im(\mathbf{W} * \mathbf{h}) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} * \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

In figure 3.4 we can observe a visual representation of the complex convolution proposed by [1].

In principle, however, even tho complex convolution should have the same computational cost as its real-valued counterpart, in practice it is four times more expensive, just because complex multiplication in the worst case requires four products and two additions.

## Pooling Layers

The pooling operation involves sliding a n-dimensional filter over each channel of the feature map and summarizing the features lying within the region covered by the filter. Pooling layers are essentially dimensionality reduction levels inside the network, with the purpose of making the model more robust to positional variations in the input.

There are mainly two kinds of pooling operations that we want to extend to the complex domain:

- **Average Pooling**: replaces the elements in the filtered region of the feature map with their mean. The average of a set of complex number is a well defined operation and so no further work is needed in this case.
- **Max Pooling**: replaces the elements in the filtered region of the feature map with their maximum. In this case, instead, we have that the maximum operation is ambiguous in  $\mathbb{C}$ , and so we must establish an ordering to re-define this layer. Our choice that was to setup the max pooling operation to return the complex number with the highest magnitude, inside the region covered by the filter. Namely:

$$\text{MaxPool}(\{z_k\}) = z_n \quad \text{where } n = \underset{k}{\operatorname{argmax}} \|z_k\|$$

This choice is not unique, since, as repeated several times, there are more than one total ordering for  $\mathbb{C}$ .

## Normalization Layers

Normalization layers are usually inserted inside the architecture of a neural network with the purpose of improving the stability of the network optimization, especially in cases of an high depth, and for

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<sup>1</sup>As explained in this brief Wikipedia section, we can reduce the cost of complex multiplication, in some cases.

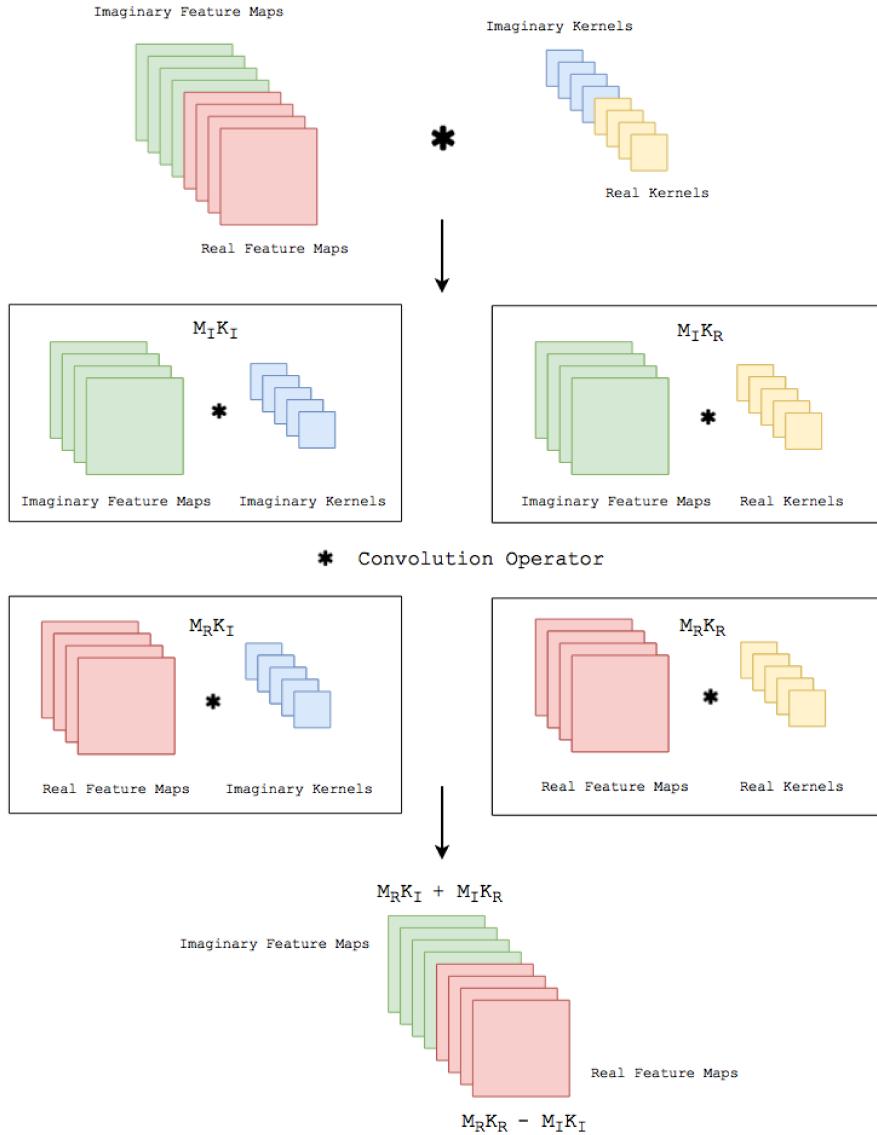


Figure 3.4: Implementation details of the Complex Convolution (by [1]).

a better control of the gradients. In this case the extension not so straightforward, since you need to re-define several statistical objects.

In standard **Batch-Normalization** we train two internal parameters, a scale and an offset, such that each batch of data has zero mean and standard deviation one. This approach, however, is valid only for real data, since it does not guarantee equal variance for both the real and imaginary components, with a resulting distribution turning out to be non-circular. Following the method proposed by [1], the idea is to normalize data to obtain a standard complex normal distribution (basically the one in figure ??), achieved by multiplying the zero-centered data ( $\mathbf{z} - \mathbb{E}[\mathbf{z}]$ ) by the inverse square root of their  $2 \times 2$  covariance matrix  $\mathbf{V}$ <sup>2</sup>:

$$\tilde{\mathbf{z}} = (\mathbf{V})^{-\frac{1}{2}} (\mathbf{z} - \mathbb{E}[\mathbf{z}]) \quad \text{where} \quad \mathbf{V} = \begin{pmatrix} \text{Cov}(\text{Re}\{\mathbf{z}\}, \text{Re}\{\mathbf{z}\}) & \text{Cov}(\text{Re}\{\mathbf{z}\}, \text{Im}\{\mathbf{z}\}) \\ \text{Cov}(\text{Im}\{\mathbf{z}\}, \text{Re}\{\mathbf{z}\}) & \text{Cov}(\text{Im}\{\mathbf{z}\}, \text{Im}\{\mathbf{z}\}) \end{pmatrix} \quad (3.3)$$

From a practical point of view, the implementation is the same of the real case. You have again two trainable parameters:  $\beta \in \mathbb{C}$ , i.e. the complex mean, and  $\gamma \in \mathbb{C}^2$ , the complex-valued positive-defined covariance matrix. The **complex batchnorm** operation is then defined as

$$BN(\mathbf{z}) = \gamma \mathbf{z} + \beta$$

<sup>2</sup>The existence of the inverse matrix is guaranteed by the positive (semi-) definiteness of  $\mathbf{V}$ . Eventually, you can enforce this condition by adding a small quantity  $+\varepsilon \mathbb{I}$  to the matrix (Tikhonov regularization).

We need, however, to be careful when relying on batch normalization. This procedure allows, in fact, to avoid co-adaptation between real and imaginary parts of data, effectively reducing the risk of overfitting. But there is a cost to this, since you are basically decorrelating your complex data, partially losing the advantage over two-channels networks [14].

For this reason, we sometimes prefer to rely on other kind of normalization layers that, instead, allows to preserve complex data correlations.

In simple **Complex Normalization** we scales a complex scalar input  $\mathbf{z}$  such that its magnitude is set to one, while the phase remains unchanged. In practice we project  $\mathbf{z}$  onto the unit circle. The forward pass is then

$$\hat{\mathbf{z}} = e^{i\angle \mathbf{z}} = \frac{\mathbf{z}}{\|\mathbf{z}\|} = \frac{\mathbf{z}}{(\mathbf{z}\bar{\mathbf{z}})^{1/2}}$$

## Other Layers

There are many layers that do not need any further re-definition to work also in the complex domain: **Dropout**, **Pad** or **Attention** layer, for example. There are also many other structures that should be re-derived (e.g Recurrent layers, LSTM, etc.), but that were out of our scope and so we haven't examined. This should be interpreted just as a starting point in the development of an higher level complex-valued deep learning framework.

## 3.4 Complex-Valued Activation Functions

One of the main issues encountered in the last 30 years in the developing of a complex-valued deep learning framework was exactly the definition of reliable activation functions. The extension from the real-valued domain turned out to be quite challenging: because of the Liouville's theorem 2.3.1, in fact, stating that the only complex-valued functions that are bounded and analytic everywhere are constants, one have necessarily to choose between boundedness and analyticity, in the design of those activations. Furthermore, before the introduction of ReLU, almost all the activation functions known in the real case were bounded. And for the same ReLU the extent was not trivial, since operations like *max* are not defined in the complex domain. Additionally, with complex-valued outputs, we have lost the probabilistic interpretations that functions like **sigmoid** and **softmax** used to provide.

We have to say, however, that most of the candidate functions that have been proposed, have been developed in a split fashion, i.e. by considering the real and imaginary parts of the activation separately. But, as discussed also in the previous chapter, this approach should be abandoned, since you risk losing the complex correlations stored in those variables.

In this section, we will explore a few complex-valued activations proposed during the years: first with the ones that are direct extensions of their real counterparts, and then with more "abstract" candidates, that have more reasons to live and work in the complex domain.

### Extent from the real case

The first class of viable approaches consists into barely extending real-valued activations in the complex domain, like the **sigmoid** and the **hyperbolic tangent**:

$$\sigma_C(z) = \frac{1}{1 + e^{-z}} \quad \tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

These functions are fully-complex, and also analytic and bounded almost everywhere, at the cost of introducing a set of singular points: both function, in fact, seems to diverge periodically in the imaginary axis of the complex plane. Limiting and scaling carefully the input values seems to help avoiding then those singularities and so partially containing the instability. However, I believe that there are simpler and more efficient alternatives.

## Separable Activations

As already explained, the main tendency in the development of complex-valued activation functions was basically getting back the "old" designs for real-valued models and using them independently on the real and imaginary components of the input.

This can be done easily with both the sigmoid and the hyperbolic tangent, mapping real and imaginary parts among input and output as they were independent channels:

$$f(\mathbf{z}) = g(\operatorname{Re}(\mathbf{z})) + ig(\operatorname{Im}(\mathbf{z})), \quad \text{where } g(x) = \frac{1}{1+e^{-x}} \text{ or } g(x) = \tanh(x)$$

Notice that this approach maps the phase of the signal into  $[0, 2\pi]$ , since the function  $g$  returns always a positive value.

There are also interesting variations to the separable sigmoid, properly designed to work using a complex-valued network on real-valued data. But, for this reason, they are functions with values in  $\mathbb{R}$  and not in  $\mathbb{C}$ , and so we won't go through them in this work.

After the advent of the ReLU activation functions, two designs where developed in this fashion, the  $\mathbb{C}\text{ReLU}$  and the  $z\text{ReLU}$ :

$$\mathbb{C}\text{ReLU}(z) = \operatorname{ReLU}(\operatorname{Re}(z)) + i\operatorname{ReLU}(\operatorname{Im}(z)) \quad z\text{ReLU} = \begin{cases} z & \text{if } \theta_z \in [0, \pi/2] \\ 0 & \text{otherwise} \end{cases}$$

These functions also share the nice property of being holomorphic in some regions of the complex plane:  $\mathbb{C}\text{ReLU}$  in the first and third quadrants, while  $z\text{ReLU}$  everywhere but the set of points  $\{\operatorname{Re}(z) > 0, \operatorname{Im}(z) = 0\} \cup \{\operatorname{Re}(z) = 0, \operatorname{Im}(z) > 0\}$ .

## Phase-preserving Activations

Phase-preserving complex-valued activations are those functions that usually act only on the magnitude of input data, preserving the pre-activated phase during the forward pass. Those are usually non-holomorphic, but at least bounded in the magnitude. They are all based on the intuition that, altering the phase could severely impact the complex representation.

The first proposal is the so called **siglog** activation function: It is called in this way because it is equivalent to applying the sigmoid to the log of the input magnitude and restoring the phase:

$$\text{siglog}(z) = g(\log \|z\|) e^{-i\angle z} = \frac{z}{1 + \|z\|}, \quad \text{where } g(x) = \frac{1}{1 + e^{-x}}$$

Unlike the sigmoid and its separable version, the siglog projects the magnitude of the input from the interval  $[0, \infty)$  to  $[0, 1]$ . The authors of this proposal suggested also the addition of a couple of parameters to adjust the *scale*,  $r$ , and the *steepness*,  $c$ , of the function:

$$\text{siglog}(z; r, c) = \frac{z}{c + \frac{1}{r} \|z\|}$$

The main problem with *siglog* is that the function has a nonzero gradient in the neighborhood of the origin of the complex plane, which can lead to gradient descent optimization algorithms to continuously stepping past the origin rather than approaching the point and staying here.

For this reason, an alternative version have been proposed, this time with a better gradient behavior (approaching zero as the input approaches zero), that goes under the name of **iGaussian**.

$$i\text{Gauss}(z; \sigma^2) = g(z; \sigma^2)n(z) \quad \text{where } g(z; \sigma^2) = 1 - e^{-\frac{z\bar{z}}{2\sigma^2}}, \quad n(z) = \frac{z}{\|z\|}$$

This activation is basically an inverted gaussian (and this is the reason for which it is more smooth around the origin) and so depends only on one parameter, i.e. its standard deviation  $\sigma$ .

The last activation that we want to consider for this class is another variation of the rectified linear unit, this time called **modReLU**:

$$\text{modReLU}(z) = \operatorname{ReLU}(\|z\| + b)e^{i\theta_z} = \begin{cases} (\|z\| + b) \frac{z}{\|z\|} & \text{if } \|z\| + b \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where  $z \in \mathbb{C}$ ,  $\theta_z$  is the phase of  $z$ , and  $b \in \mathbb{R}$  is a learnable parameter. The idea of this activation is to create a "dead zone" of radius  $b$  around the origin, where the neuron will be inactive, while it will be active outside. We decided to consider this function, even if apparently was designed for Recurrent neural networks.

## Complex Cardioid

Even if the iGaussian has nice gradients properties around the origin in the complex plane, the same cannot be said when in input are given large values: in that situation there is high risk of vanishing gradients, the same that have historically hindered the performances of sigmoid-like activations. Because of this, recently a new complex activation function have been proposed: the **Complex Cardioid** [9]. The cardioid acts as an extension of the ReLU function to the complex domain, rescaling from one to zero all the values with non-zero imaginary component, based on how much the same input is rotated in phase with respect to the real axis. Also, the cardioid is sensitive to the input phase, rather than the modulus: the output magnitude is, in fact, attenuated based on the input phase, while the output phase remains equal to the original one.

The analytical expression for this activation function is:

$$\text{cardioid}(z) = \frac{1}{2} (1 + \cos(\angle z)) z$$

Another very nice property is that when the inputs are restricted to real values, this function becomes simply the ReLU activation.

We will see, in our applications, that the cardioid effectively allows complex networks to converge in a fast and also stable way.

## A brief recap

Activation	Analytic Form	Reference
Sigmoid	$\sigma(z)$	//
Hyperbolic Tangent	$\tanh(z)$	//
Split-Sigmoid	$\sigma(\operatorname{Re}(z)) + i\sigma(\operatorname{Im}(z))$	[7]
Split-Tanh	$\tanh(\operatorname{Re}(z)) + i\tanh(\operatorname{Im}(z))$	[7]
CReLU	$\operatorname{ReLU}(\operatorname{Re}(z)) + i\operatorname{ReLU}(\operatorname{Im}(z))$	[1]
$z$ ReLU	$z$ if $z \in [0, \pi/2]$ , 0 otherwise	[15]
Siglog	$\sigma(\log \ z\ )e^{-i\theta_z}$	[16]
iGauss	$\left(1 - e^{-\frac{z\bar{z}}{2\sigma^2}}\right) \frac{z}{\ z\ }$	[9]
modReLU	$\operatorname{ReLU}(\ z\  + b)e^{i\theta_z}$	[17]
Cardioid	$\frac{1}{2} (1 + \cos(\angle z)) z$	[9]

Table 3.2: Recal of the most popular complex-valued activation functions.

## 3.5 JAX Implementation

From a practical perspective, in order to setup and realized all the studies and analysis we are going to present, I had to realize a dedicated **Python** library.

In the previous sections we have analyzed all the theoretical obstacles that researchers had to overcome in order to develop a working complex-valued deep learning framework. But, in reality, there are many drawbacks also at the implementation level: first of all the most popular hardware acceleration architectures, CUDA and CuDNN doesn't own a native support for complex-valued data types. And this is by itself a huge limitation, since we cannot train our networks efficiently, and we will have to rely on simple models with few parameters.

Regarding existing deep learning libraries, like Keras, TensorFlow and Pytorch, we have to say

that they provide a lot of interesting high-level architectures to setup deep learning algorithms, and also they "officially" support complex inputs. Even better, they support complex derivatives. But when you try to implement an effective complex-valued network, you understand how much are they far from an effective comprehensive support (just some known issues: 1, 2, 3): regardless of the many errors you can get from structures that should work, from an accurate analysis of the source code of the main layers, you notice that many operations are ambiguous or at least not compatible with the reasoning we adopted to develop the layers extents in this chapter. We could have probably found a way to redefine or override those structures, but we felt that modifying a so large and complex library, without getting undesired side effects, would have been too much work. For this reason we had to rely on a more niche library, called **JAX**, and recently developed by Google DeepMind. JAX is **Autograd** and **XLA** (a domain-specific compiler for linear algebra designed for TensorFlow models), brought together for high-performance numerical computing and machine learning research. It provides composable transformations of Python and NumPy programs: differentiate, vectorize, parallelize, Just-In-Time compile to GPU/TPU, and more. The main advantages that we found using JAX were:

- it supports and optimize complex differentiation, for holomorphic and non functions;
- it is extremely optimized, with XLA + JIT that partially compensate the lack of native hardware acceleration;
- many complex operations/layers are already supported and well defined.

More specifically, for building our complex-valued neural networks we will rely on **dm-haiku**, another library built on top of JAX with the purpose of covering the same role that **Sonnet** (widely used ad DeepMind) has for Tensorflow, and to simplify the approach of users that are familiar with object oriented programming.

Since these are quite recent libraries, and also a definitive approach to complex-valued deep learning does not exists yet, I made a careful and complete analysis of the source code of Haiku and of the most important JAX functions. Thanks to this I effectively noticed that much work is still needed: even if in a small quantity respect to Tensorflow/Pytorch, also in this case many operations turns out to be ambiguous or bad-defined for complex-valued data types (e.g. the **square** or the **max** operators). Many of them are also completely undefined (e.g. initialization or, more in general, random complex distributions). We still decided to proceed with JAX mainly because of its flexibility, and many new functions/operations can be redefined without caring of implicit undesired side effects. This is possible also thanks to the design of the training loop, that is quite "explicit" and customizable.

From a practical point of view, I had basically to realize small a '**complex\_nn**' library on top of Haiku, containing the definitions (and re-definitions) of all the necessary components of a complex-valued neural network:

- **layers**: the adaptations derived before for linear, convolutional, pooling and normalization operations;
- **activations**: all functions listed in table 3.2;
- **initializers**: weights initializers following uniform or truncated random normal distributions, together with the modified approaches described above by Xavier and He;
- **metrics**: categorical accuracy and categorical cross-entropy, useful for the successive classifiers designs;
- **optimizers**: a *complex-Adam* algorithm;
- a **classifier wrapper** with the purpose of collecting all the necessary functions to setup a training loop with JAX and Haiku;

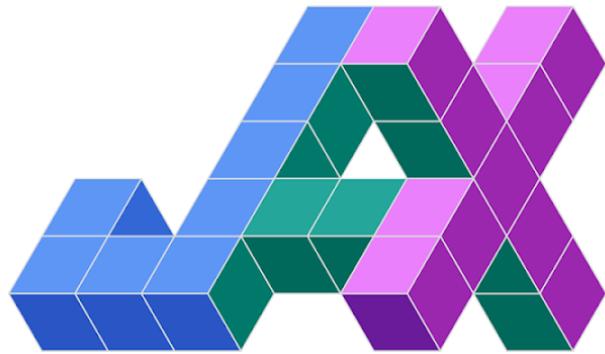


Figure 3.5: JAX logo.

- for completeness, also some utility functions, mainly to realize plots or wrapping more complex structures.

This code has been tested over several datasets. There are in fact a few `Jupyter Notebooks` providing a detailed explanation of analysis we are going to see, together with some basic setup of some learning procedures.

All the implementation and complete analysis is actually available at my gitlab page <sup>3</sup>.

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<sup>3</sup><https://gitlab.fbk.eu/mpujatti/complex-valued-deep-learning-for-condition-monitoring>

# Chapter 4

## Complex-Valued vs Real Neural Networks

### Introduction and Chapter Organization

#### 4.1 Complex vs Real Models

#### 4.2 Impact of the Circularity

In chapter 2.4 we dedicated a whole section to the discussion about the circularity property of a complex random variable: we provided rigorous definitions of the circularity and the correlation coefficients, together with a brief explanation of their geometrical interpretation. We also cited a recent work by Baracchina et al. [8], suggesting the impact that datasets with such properties could have in machine learning applications. The authors, in fact, set up a classification task with two different distributions at a given circularity value, and then they use these to train real and complex-valued models. However they focused mostly on the training stability rather than on the difference in term of performances, that we, instead, believe could bring to very interesting results. For this reason, we decided to set up a similar problem in order to confirm first what they obtained, and then to come up with a more quantitative analysis of the accuracy that complex models can effectively achieve.

### Data Generation

The idea behind this analysis is to understand how much complex-valued models can effectively benefit of data with good circularity properties. For this reason we will generate two distinct complex-valued datasets, one with high correlations among real and imaginary components, and one with poor correlations.

An easy way to generate data with pre-determined circularity is to rely on the *complex normal distribution*, that characterizes complex random variables whose real and imaginary parts are jointly normal. So, we can construct the samples we need simply exploiting the *multivariate random normal distribution*, supported by `numpy`: we just need to generate two-dimensional values that will constitute the real and imaginary parts of our data. Recalling the definition of the circularity coefficient

$$\rho_z = \frac{\sigma_x^2 - \sigma_y^2}{\sigma_x^2 + \sigma_y^2} + i \frac{2\sigma_{xy}}{\sigma_x^2 + \sigma_y^2}$$

we can then tune the covariance matrix  $\Gamma$  of the generating distribution to regulate the two main sources of non-circularity:

$$\Gamma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \quad (I) \quad \sigma_x \neq \sigma_y \quad (II) \quad \sigma_{xy} \neq 0$$

As in the paper discussed above, we want to setup a classification problem among data following two different distributions:

- with label "0", a perfectly circular complex normal ( $\rho_z = 0$ ), with zero-centered data and the identity as covariance matrix;
- with label "1", still a complex normal with mean zero, but with an arbitrary covariance matrix (variable  $\rho_z$ ).

In practice, we will construct a dataset made of 5000 samples for each distribution, with each of them being constituted by 128 complex values. In figure 4.1 we represent an example for each class of data. Successively the dataset will be partitioned in a train and a test set with proportions 80% – 20%.

It is important to note that the distinction between the classes is entirely contained in the relation-

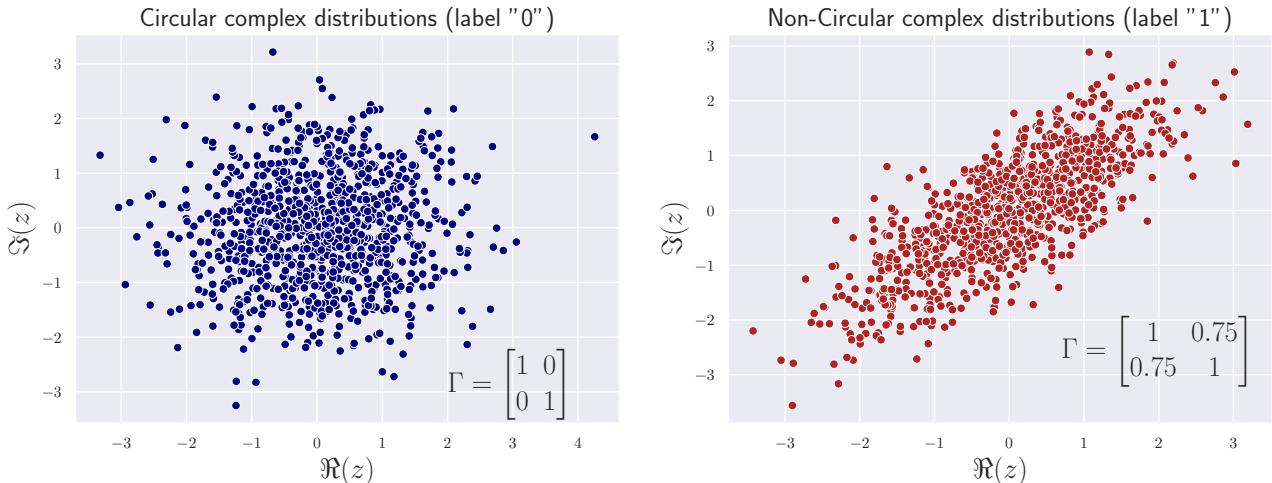


Figure 4.1: Samples coming from a perfectly circular distribution (left) and from a non-circular one (right).

ship between the real and imaginary parts. This means that removing, for example, the imaginary part of the dataset will result in both classes being statistically identical, and therefore, making the classification impossible.

## Classification Task

To complete our setup, we just need to define a couple of neural networks to train: a complex-valued model, and its equivalent real-valued counterpart (treating the real and imaginary parts of the input as two independent channels) with double parameters respect to the first one. Datasets are quite simple by themselves, and so we used very elementary classifiers: we used a complex network with a couple of hidden layers (64 and 16 units respectively), and a small dropout (20%). The same structure has been duplicated over two independent channels to obtain the real-valued network.

We made our tests for different label 1 distributions (changing the second covariance matrix while leaving the first equal to the identity), seeing that both models are able to learn easily from the dataset reaching also high accuracies (> 95%). But we also noticed that there are some configurations for which one of them outperform the other, sometimes even by a lot. In order to better investigate this situation, we proceed examining the two sources of non-circularity separately. Just to remark, the samples labeled with "0" have:

$$\sigma_x = \sigma_y = 1 \quad \sigma_{xy} = 0 \quad \Rightarrow \rho_z = \rho = 0$$

Let's examine the first source of non-circularity in a complex distribution, i.e. data with a non-zero covariance factor ( $\sigma_{xy} \neq 0$ ). Considering the distribution "1", let's fix, for simplicity,  $\sigma_x = \sigma_y = \sigma = 1$  and leave the covariance  $\sigma_{xy}$  free to space the interval  $(-1, 1)$  (to keep the covariance matrix positive defined). In this case we will have:  $\rho_z = i\sigma_{xy}/\sigma^2$ ,  $\rho = \sigma_{xy}/\sigma^2$  and so  $|\rho_z| = |\rho|$ .

We generated several datasets with the setup just described (fixed variances but different  $\sigma_{xy}$ ) and trained both models over all of them for 60 epochs. As final "performance score" we kept the average

classification computed over the last 10 epochs for the test set. To improve the reliability of the statistics found, we repeated the entire procedure for different random seeds (taken into account in the errorbars of the plots) and represented the results in figure 4.2 (left). The behavior of the points is quite clear and regular: for this kind of non-circularity, the complex model outperforms the real one, also by a significative percentage, as the modulus of the circularity coefficient  $\rho_z$  approaches 1 (and so does the correlation  $\rho$ ).

Then we examined also the second source of non-circularity, i.e. different variances among real and imaginary parts of the variables ( $\sigma_x \neq \sigma_y$ ). This time we started fixing  $\sigma_{xy} = 0$  and also, for simplicity,  $\sigma_x = 1$  (because, in the end, it is just the shape of the distribution that matters). We leaved, instead, the second variance  $\sigma_y$  free to move in an interval  $(0, 3]$ . This time we have  $\rho_z = (\sigma_x^2 - \sigma_y^2)/(\sigma_x^2 + \sigma_y^2)$  and  $\rho = 0$ .

We repeated again the same procedure as for the first source and obtained the results in figure 4.2 (right). The behavior of the accuracies is again quite clear and regular, but this time we see the real model performing better than its complex-valued counterpart; however, this advantage is relatively small, and seems to completely vanish as  $|\rho_z|$  approaches 1. A possible justification to this behavior stands on the fact that the real and imaginary part of the data are statistically not correlated, since  $\rho = 0$ . We believe, in fact, that the two independent channels of the real model are probably more adapt to learn such internal structure with respect to the complex-valued architecture, whose parameters still preserve some local correlations, even after the training loop.

In the previous analysis we found some really interesting regular behaviors, with situations in which

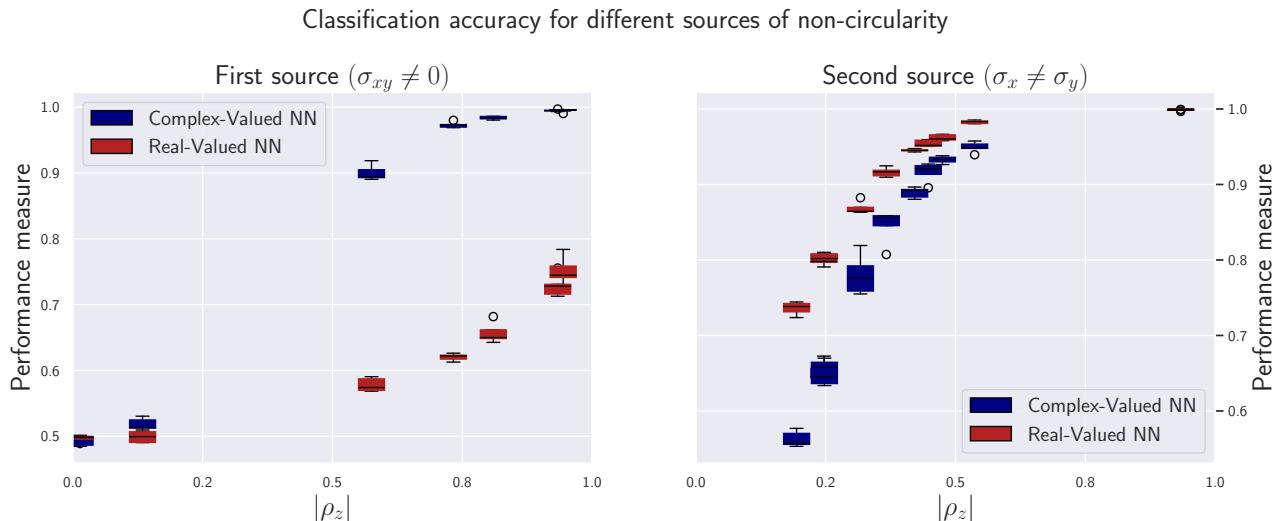


Figure 4.2: Classification accuracy for real and complex-valued networks as function of different source of non-circularity.

complex models are dominating and others in which are the real ones performing better. But, is there any general law, or rule, that we can find, in order to understand which characteristics should our data have, in order to guarantee nice performances for complex networks?

Let's proceed in the same way as before, fixing a distribution "0" that is perfectly circular, again with the identity as covariance matrix, and let's try to distinguish its samples from the ones coming from a second distribution "1", in which we add some non-circularity from both sources. We decide again to fix  $\sigma_x = 1$  for the second distribution, while the two remaining degrees of freedom  $\sigma_y$  and  $\sigma_{xy}$  are let free. Let's vary those parameter in order to space all the possible values that  $\rho_z$  can assume, i.e. all the points in the unitary complex circle, and train again the models.

Because of the finite precision and the low amount of points in our samples (128), we cannot construct an accurate and uniform grid spacing all the possible values, and so we decided to adopt a stochastic method: sampling random combinations of  $(\sigma_y, \sigma_{xy})$ , computing the relative performance measure for the architectures, and reconstructing the accuracy distribution with the functionality `tri.LinearTriInterpolator`, provided by `matplotlib`. The final results of this analysis are presented in figure 4.3

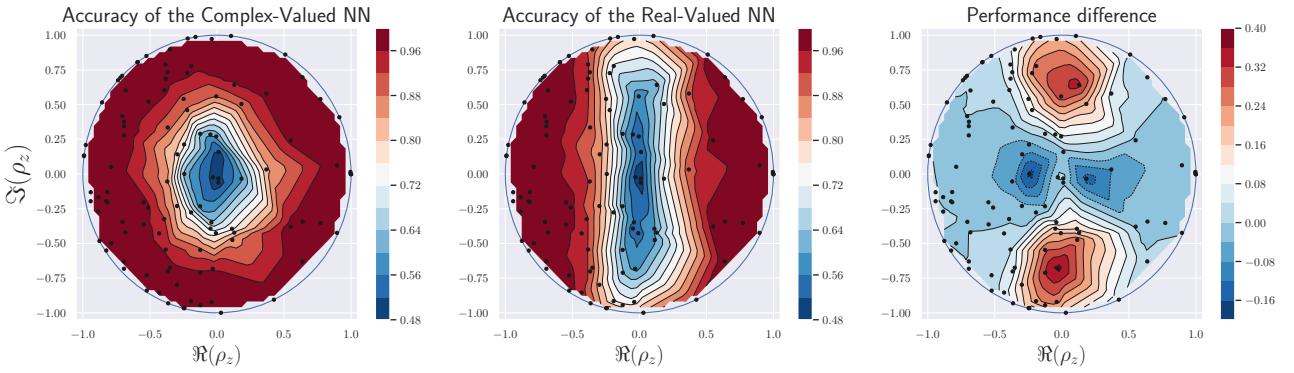


Figure 4.3: Comparison of the accuracy achieved by a complex-valued model (left) and its real equivalent counterpart (center) in distinguishing a perfectly circular distribution from a second one as function of its circularity quotient. The third plot (right), instead, represents simply the differences among the first two.

As we can see, there are effectively some regularities in the behavior of those models. For the complex-valued network, we can see that its discrimination performances raises uniformly with the modulus of the circularity quotient, achieving the maximum accuracy, and training stability, on the edge of unitary circle. For their real-valued counterpart, instead, the situation is a bit different: this models, in fact, seems to not handle very well the "first source" of non-circularity ( $\sigma_{xy} \neq 0$ ), probably because it coincide with a non-zero correlation coefficient among real and imaginary parts of data. Looking at the right figure, the one with the effective comparison among the models, we can clearly see that there are still some configurations in which real networks seems to perform better (-16%), but the theoretical advantage (+40%) guaranteed by the complex model in the two red regions is too high to not be taken into account.

## **Part II**

# **Condition Monitoring**



## Chapter 5

# Healthy-Faulty Signal Classification

In the second part of this thesis, we will apply the complex-valued deep learning framework, developed in part 1, to a real world problem of condition monitoring in industrial applications.

Condition Monitoring is defined as *the process of monitoring a parameter of condition in machinery (vibration, temperature, etc.), in order to identify a significant change which is indicative of a developing fault.*

In our technological and industrial society, the use and mastery of condition monitoring turned out to be of extreme importance, since it allows to be schedule maintenance, or other actions to be taken in order to prevent consequential damage and avoid its, often expensive, consequences. Furthermore, it can help in increasing the lifespan of many engines, preventing faults that could develop in major failures.

Condition Monitoring techniques are normally used on rotating equipment, auxiliary systems and other dynamic machinery (compressors, pumps, electric motors, etc.) while for static devices, usually, periodic inspections are sufficient.

In our analysis, we will propose a modification (or, better, an extension) to the existing approach to the strategy that is usually followed for rotating machines, i.e. **vibration analysis**.

What engineers do, is usually taking measurements on machine bearing casings with *accelerometers* to measure the casing vibrations, sometimes provided also of electric transducers able to directly observe the rotating shafts and detect their radial (and axial) displacements. Data collected are then set of vibrational signals that can be analyzed and studied to detect clues of something bad that is happening. Vibration levels can then be compared with some historical baseline values (a "ground truth") derived after long periods of experiments, and in some cases compared with established standards such as load changes, to keep high the attention. Vibration limits can also be defined based on the machine design or components, knowing their fault frequencies of bearings.

Interpreting the vibration signal obtained is an elaborate procedure that requires specialized training and experience. With the technology progresses and increasingly complicated machines, simple comparisons and vibration limits are not anymore sufficient to guarantee the performances and accuracies needed by the companies. Luckily, also the techniques available have widely improved: thanks especially to the recent advances of machine and deep learning, signal analysis and classification is strongly simplified. Several state-of-the-art approaches have been developed, able to provide the vast majority of data analysis automatically, retrieving information instead of raw data.

In general one commonly employed technique is to examine the individual frequencies present in the signal. These frequencies, in fact, can correspond to certain mechanical components or specific malfunctions (e.g. unbalance or misalignment). So, by analyzing these frequencies and their harmonics, a specialist can in principle be able to identify the location or the type of a problem, sometimes even the cause. And this is possible weeks, even months, before the effective failure or damage of the apparatus, giving ample time to the technicians for replacing or repairing the machine.

But the interesting part in this approach is that we have a mathematical instrument that allow us

to move from the time to the frequency domain, with all the consequent advantages: the **Fourier Transform**. We will see, in the next section, that this operator not only permits the analysis of the frequency spectrum of the signal, but also represents the connection among the real-valued measurements of physical quantities and the complex-valued domain in which we can effectively put to test the framework we have developed.

It is however better to specify that frequency analysis tends to be most useful on machines that employ rolling element bearings and whose main failure modes tend to be the degradation of those bearings, which typically exhibit an increase in characteristic frequencies associated with its geometries and constructions.

Tons of different "manual" analysis are possible for those kind of signals in the frequency domain, a study of its phase spectrum for example, but in this thesis we are not really interested in them since we will rely on modern deep learning approaches, providing also efficient alternatives to the actual state-of-the-art techniques.

## 5.1 State-of-the-Art

In chapter ??, we were complaining about the fact that literature does not provide almost any kind of complex-valued dataset to effectively test the models we have developed. There, we were forced to "manually construct" our data, with simple distributions of points satisfying certain properties ??, or clever modifications of existing real collections ???. Relying on them could have been fine for that preliminary analysis, mainly focused on the convergence and the stability of the training process, rather than on the final performances. However, we believe that, in order to prove the efficacy of our method, we should stress our complex-valued architectures on more concrete and reliable datasets.

But, is there the possibility of measuring in some way physical quantities that are inherently complex-valued? From this point of view, the answer is yes, and we just need to think about magnetic momenta. The PHD thesis we have based our work on ??, is constructed exactly on these kind of data: Magnetic Resonance Images contains, in fact, much information in their phase, that, with a complex neural network the author managed to outperform many existing approaches that used to discard such information.

But we should not limit our horizons when we have a powerful instrument like the Fourier transform  $\mathcal{F}$ , that allow us to associate a complex-valued frequency domain representation to a function of time. In this new representation, the magnitude represents the amount of a certain frequency present in the original signal, while the argument is its phase offset with respect to the basic sinusoid. All of this means that, given any dataset composed of waves in the time domain, the same can be studied in the frequency domain exploiting the new power of complex-valued models, keeping all the physical information without need of expedients like before. Several papers have been published, suggesting complex-valued models to study electric, seismic ?? and vibrational signals in the complex domain. So, in principle, for any signal registered in the time domain, we can recover, and maybe exploit, the phase information thanks exactly to the Fourier transform.

Studying the problem in the frequency domain is a quite common approach to signal processing, since many operations in the time domain have a complex counterpart that sometimes turns out to be easier to perform (eg. differentiation and convolution). Furthermore, also from a purely theoretical point of view, several concepts are easier to derive and understand with complex notations.

### 5.1.1 Baseline Machine Learning

But how can we effectively distinguish among vibration signals? According to the traditional machine learning approach we should rely on some hand-crafted features extractable from the wave in the time domain; those features can then face a dimensionality reduction step (PCA, t-SNE, etc.) and then sent through baseline classifiers like Support Vector Machines or Multi-Layer Perceptrons. This was the fundamental strategy for years: in term of memory and time complexity it is very efficient, because each sample is reduced to a handful of meaningful values, but, in general, it lacks of generalization capabilities, since the high-level features to be used are chosen every time by an expert, depending

on the situation, and also from the point of view of the accuracy reached, more modern technologies employing deep learning are already able to outperform it.

### 5.1.2 Study in the frequency domain

However, even if condition monitoring is not one of the main tasks over which machine learning researchers focused mostly in the last years, several techniques and improvements could be recovered and re-adapted from the popular area of audio applications. In the end, in fact, sounds are vibrations of the air, and so their analysis can present interesting similarities with our problem.

It is exactly from the widely developed theory of deep learning applied to audio processing, that we inherit techniques suitable to study a wave signal in the frequency domain. The computational instrument that allow us to do so is the **Short-Time Fourier Transform (STFT)**. Discrete STFT is a method of partitioning continuous signals (in the time-dependent representation) over a long period into shorter segments (usually overlapped) at short time intervals, and applying a Fourier transform to each of those segments. An additional step can be taken, at this point, deriving the power spectrum of the signal: taking the square modulus of the STFT we can construct a 2D sample, i.e. an image, representing the amplitude of the signal for a particular frequency at a particular time.

In figure 5.1 we took a random signal from the dataset we are going to study, and we transformed it in the Fourier representation: the leftmost picture represents the original wave, the central one is the result after the application of the Fast-Fourier transform (so the signal in the frequency domain), while the rightmost is its power spectrum. After the computation of the power spectrum for each

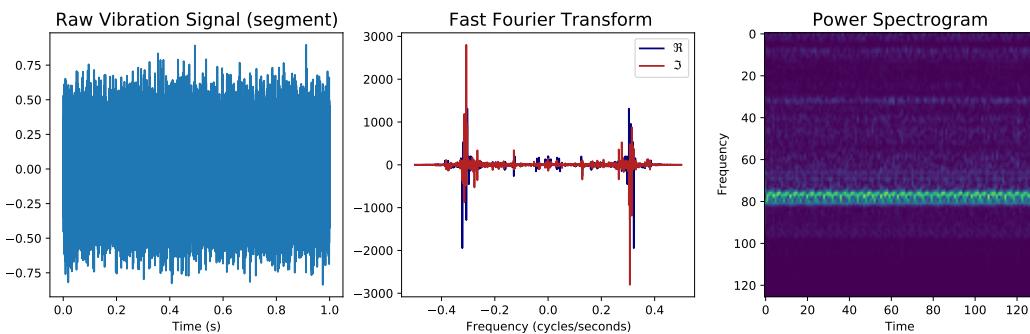


Figure 5.1: Representation of a vibration signal in the time domain (left), in the frequency domain via FFT (center) and with its power spectrum (right).

vibration signal at this point, there are two directions that the researcher can decide to follow:

- keeping the two-dimensional input and building a machine learning classifier able to work directly with the spectrograms (usually a Convolutional Neural Network);
- taking a further step and extracting from the spectrogram the so called *Mel Frequency Cepstral Coefficients* (MFCC): these coefficients, widely used in speech recognition tasks, permit an efficient compression of the "important information" carried by the signal, and can be simply given in input to an ordinary classifier.

Even if they were not properly designed for vibration analysis applied to condition monitoring, the usage of MFCCs can bring very good results together with an high efficiency in term of space and time complexity, because of the information compressed.

In figure 5.2 are represented schematically the two main state-of-the-art approaches (high-level features extraction vs power spectrograms) for healthy-faulty vibration signal classification, that we just described.

In the successive analysis, we will propose an alternative to the approaches listed above, that is more suitable to complex-valued deep learning: the coefficients of a Fourier transform are, in fact, inherently complex, and so, instead of losing all the phase information taking the power spectrum (that is constituted only by their amplitudes), we stop one step before keeping the original complex-valued variables. In the end, deriving the complex spectrum is equivalent to computing a discrete Fourier transform over short overlapping windows of the signal.

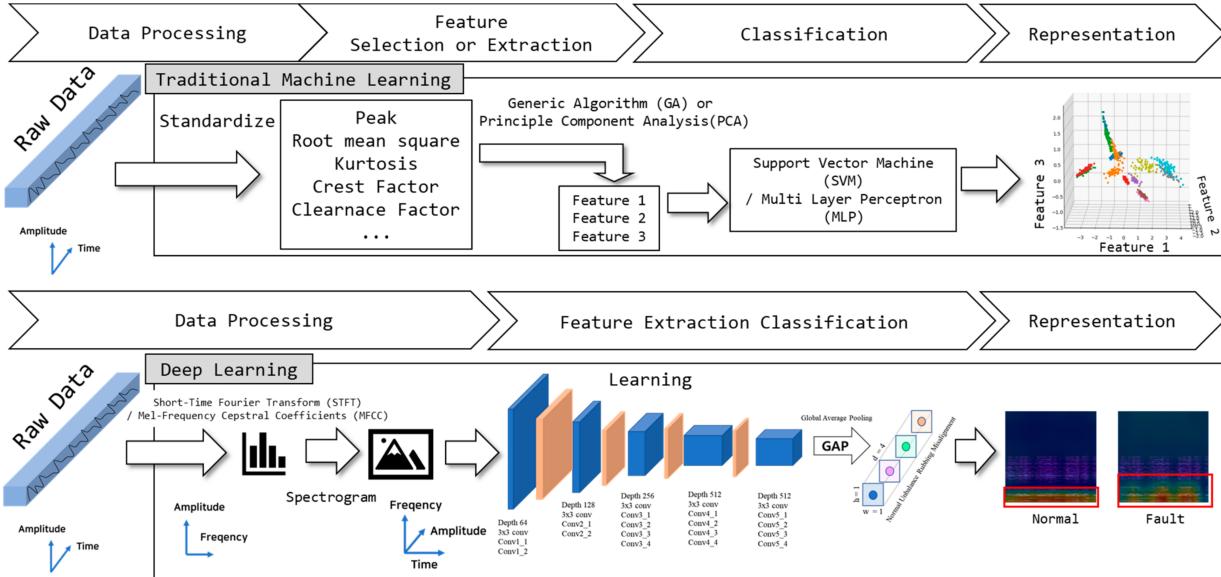


Figure 5.2: State-of-the-art approaches for vibration data classification applied to predictive maintenance (source: [18]).

## 5.2 Bonfiglioli

The main motivation of this thesis work was the development and testing of an unconventional deep learning approach, based on complex-valued neural networks, on a series of datasets provided to FBK by Bonfiglioli 5.3, a worldwide designer, manufacturer and distributor of a complete range of gearmotors, drive systems, planetary gearboxes and inverters.

The company provided a large number of datasets collected from as many experiments, with the purpose of simulating artificially the most frequent faults, that usually occur over the bearings of different kind of engines. Our objective will be then to find an efficient approach for failure detection and classification.



Figure 5.3: Bonfiglioli logo.

### 5.2.1 Simulation Environment

Data have been collected at the Bonfiglioli headquarters of Rovereto (TN), with the experimental setup schematized in figure 5.4. The laboratory is constituted of four different workbenches in which different experiments can be conducted. In each workbench, a lab-scale rotating simulation equipment is installed, constituted

In the experiments, faults were artificially simulated using mechanical tricks (e.g. unbalance is simulated attaching a small object to the shaft)

In order to have data more heterogeneous, able to represent the engine behavior under different conditions, and to improve the generalization capabilities of future models trained over them, Bonfiglioli setup a *working cycle*, during which measurements were taken. This cycle is periodic with a period of 24 hours, in general repeated up to five times to collect most of the datasets. It is characterized by the engine varying its speed and torque with a quite regular behavior following a piecewise constant function, also to guarantee stationary conditions during the acquisition phases. Data were not, in fact, collected continuously during this cycle: 60 seconds long windows were fixed along this period, during which sensors were actively measuring. An example of working cycle has been reported in figure 5.5.

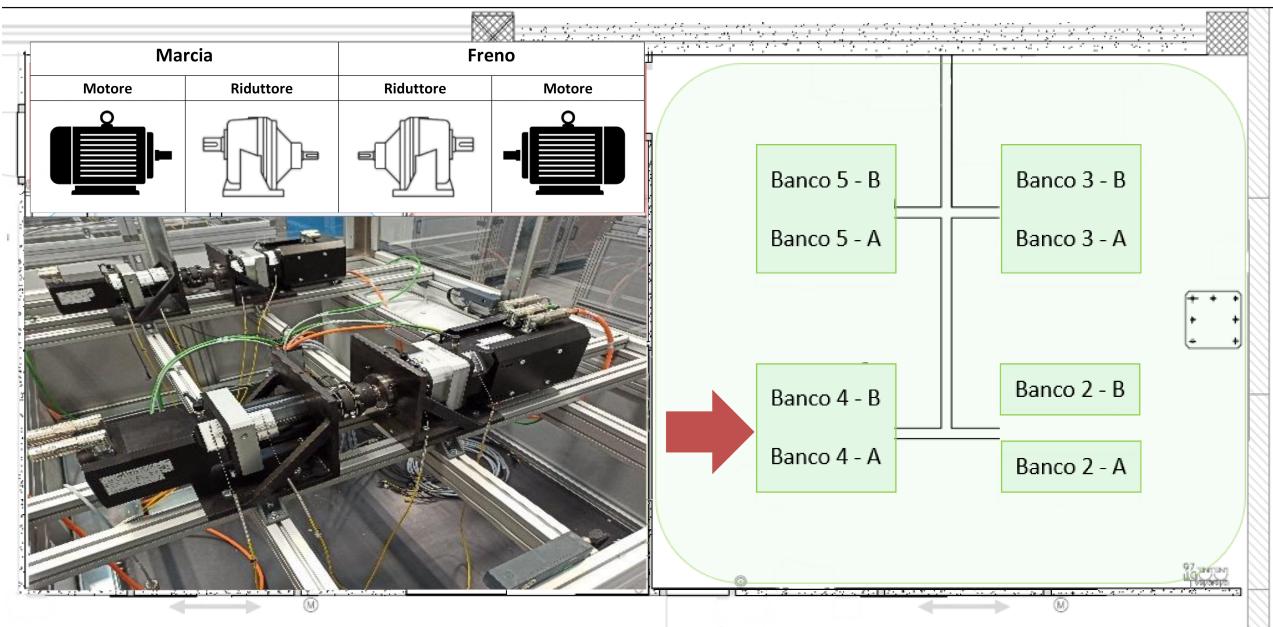


Figure 5.4: Schematic representation of the experimental workbench used by Bonfiglioli.

Vibrations signals, the ones we are mainly interested in, were collected thanks to accelerometers at-

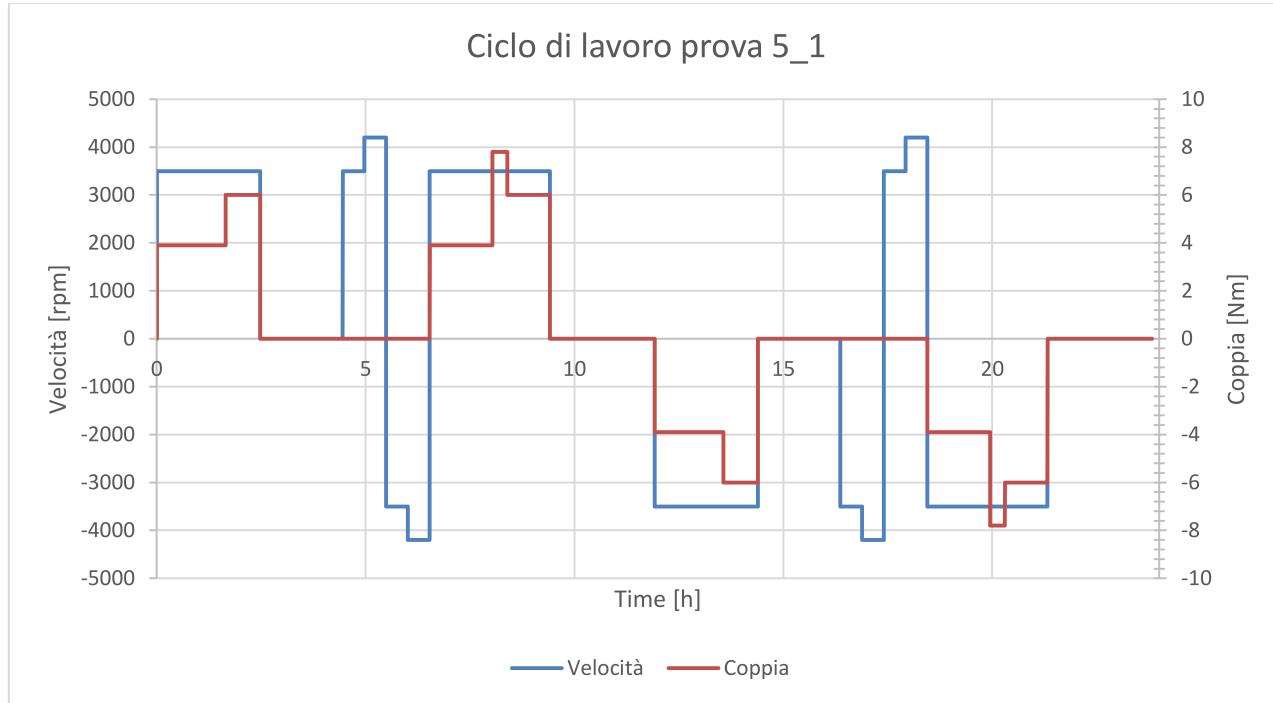


Figure 5.5: Example of working cycle (from Prova-5) of the Bonfiglioli experimental setup.

tached to the components of the engine (motor, brake and multiplier) they wanted to test; as explained above, each acquisition was 60 seconds long, with a sample rate of 25600 Hz: sampling was performed because a vibration is a periodic signal in the time domain, and most fault signals have periodicity. Therefore, sampling can be used to examine the consistency and continuity of each condition [18]. The signal segmentation for sampling is usually based on the rotational frequency of the rotor. Generally, in fact, in rotating equipment, the rotational frequency is the most dominant component, and the majority of fault contributions appear in its harmonic form.

### 5.2.2 Datasets

In these last months, the Bonfiglioli laboratory have setup different experiments, in order to provide data relative to different kind of engines and for different types of simulated faults. For each of those runs, containing 120 hours of measurements, a dataset in .h5 format has been provided, with the values registered during a working cycle. Those datasets had an internal structure quite mazy, and so we had to rely on a custom library, provided by FBK researchers, to load and process the data. Each sample contains several entries, related to the different variables measured during the process: beyond vibration signals collected through the accelerometers, Bonfiglioli decided to monitor also the temperature of some components, voltage and current flowing through them, together with the phase information of the inverter. Additionally, for the experiments realized more recently, they used an additional component in their setup, a *multiplier*, providing a second vibration signal.

For what concern our work, we are interested only in the vibration signals.

N	Name	Subname	Engine A	Duration	Fault
4	Prova 4 - BN90 - HF Sbilanciato	4_1	BN90	24	Unbalance
4	Prova 4 - BN90 - HF Sbilanciato	4_2	BN90	120	Unbalance
5	Prova 5 - MP105 - HF Cuscinetto	5_1	BMD118	24	Bearing
5	Prova 5 - MP105 - HF Cuscinetto	5_2	BMD118	?	Bearing
5	Prova 5 - MP105 - HF Cuscinetto	5_3	BMD118	120	Bearing
5	Prova 5 - MP105 - HF Cuscinetto	5_4	BMD118	100	Bearing
6	Prova 6 - BN90 - HF Cuscinetto	6_3	BN90	100	Bearing
6	Prova 6 - BN90 - HF Cuscinetto	6_4	BN90	100	Bearing
10	Prova 10 - MP105 - HF Solare	10_1	BMD118	120	Solar
10	Prova 10 - MP105 - HF Solare	10_2	BMD118	24	Solar
10	Prova 10 - MP105 - HF Solare	10_3	BMD118	24	Solar
10	Prova 10 - MP105 - HF Solare	10_4	BMD118	120	Solar
11	Prova 11- S30 - HF Cuscinetto	11_1	BN112	120	Bearing
13	Prova 13 - S30 - HF Pignone	13_1	BN112	120	Pinion
15	Prova 15 - S30 - HF Corona	15_1	BN112	78	Crown
15	Prova 15 - S30 - HF Corona	15_2	BN112	48	Crown

Table 5.1: Summary of the main technical features of the datasets provided by Bonfiglioli and used in our analysis.

### 5.2.3 Baseline Classification Approach

## 5.3 Mendelay Data

The last dataset we are going to analyze for this chapter is the Mendelay Dataset [19]: a collection of vibration signals collected from bearings with different health conditions and under different time-varying speed conditions. The problem is again an healthy-faulty classification task with two different "bad" classes: faulty with an inner race defect and faulty with an outer race defect.

The experiments have been realized over a machinery fault simulator with the setup shown in 5.6. The shafts is driven by a motor which rotational speed is controlled by an AC Drive. Two ball bearings are installed to support the shaft: the left one is healthy, while the one on the right is the experimental bearing, replaced during each experiment with a component of a different health condition. Again, vibration data are collected thanks to an accelerometer fixed on the housing of the experimental bearings, while an incremental encoder is used to collect the rotational speed values.

These datasets were collected and reorganized with the purpose of evaluating the effectiveness of methods developed for bearing fault diagnosis under time varying speed conditions. This last point is the main difference with most of vibration datasets available from actual literature, that have been collected under constant speed condition (as Bonfiglioli's). The complex-valued models we have developed so far, however, were not properly designed to work on such dynamic signals, but this still

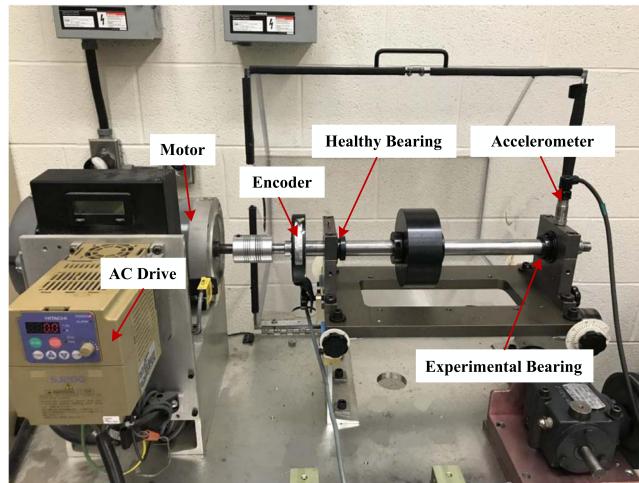


Figure 5.6: Experimental setup for Mendeley data.

represents an opportunity to stress and push to the limit the approach we have designed.

### 5.3.1 Dataset Structure

Given the various general purposes for which this dataset was designed (i.e. condition monitoring and analysis of bearings' frequency characteristics under time-varying rotational speed conditions), its internal structure is slightly intricate. First of all, each sample has two channels, one for vibration data and the other for rotational speed data. Each signal is sampled at 200000 Hz and the sampling duration is 10 s. There whole collection of signals is divided into 36 different datasets, depending on two experimental settings:

- *health condition* (healthy, faulty with inner race defect, faulty with outer race defect);
- *speed varying condition* (increasing speed, decreasing speed, increasing then decreasing speed and decreasing then increasing speed).

Therefore, there are 12 different cases for configuration, each of them constituted of 3 trials, to ensure the authenticity of the data, that explain the 36 subsets. Raw signals and corresponding spectrograms are, in the end, visually similar to the ones of Bonfiglioli ???. Data in the second channel, containing speed information, can be properly visualized via their spectrograms, looking at the frequency changes in the signal through the time (figure 5.7). For the data generation part, two parameters still needs

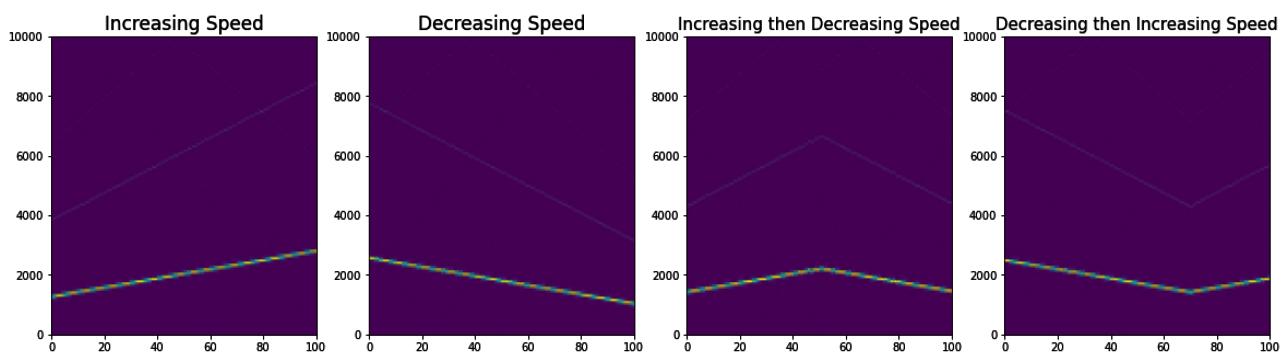


Figure 5.7: Spectrograms of the four varying-speed conditions of Mendelay data.

to be fixed:

- the `n_fft` of `torchaudio.transform.Spectrogram`, representing the resolution of the spectrogram;
- the "split rate" parameter, determining the number of subsegments in which each signal from the dataset has to be split.

The latter turns out to be quite important in this problem: recall, in fact, that each vibration registered in the dataset is 10 seconds long, with a sample rate of 200000 Hz, and so it is constituted by a total of  $2 \cdot 10^6$  points, and so is quite long. Furthermore, the dataset itself is quite small: only  $36 \cdot 3 = 108$  signals, number that could be increased just splitting them.

### 5.3.2 Mendeley Classification

We start by fixing the length of the subsignals equal to the sample rate (200000 points): as discussed in ??, it is quite common that fault signals have periodicity, and we believe that the sampling rate provided has been chosen to properly detect this feature. Now, we have a dataset of 360, one second long, vibrations. Train-test splitting is again 75% – 25%.

Even if they were not designed to deal with vibration signals registered in non-stationary conditions (with varying speed), we decided to rely on the same architecture built for Bonfiglioli analysis ??.

We trained both models on the Mendelay dataset with all the setup described above, and the results are briefly summarized in picture 5.8. Those are very bad results: the training errors and accuracy,

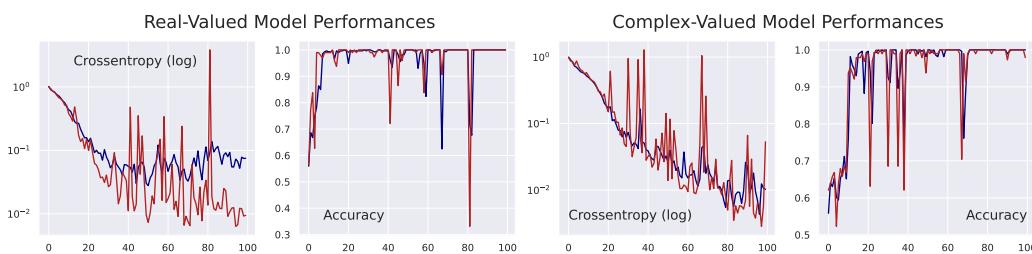


Figure 5.8: Performances achieved over the Mendelay dataset constructing one-second long sub-signals.

especially in the real-valued case, behaves nicely, but the models look like having bad generalization performances, given the high instability that we can notice on the validation lines. Now, since the architecture is already quite small, and we have already exploited the canonical techniques to prevent overfitting (normalization, dropout) what we can do is acting directly on the dataset.

Other tests realized have proved that using only the files that belong to the same speed class (increasing, decreasing, increasing then decreasing or decreasing then increasing), so keeping only monotonic or non-monotonic speed samples together, we noticed that the overfit and the instability were more limited.

Under this perspective, we believe that further reducing the length of input signals, even before constructing the spectrograms, we can manage to reduce also the effects due to high variations in rotational speed. To prove it, we repeated the train considering samples 0.25 seconds long. In this way we have much more samples (3600) in the dataset. For completeness, it is better to specify that to obtain the results shown in figure 5.9 we established a decay schedule for learning rate, in the hope of a faster, but more accurate, convergence.

Finally, this is a really nice behavior, with both models reaching the convergence and the perfect classification (100%) accuracy. This means that our supposition was probably right, and a suitable approach to signal classification in case of time-varying speed conditions, is to fragment the original wave into smaller samples to reduced the effects of this non-stationarity. Of course, we believe that a necessary hypotheses for this to happen is some kind of monotony in the rotational speed variation: also in the third and fourth cases 5.7, we could clearly distinguish two regions in which the velocity behavior was monotonic and amenable to the first two. Probably, an higher irregularity in the rotational speed would have made also the classification more unstable and challenging, but we would need further more datasets to verify it.

To conclude this part, we want to try stressing the generalization performances of our models. This can be done first dividing the data files based on the four speed conditions (increasing, decreasing, increasing then decreasing, decreasing then increasing), training the networks over just one of them, and computing the classification accuracy over the others. In this way we can effectively verify which is

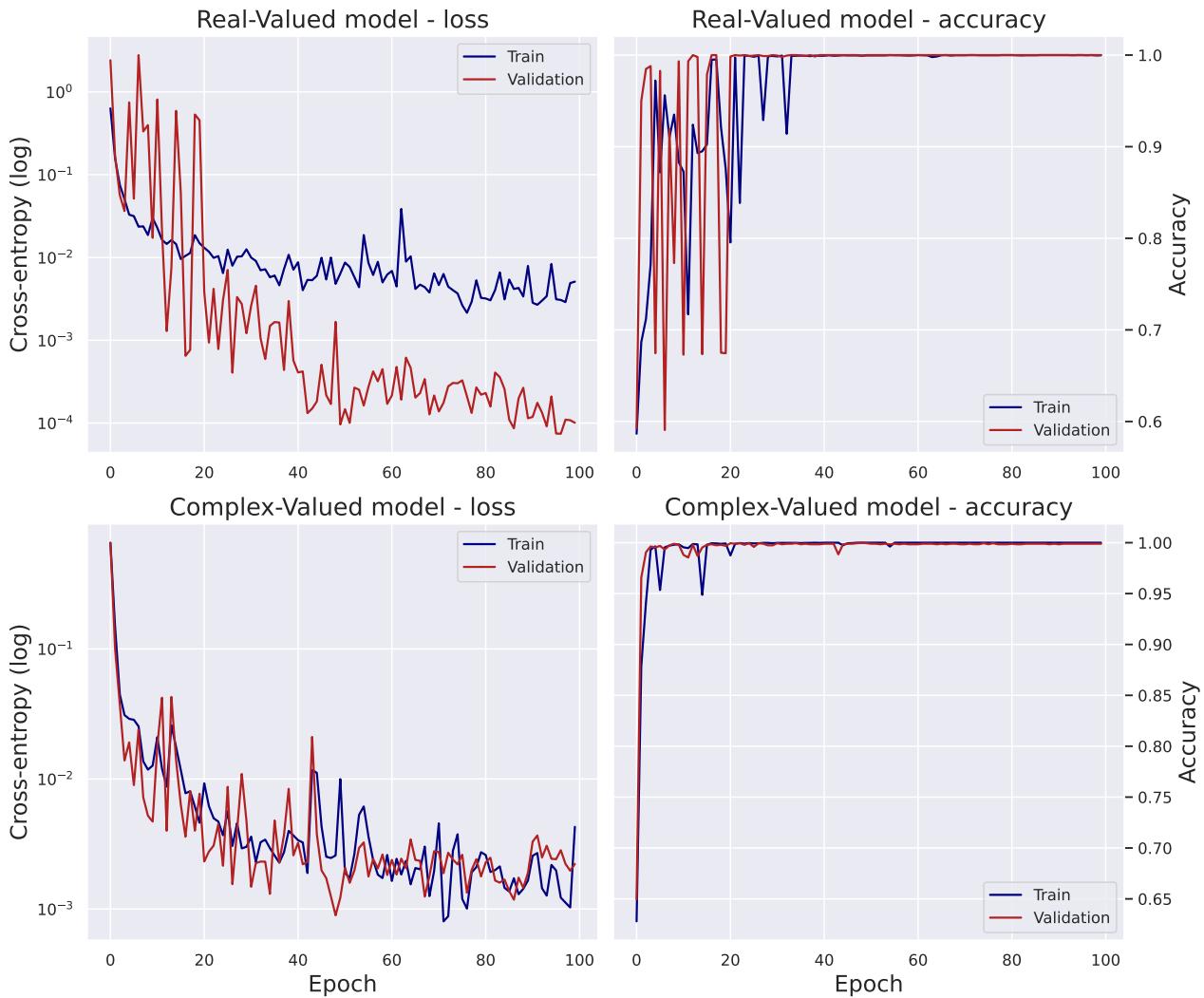


Figure 5.9: Performances achieved over the Mendelay dataset constructing 0.25 seconds long sub-signals.

the impact of non-stationary rotational speed conditions in the training process. The results, reported

	100.00	99.72		98.89	97.18		99.94	99.00		99.50	99.34	
	100.00	99.94		79.70	77.93		99.89	96.18		96.29	73.56	
	100.00	100.00		88.05	82.13		98.34	98.23		100.00	89.77	
	100.00	99.67		88.38	94.63		97.90	94.08		99.94	98.56	

Table 5.2: Generalization performances of real and complex-valued models over the Mendelay dataset.

in table 5.2 are generally good: for almost all the combinations both networks managed to converge, obtaining nearly the perfect classification accuracy.



# Bibliography

- [1] C. Trabelsi, O. Bilaniuk, Y. Zhang, D. Serdyuk, S. Subramanian, J. F. Santos, S. Mehri, N. Ros-tamzadeh, Y. Bengio, and C. J. Pal, “Deep complex networks.” <https://arxiv.org/abs/1705.09792>, 2018.
- [2] D. G. Messerschmitt, “Stationary points of a real-valued function of a complex variable,” Tech. Rep. UCB/EECS-2006-93, EECS Department, University of California, Berkeley, Jun 2006.
- [3] H. Li and T. Adali, “Complex-valued adaptive signal processing using nonlinear functions,” *EURASIP J. Adv. Sig. Proc.*, vol. 2008, 12 2008.
- [4] J. S. Dramsch, M. Lüthje, and A. N. Christensen, “Complex-valued neural networks for machine learning on non-stationary physical data,” vol. 146, p. 104643, Jan 2021.
- [5] R. S. Elias M. Stein, *Complex analysis Vol.2*. Princeton lectures in analysis 2, Princeton University Press, 2003.
- [6] K. Kreutz-Delgado, “The complex gradient operator and the cr-calculus.” <https://arxiv.org/abs/0906.4835>, 2009.
- [7] T. Nitta, “An extension of the back-propagation algorithm to complex numbers,” *Neural Networks* vol. 10 iss. 8, vol. 10, nov 1997.
- [8] J. A. Barrachina, C. Ren, C. Morisseau, G. Vieillard, and J.-P. Ovarlez, “Complex-valued vs. real-valued neural networks for classification perspectives: An example on non-circular data.” <https://arxiv.org/abs/2009.08340v2>, 2021.
- [9] P. Virtue, *Complex-valued Deep Learning with Applications to Magnetic Resonance Image Synthesis*. PhD thesis, EECS Department, University of California, Berkeley, Aug 2019.
- [10] H. Akira, *Complex-Valued Neural Networks: Advances and Applications*. Wiley-IEEE Press, 2013.
- [11] F. Amin, M. Amin, A. Y. H. Al Nuaimi, and K. Murase, “Wirtinger calculus based gradient descent and levenberg-marquardt learning algorithms in complex-valued neural networks,” vol. 7062, pp. 550–559, 11 2011.
- [12] X. Glorot and Y. Bengio, “Understanding the difficulty of training deep feedforward neural networks,” *Journal of Machine Learning Research - Proceedings Track*, vol. 9, pp. 249–256, 01 2010.
- [13] K. He, X. Zhang, S. Ren, and J. Sun, “Delving deep into rectifiers: Surpassing human-level performance on imagenet classification,” 2015.
- [14] M. Cogswell, F. Ahmed, R. Girshick, L. Zitnick, and D. Batra, “Reducing overfitting in deep networks by decorrelating representations,” 2016.
- [15] N. Guberman, “On complex valued convolutional neural networks.” <https://arxiv.org/abs/1602.09046>, 2016.
- [16] G. Georgiou and C. Koutsougeras, “Complex domain backpropagation,” *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 39, no. 5, pp. 330–334, 1992.

- [17] M. Arjovsky, A. Shah, and Y. Bengio, “Unitary evolution recurrent neural networks,” *CoRR*, vol. abs/1511.06464, 2015.
- [18] S. Lee, H. Yu, H. Yang, I. Song, J. Choi, J. Yang, G. Lim, K.-S. Kim, B. Choi, and J. Kwon, “A study on deep learning application of vibration data and visualization of defects for predictive maintenance of gravity acceleration equipment,” *Applied Sciences*, vol. 11, p. 1564, 02 2021.
- [19] H. Huang and N. Baddour, “Bearing vibration data collected under time-varying rotational speed conditions,” *Data in Brief*, vol. 21, pp. 1745–1749, 2018.
- [20] A. Ziller, D. Usynin, M. Knolle, K. Hammernik, D. Rueckert, and G. Kaassis, “Complex-valued deep learning with differential privacy.” <https://arxiv.org/abs/2110.03478>, 2021.
- [21] M. Ragab, Z. Chen, M. Wu, H. Li, C.-K. Kwoh, R. Yan, and X. Li, “Adversarial multiple-target domain adaptation for fault classification,” *IEEE Transactions on Instrumentation and Measurement*, vol. PP, pp. 1–1, 07 2020.
- [22] E. Ollila, “On the circularity of a complex random variable,” *IEEE Signal Processing Letters*, vol. 15, pp. 841–844, 2008.
- [23] P. Virtue, S. X. Yu, and M. Lustig, “Better than real: Complex-valued neural nets for mri fingerprinting.” <https://arxiv.org/abs/1707.00070>, 2017.
- [24] P. Gardner, L. Bull, N. Dervilis, and K. Worden, “Overcoming the problem of repair in structural health monitoring: Metric-informed transfer learning,” *Journal of Sound and Vibration*, vol. 510, p. 116245, 2021.
- [25] H. Ajakan, P. Germain, H. Larochelle, F. Laviolette, and M. Marchand, “Domain-adversarial neural networks.” <https://arxiv.org/abs/1412.4446>, 2015.
- [26] Y. Ganin, E. Ustinova, H. Ajakan, P. Germain, H. Larochelle, F. Laviolette, M. Marchand, and V. Lempitsky, “Domain-adversarial training of neural networks.” <https://arxiv.org/abs/1505.07818>, 2016.
- [27] D. P. Reichert and T. Serre, “Neuronal synchrony in complex-valued deep networks.” <https://arxiv.org/abs/1312.6115>, 2014.
- [28] S. Scardapane, S. V. Vaerenbergh, A. Hussain, and A. Uncini, “Complex-valued neural networks with non-parametric activation functions.” <https://arxiv.org/abs/1802.08026>, 2018.
- [29] J. Shen, Y. Qu, W. Zhang, and Y. Yu, “Wasserstein distance guided representation learning for domain adaptation.” <https://arxiv.org/abs/1707.01217>, 2018.
- [30] N. Schröder, “cplot: Plot complex functions,” Nov. 2021. If you use this software, please cite it as below.
- [31] T. Farris, Frank A.; Needham, “Visual complex analysis.,” *American Mathematical Monthly* vol. 105 iss. 6, vol. 105, jun 1998.

# Appendix A

## Mathematical Proofs

### A.1 Complex Weights Initialization [1]

For complex-valued deep learning traditional approaches of Glorot [12] and He [13], are no more suitable to be used for weights initialization. So we need to re-derive, or at least adapt, those efficient methods also for the complex domain.

Given a generic neural network layer, let's call  $\mathbf{w} \in \mathbb{C}$  its set of complex-valued weights, that we prefer written in polar form:

$$\mathbf{w} = \|\mathbf{w}\| e^{i\theta}$$

The variance of this set is defined as

$$\text{Var}(\mathbf{w}) = \mathbb{E}[\mathbf{w}\bar{\mathbf{w}}] - (\mathbb{E}[\mathbf{w}])^2 = \mathbb{E}[\|\mathbf{w}\|^2] - (\mathbb{E}[\mathbf{w}])^2$$

that, if the parameters are symmetrically distributed around 0, reduces to  $\mathbb{E}[\|\mathbf{w}\|^2]$ .

In order to compute these quantities, we rely on the fact that the magnitude of a standard complex normally distributed variable follows the Rayleigh distribution<sup>1</sup>, i.e. basically a Chi-Square with two degrees of freedom. Just for knowledge, its probability density function depends only on a single parameter and writes

$$f(x; \sigma) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}$$

So we can find a relation among the variances of  $\mathbf{w}$  and its magnitude:

$$\text{Var}(\|\mathbf{w}\|) = \text{Var} \mathbf{w} - (\mathbb{E}[\|\mathbf{w}\|])^2 \quad \longrightarrow \quad \text{Var}(\mathbf{w}) = \text{Var}(\|\mathbf{w}\|) + (\mathbb{E}[\|\mathbf{w}\|])^2$$

But now, that we know the analytical distribution followed by  $\|\mathbf{w}\|$ , we can derive also the two addends of the sum above:

$$\mathbb{E}[\|\mathbf{w}\|] = \sigma \sqrt{\frac{\pi}{2}}, \quad \text{Var} \|\mathbf{w}\| = \frac{4-\pi}{2} \sigma^2$$

The variance of  $\mathbf{w}$  can thus be expressed in terms of its generating Rayleigh distribution's single parameter  $\sigma$ :

$$\text{Var}(\mathbf{w}) = \frac{4-\pi}{2} \sigma^2 + \left( \sigma \sqrt{\frac{\pi}{2}} \right)^2 = 2\sigma^2$$

How can we determine  $\sigma$ ?

- Following the Xavier initialization [12], we would exploit a normal distribution (or a truncated-normal) with variance  $\text{Var}(\mathbf{w}) = 2/(n_{in} + n_{out})$ , with  $n_{in}$  and  $n_{out}$  being the number of input and output units, respectively. For continuity, we have now to set

$$\sigma = 1/\sqrt{n_{in} + n_{out}}$$

---

<sup>1</sup>source

- With the He initialization [13], instead, we used still a normal distribution, but with a variance depending only on input units, i.e.  $\text{Var}(\mathbf{w}) = 2/n_{in}$ , for which we have to set correspondingly

$$\sigma = 1/\sqrt{n_{in}}$$

The magnitude of the complex parameters is then initialized using a Rayleigh distribution with an appropriate  $\sigma$ , while their phase (that never appeared in the equations) can be set uniformly in  $[-\pi, \pi]$ .

## A.2 Stationary points of a real-valued function of a complex variable [2]

Let  $f(z) : \mathbb{C} \rightarrow \mathbb{R}$  be a real-valued function of a complex variable  $z$ , and let's say that we want to find the extreme points of  $f$  (i.e. the values of  $z$  for which  $f$  is maximum or minimum) exploiting the differential calculus.

As explained also in 2.2, differentiability in the complex plane is a quite strong assumption, and there are many function for which the stationarity condition in a point  $z_0$ ,

$$\left. \frac{\partial f}{\partial z} \right|_{z=z_0}$$

cannot even be computed, because the limit in 2.1.3 is not the same from any direction.

A first approach to avoid the complex differentiability is to reformulate the problem in terms of two real variables, i.e. the real and imaginary parts of  $z = x + iy$ . Writing (with a minor abuse of notation)  $f(z) = f(x, y)$ , now the stationarity condition for a point  $z_0$  becomes:

$$\left. \frac{\partial f}{\partial x} \right|_{z=z_0} = \left. \frac{\partial f}{\partial y} \right|_{z=z_0} = 0$$

While this method works, it is cumbersome because it involves the extra step of substituting  $x + iy$  for  $z$ . Thus, we seek an equivalent method that works directly with the complex variable.

Assume now that  $f$  can be represented as  $f(z) \equiv g(z, \bar{z})$ , where  $g$  is an analytic function of two complex variables  $z$  and  $\bar{z}$  that can be freely differentiated with respect to its arguments (because of the assumption of analyticity). Now, substituting  $g(z, \bar{z})$  for  $f(z)$ , we can apply the chain rule of differential calculus to the stationarity conditions:

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial g}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial y} = 0$$

Evaluating the four derivatives, this becomes

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial z} + \frac{\partial g}{\partial \bar{z}} = 0 \quad \frac{\partial f}{\partial y} = i \frac{\partial g}{\partial z} - i \frac{\partial g}{\partial \bar{z}} = 0$$

which has a unique solution

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \bar{z}} = 0$$

We have uncovered that the stationary point can be found by taking the partial derivatives of  $f(z)$  with respect to both  $z$  and  $\bar{z}$ , considering them as independent variables, and setting those derivatives to zero.

Notice that, when  $f$  is real-valued, the complex condition yield redundant information. In this case, in fact, both  $\partial f / \partial x$  and  $\partial f / \partial y$  must be real-valued, and so

$$\frac{\partial f}{\partial x} = \overline{\left( \frac{\partial f}{\partial x} \right)} \quad \frac{\partial f}{\partial y} = \overline{\left( \frac{\partial f}{\partial y} \right)}$$

Plugging in the earlier expressions for these derivatives, we can solve them arriving at the conclusion that  $\partial f/\partial z = \partial f/\partial \bar{z}$ . This establish the redundancy and allows to rewrite the stationarity condition for a real-valued complex function:

$$\frac{\partial f}{\partial \bar{z}} = 0$$

These results are easily extended to the case of vectors of complex variables. Let  $\mathbf{z} = [z_1, z_2, \dots, z_n]^T$  and assume that  $g(\mathbf{z}, \bar{\mathbf{z}})$  is an analytic function of the complex vector  $\mathbf{z}$  as well as its conjugate. Then the condition fo a stationary points becomes

$$\nabla_{\mathbf{z}} g = \mathbf{0} \quad \nabla_{\bar{\mathbf{z}}} g = \mathbf{0}$$

or only the latter if  $g$  is real-valued.

### A.3 Steepest complex gradient descent [3]

Once derived necessary and sufficient conditions for a certain  $z_0 \in \mathbb{C}$  to be a stationary point of a real-valued function  $f(z) : \mathbb{C} \rightarrow \mathbb{R}$ , it is time to derive also the best optimization procedure. Using Wirtinger calculus we managed to overcome the strong requirements necessary to complex differentiability, but now we have twice the derivatives to compute ( $\partial f/\partial z$  and  $\partial f/\partial \bar{z}$ ). So, which direction should our gradient descent algorithm follow, in order to properly optimize  $f$ ?

Let's start defining the gradient vector  $\nabla_{\mathbf{z}} = [\partial/\partial z_1, \partial/\partial z_2, \dots, \partial/\partial z_n]$  for the vector  $\mathbf{z} = [z_1, z_2, \dots, z_n]$  with  $z_k = z_{k,Re} + iz_{k,Im}$  in order to write the first order Taylor series expansion for a function  $g(\mathbf{z}, \bar{\mathbf{z}}) : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}$ ,

$$\Delta g = \langle \Delta \mathbf{z}, \nabla_{\bar{\mathbf{z}}} g \rangle + \langle \Delta \bar{\mathbf{z}}, \nabla_{\mathbf{z}} g \rangle = 2 \operatorname{Re} \{ \langle \Delta \mathbf{z}, \nabla_{\bar{\mathbf{z}}} g \rangle \}$$

where  $\langle \cdot, \cdot \rangle$  is the canonical *inner product* in  $\mathbb{C}^n$ , and the last equality holds because  $g$  is real-valued. Using the Cauchy-Schwarz inequality, it is easy to show that the first-order change in  $g$  will be maximized when  $\Delta \mathbf{z}$  and the cogradient  $\nabla_{\bar{\mathbf{z}}}$  are collinear. Hence, it is the gradient with respect to the conjugate of the variable that defines the direction of the maximum rate of change in the function with respect to  $\mathbf{z}$ , and not the ordinary gradient  $\nabla_{\mathbf{z}}$ .

Thus, the gradient optimization of  $g$  should use the update rule

$$\Delta \mathbf{z} = -\alpha \nabla_{\bar{\mathbf{z}}} g$$

as this form leads to a non-positive increment given by  $\Delta g = -2\alpha \|\nabla_{\bar{\mathbf{z}}} g\|^2$ , while the same rule but exploiting the other gradient would result in an update of  $\Delta g = -2\alpha \operatorname{Re} \{ \langle \nabla_{\bar{\mathbf{z}}} g, \nabla_{\mathbf{z}} g \rangle \}$ , which are not guaranteed to be non-positive.