On the Circularity of a Complex Random Variable

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***Abstract*—An important characteristic of a complex random variable z is the so-called circularity property or lack of it. We study the properties of the degree of circularity based on second-order moments, called circularity quotient, that is shown to possess an intuitive geometrical interpretation: the modulus and phase of its principal square-root are equal to the eccentricity and angle of orientation of the ellipse defined by the covariance matrix of the real and imaginary part of z. Hence, when the eccentricity approaches the minimum zero (ellipse is a circle), the circularity quotient vanishes; when the eccentricity approaches the maximum one, the circularity quotient lies on the unit complex circle. Connection with the correlation coefficient ρ is established and bounds on ρ given the circularity quotient (and vice versa) are derived. A generalized likelihood ratio test (GLRT) of circularity assuming complex normal sample is shown to be a function of the modulus of the circularity quotient with asymptotic χ^2_2 distribution.**

***Index Terms*—Circularity coefficient, complex random variable, correlation coefficient, eccentricity, EVD, noncircular random variable.**

I. Introduction

COMPLEX-VALUED (I/Q) signals play a central role in many application areas including communications and array signal processing. An important statistical characterization of a complex random variable (r.va.) is the so-called circularity property (or properness) or lack of it (noncircularity, nonproperness); see, e.g., [1]–[3]. Circular r.va. has vanishing pseudo-variance, namely, r.va. is statistically uncorrelated with its complex-conjugate. For example, M-QAM with $M = 4^k$ and 8-PSK modulated communications signals are circular, but some other commonly used modulation schemes (such as BPSK, AM, or PAM) lead to noncircular signals. Transceiver imperfections or interference from other signal sources may also lead to noncircular observed signals. Commonly, the additive sensor noise is modeled as circular complex Gaussian, but alternative (more flexible) models exist [4], [5]. The circularity/noncircularity property of the signals can be exploited in designing wireless transceivers or array processors such as beamformers, DOA algorithms, blind source separation methods, etc. See [2], [3], and [6]–[10] to cite only a few. Hence, statistical tests of circularity are also of great interest; see [5] and [11].

In this letter, the complex-valued measure of circularity based on second-order moments of a complex random variable $z = x+\jmath y$, called the circularity quotient ϱ_z, is studied. This measure has appeared with different names in the literature (cf. [3], [8], and [11]), but a detailed study of its properties is still lacking. We show that ϱ_z possesses an intuitive geometrical interpretation (Th. 1): its modulus $|\varrho_z|$ equals the squared eccentricity of the ellipse defined by the covariance matrix of $v = (x,y)^T$, while its argument (phase) $\arg[\varrho_z]$ is twice the orientation angle of the ellipse. The connection with the correlation coefficient $\rho = \mathrm{cor}(x,y)$ is established and bounds on ρ given ϱ_z (and vice versa) are derived (cf. Theorem 2, 3). Finally, a generalized likelihood ratio test assuming complex normal sample is shown to be a function of the modulus of the circularity quotient with asymptotic χ^2_2 distribution (chi-squared distribution with two degrees of freedom). Throughout, geometrical aspects are emphasized.

Notations: Symbol $|\cdot|$ denotes the modulus $|z| = \sqrt{zz^*}$, where $z^* = x - \jmath y$ is the *complex conjugate* of z and $\jmath = \sqrt{-1}$ the imaginary unit. Recall that any nonzero complex number has a unique (polar) representation, $z = |z|\exp(\jmath\theta)$, where $-\pi < \theta \leq \pi$ is called the *(principal) argument* of z and denoted by $\theta = \arg[z]$; if $z = 0$, then $\arg[z] = 0$ by convention. Then $\sqrt{z} \triangleq \sqrt{|z|}e^{\jmath\theta/2}$ is called the *principal square-root* of z. Let $\Omega = \{z \in \mathbb{C} : |z| \leq 1\}$ denote the *closed unit disk* and $\partial\Omega$ its boundary, the *unit circle*. Let $\Omega^+ = \{z \in \mathbb{C} : |z| < 1 \text{ and } \arg[z] \in (0,\pi)\}$ denote the *open unit upper half-disk* and $\Omega^- = \{z \in \mathbb{C} : |z| < 1 \text{ and } \arg[z] \in (-\pi, 0)\}$ the *open unit lower half-disk*. Sign function $\mathrm{sign}[\cdot]$ is defined as $\mathrm{sign}[x] = -1, 1, 0$ if $x < 0, > 0, = 0$.

II. Complex Random Variables: Preliminaries

Denote by $v \triangleq (x,y)^T$ the composite real random vector (r.v.) formed by stacking the real part $x = \mathrm{Re}[z]$ and imaginary part $y = \mathrm{Im}[z]$ of z. The distribution of z is identified with that of v, i.e., $F_z(a + jb) \triangleq P_v(x \leq a, y \leq b)$. Hence, the probability density function (p.d.f.) of z is identified with the p.d.f. $f(x,y)$ of v, so $f(z) \equiv f(x,y)$. The *mean* of z is defined as $E[z] = E[x] + \jmath E[y]$. For simplicity of presentation, we assume that $E[z] = 0$ (otherwise, replace z by $z - E[z]$). We assume that z is nondegenerate, i.e., z is not a constant equal to zero.

The most commonly made symmetry assumption in the statistical signal processing literature is that of *circular symmetry* [1]. Complex r.va. z is said to be *circular* if z has the same distribution as $e^{\jmath\theta}z, \forall\theta \in \mathbb{R}$. The p.d.f. then satisfies $f(z) = cg(|z|^2)$ for some nonnegative function $g(\cdot)$ and normalizing constant $c > 0$. Hence, the regions of constant contours are circles in the complex plane.

Denote the 2×2 real covariance matrix of the composite real r.v. v by

$$\Sigma \triangleq E\left[\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} x & y \end{pmatrix}\right] = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}. \tag{1}$$

The *variance* $\sigma_z^2 \equiv \mathrm{var}(z) > 0$ of a complex r.va. z

$$\sigma_z^2 \triangleq E\left[|z|^2\right] = \sigma_x^2 + \sigma_y^2 \tag{2}$$

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does not bear any information about the correlation between the real and the imaginary part of z, but this information can be retrieved from the *pseudo-variance* $\tau_z \equiv \text{pvar}(z) \in \mathbb{C}$ of z

$$\tau_z \triangleq E[z^2] = \sigma_x^2 - \sigma_y^2 + \jmath 2\sigma_{xy}.$$

Variance *together with* pseudo-variance carry all the second-order information since

$$\sigma_x^2 = \frac{\sigma_z^2 + \text{Re}[\tau_z]}{2}, \sigma_y^2 = \frac{\sigma_z^2 - \text{Re}[\tau_z]}{2}, \sigma_{xy} = \frac{\text{Im}[\tau_z]}{2}. \quad (3)$$

Circular r.va. z has the property that its pseudo-variance vanishes, $\tau_z = 0$ (i.e., $\sigma_x^2 = \sigma_y^2$ and $\sigma_{xy} = 0$), i.e., it is *proper*.

R.va. $z = x + \jmath y$ has zero-mean *circular complex normal distribution* if $v \sim N_2(0, \sigma^2 I)$, i.e., x and y are zero-mean independent identically distributed (i.i.d.) real normal variates with variance σ^2. Thus, $\sigma_z^2 = 2\sigma^2$, $\tau_z = 0$, and the p.d.f. $f(x, y|\sigma^2) \equiv f(z|\sigma_z^2) = (\pi\sigma_z^2)^{-1} e^{-|z|^2/\sigma_z^2}$. R.va. z is said to have *complex normal (CN) distribution* if $v \sim N_2(0, \Sigma)$, i.e., no structure on Σ is assumed. The bivariate normal density $f(x, y|\Sigma)$ can be written neatly in complex form [4] via σ_z^2 and τ_z. We shall write $z \sim CN(0, \sigma_z^2, \tau_z)$. Thus, circular CN distribution is a special case of CN distribution with $\tau_z = 0$.

Denote the eigenvalue decomposition (EVD) of the covariance matrix Σ of v by $\Sigma = E\Lambda E^T$, where $E = (e_1\ e_2)$ denotes the orthogonal matrix of eigenvectors of Σ and $\Lambda = \text{diag}(\lambda_1, \lambda_2)$ denotes the diagonal matrix of respective ordered eigenvalues, i.e., $\lambda_1 \geq \lambda_2 \geq 0$. To avoid the sign ambiguity of eigenvectors, we define the first (resp. second) eigenvector to have positive first coordinate (resp. positive second coordinate), i.e.

$$e_1 = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}, \quad e_2 = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix}, \quad \alpha \in [-\pi/2, \pi/2].$$

The triple $(\alpha, \lambda_1, \lambda_2)$ thus determines the EVD of Σ. If $\lambda_1 > \lambda_2$, then EVD is unique; if $\lambda_1 = \lambda_2$, i.e., $\sigma_x^2 = \sigma_y^2$ and $\sigma_{xy} = 0$, then α cannot be determined and is arbitrary. Variance and pseudo-variance can be linked with the EVD as is shown next.

Lemma 1: In terms of the EVD triple $(\alpha, \lambda_1, \lambda_2)$, we can express the variance as $\sigma_z^2 = \lambda_1 + \lambda_2$ and the pseudo-variance as $\tau_z = (\lambda_1 - \lambda_2)e^{\jmath 2\alpha}$, i.e., $|\tau_z| = \lambda_1 - \lambda_2$ and $\arg[\tau_z] = 2\alpha$.

Proof: Clearly, $\sigma_z^2 = \text{tr}[\Sigma] = \text{tr}[\Lambda] = \lambda_1 + \lambda_2$. From the EVD $\Sigma = E\Lambda E^T$, we obtain the identities $\sigma_x^2 = \lambda_1 \cos^2(\alpha) + \lambda_2 \sin^2(\alpha)$, $\sigma_y^2 = \lambda_1 \sin^2(\alpha) + \lambda_2 \cos^2(\alpha)$, and $\sigma_{xy} = (1/2)(\lambda_1 - \lambda_2)\sin(2\alpha)$, where we used that $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$. Hence, $\tau_z = \sigma_x^2 - \sigma_y^2 + \jmath 2\sigma_{xy} = (\lambda_1 - \lambda_2)\{\cos(2\alpha) + \jmath \sin(2\alpha)\}$, where we used that $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$. ■

III. Measure of Circularity

The *complex covariance* between complex r.va.'s z and w is defined as $\text{cov}(z, w) \triangleq E[zw^*]$. Thus, $\sigma_z^2 \equiv \text{cov}(z, z)$ and $\tau_z \equiv \text{cov}(z, z^*)$.

Definition 1: Circularity quotient $\varrho_z \equiv \text{cir}(z) \in \mathbb{C}$ of a r.va. z (with finite variance) is defined as the quotient between the pseudo-variance and the variance

$$\varrho_z \triangleq \frac{\text{cov}(z, z^*)}{\sqrt{\text{var}(z)}\sqrt{\text{var}(z^*)}} = \frac{\tau_z}{\sigma_z^2}.$$

Its (unique) polar representation $\varrho_z = r_z e^{\jmath\theta}$ induces quantities $r_z \triangleq |\varrho_z|$ called the *circularity coefficient* of z and $\theta \triangleq \arg[\varrho_z]$ called the *circularity angle* of z.

Note that ϱ_z can be described as a measure of correlation between z and z^*. The term circularity coefficient for r_z is coined from [3] while the terms noncircularity rate and noncircularity phase were used for r_z and θ, respectively, in [8]. Observe that $|\text{cir}(z)| = |\text{cir}(cz)|$ for all $c \in \mathbb{C} \setminus \{0\}$, meaning that circularity coefficient ϱ_z remains invariant under invertible linear transform. In fact, circularity coefficient is the canonical correlation between z and z^* [11].

For a positive definite Σ, define $\Delta(v) \triangleq v^T \Sigma^{-1} v$, and consider the ellipse (with center at the origin)

$$\mathcal{E}_\Sigma(c^2) \triangleq \{v \in \mathbb{R}^2 : \Delta(v) \leq c^2\} \quad (4)$$

that Σ defines, where the constant $c \in \mathbb{R}^+$ controls the size of the ellipse. Its major axis (resp. minor axis) has end points at $\pm c\sqrt{\lambda_1}e_1$ (resp. $\pm c\sqrt{\lambda_2}e_2$), and thus, α determines the *orientation of the ellipse*. If $v \sim N_2(0, \Sigma)$, then $Pr(v \in \mathcal{E}_\Sigma(\chi^2_{2,p})) = p$, where $\chi^2_{2,p}$ denotes the pth quantile of χ^2_2-distribution [12]. This means that if $z_1, \ldots, z_n$ is a random sample from $N(0, \sigma_z^2, \tau_z)$, then roughly 90% of the points in the complex plane will lie inside the ellipse $\mathcal{E}_\Sigma(\chi^2_{2,0.9})$.

The *eccentricity*

$$\varepsilon \triangleq \sqrt{\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}} \in [0, 1]$$

is a classical measure for the *shape of the ellipse*. A circle is a special case of an ellipse with $\lambda_1 = \lambda_2$ (i.e., $\sigma_x^2 = \sigma_y^2$ and $\sigma_{xy} = 0$) that has zero eccentricity, while as the ellipse becomes more elongated (i.e., when $\lambda_2/\lambda_1 \to 0$), the eccentricity approaches one. Note that the variance $\sigma_z^2 = \lambda_1 + \lambda_2$ measures *scale of the ellipse*. Alternatives scale measures are the geometric mean of the eigenvalues and the mean of the eigenvalues whose ratio

$$q \triangleq \frac{(\lambda_1 \lambda_2)^{1/2}}{\frac{1}{2}(\lambda_1 + \lambda_2)} \leq 1 \quad (5)$$

(with equality if and only if $\lambda_1 = \lambda_2$) can be related to eccentricity via $q = \sqrt{1 - \varepsilon^4}$. The next theorem provides a geometrical interpretation for the circularity quotient.

Theorem 1: In terms of α and ε, $r_z = |\varrho_z| = \varepsilon^2$ and $\theta = \arg[\varrho_z] = 2\alpha$. Hence, $\varrho_z \in \Omega$, i.e., the circularity quotient ϱ_z lies inside or on the unit circle, and $\varrho_z = \varepsilon^2 e^{\jmath 2\alpha} = (\varepsilon e^{\jmath\alpha})^2$.

Proof: Since $\theta = \arg[\varrho_z] = \arg[\tau_z]$, we have that $\theta = 2\alpha$ by Lemma 1. Since $|\tau_z| = \lambda_1 - \lambda_2$ and $\sigma_z^2 = \lambda_1 + \lambda_2$ (Lemma 1), we observe that $r_z = |\tau_z|/\sigma_z^2 = \varepsilon^2$. Note that $\varepsilon^2 = (\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2) \in [0, 1]$, and hence, $\varrho_z \in \Omega$. ■

Hence, eccentricity ε and orientation α of the ellipse can be calculated as $\varepsilon = \sqrt{r_z}$ and $\alpha = \arg[\varrho_z]/2$. Graphically, the shape and orientation of the ellipse is visualized by plotting $\sqrt{\varrho_z} = \varepsilon e^{\jmath\alpha}$ in the complex plane. The closer $\sqrt{\varrho_z}$ is to the unit circle the more elongated is the ellipse while the phase $\arg[\sqrt{\varrho_z}] = \alpha$ gives its orientation. Consider a random sample $z_1, \ldots, z_n$ from $CN(0, \sigma_z^2, \tau_z)$ with $\sigma_x^2 = \sigma_y^2 = 1$ (so $\rho = \sigma_{xy}$). This means that $(\sigma_z^2, \tau_z) = (2, \jmath 2\rho)$, $(\lambda_1, \lambda_2, \alpha) = (1 + |\rho|, 1 - |\rho|, \text{sign}[\rho]\pi/4)$, and $\varepsilon = \sqrt{|\rho|}$. Fig. 1 depicts such a sample of length $n = 100$ when $\rho = 0.8$. Also plotted is the ellipse $\mathcal{E}_\Sigma(\chi^2_{2,0.95})$. As we can see, approximately 95% of the points lie inside or on the ellipse. In the subplot (in the upper-left-hand corner), we have plotted $\sqrt{\hat{\varrho}_z}$, i.e., the square root of the *sample circularity quotient* $\hat{\varrho}_z = \hat{\tau}_z/\hat{\sigma}_z^2 = \sum_{i=1}^n z_i^2 / \sum_{i=1}^n |z_i|^2$ with $\hat{\varepsilon} = |\sqrt{\hat{\varrho}_z}| = 0.9106$ and $\hat{\alpha} = \arg[\sqrt{\hat{\varrho}_z}] = 0.7926$ being the ML-estimates of the eccentricity $\varepsilon \approx 0.894$ and orientation $\alpha = \pi/4 \approx 0.785$ of the ellipse.

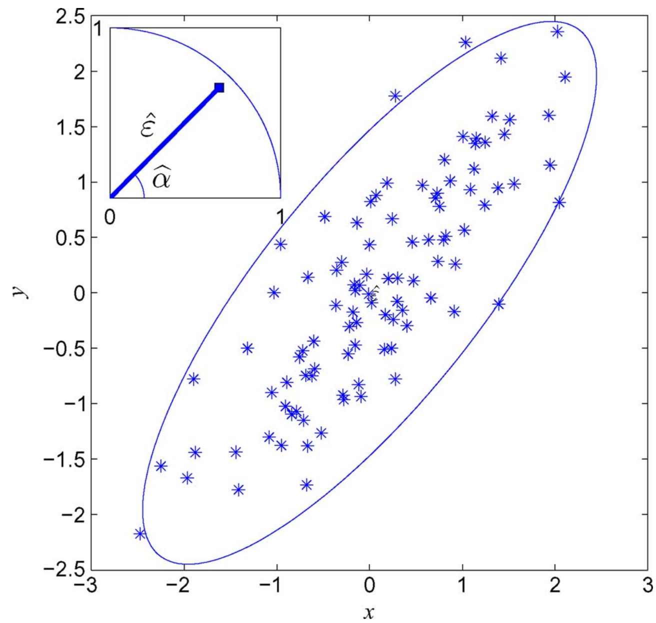

Fig. 1. Random sample of length $n = 100$ from $CN(0, 2, \jmath 2\rho)$ with $\rho = 0.8$. The EVD triple is $(\lambda_1, \lambda_2, \alpha) = (1.8, 0.2, \pi/4)$ and the ellipse $\mathcal{E}_\Sigma(\chi^2_{2,0.95})$ is shown with solid line. Subplot in the upper-left-hand corner depicts $(\hat{\varrho}_z)^{1/2} = \hat{\varepsilon}e^{\jmath\hat{\alpha}}$.

Next we link the circularity quotient of $z = x + \jmath y$ with the *correlation coefficient*

$$\rho \equiv \mathrm{cor}(x, y) \triangleq \frac{\mathrm{cov}(x, y)}{\sqrt{\mathrm{var}(x)}\sqrt{\mathrm{var}(y)}} \equiv \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

where finite nonzero variances are assumed. Recall that $|\rho| \leq 1$ with equality if and only if x is a linear function of y. Also note that there are two possible sources of noncircularity: x and y have unequal variances, and/or x and y are correlated.

Theorem 2: Circularity quotient ϱ_z of a complex r.va. $z = x + \jmath y$ satisfies:

(a) $\varrho_z = 0 \Leftrightarrow \varepsilon = 0 \Leftrightarrow x$ and y are uncorrelated ($\rho = 0$) with equal variances $\sigma_x^2 = \sigma_y^2$.

(b) $\varrho_z = 1 \Leftrightarrow y$ is equal to zero, and $\varrho_z = -1 \Leftrightarrow$ if x is equal to zero (ρ does not exist). Furthermore, $\varrho_z \in \partial\Omega \setminus \{\pm 1\} \Leftrightarrow x$ is a linear function of $y \Leftrightarrow \rho = \pm 1$.

(c) For $0 < r_z < 1$: $\varrho_z = \pm r_z \Leftrightarrow \rho = 0 \Leftrightarrow \theta = 0$, or $\theta = \pi$.

(d) For $0 < r_z \leq 1$: $\varrho_z = \pm \jmath r_z \Leftrightarrow \theta = \pm\pi/2 \Leftrightarrow \sigma_x^2 = \sigma_y^2 \Rightarrow \rho = \sigma_{xy}/\sigma_x^2 = \pm r_z$.

(e) $\varrho_z \in \Omega^+ \Leftrightarrow 0 < \rho < 1$, and $\varrho_z \in \Omega^- \Leftrightarrow -1 < \rho < 0$.

Proof:

(a) Note that $\varrho_z = 0 \Leftrightarrow r_z = 0$ which in turn by Theorem 1 holds if and only if $\varepsilon^2 = 0 \Leftrightarrow \lambda_1 = \lambda_2 \Leftrightarrow \Sigma = \sigma^2 I \Leftrightarrow \sigma_{xy} = 0$ and $\sigma^2 = \sigma_x^2 = \sigma_y^2$.

(b) Observe that $\varrho_z = 1 + \jmath 0 \Leftrightarrow \sigma_x^2 - \sigma_y^2 = \sigma_x^2 + \sigma_y^2$ and $\sigma_{xy} = 0 \Leftrightarrow \sigma_y^2 = 0$ (i.e., $y = 0$ with probability one). Similarly, $\varrho_z = -1 + \jmath 0 \Leftrightarrow \sigma_x^2 = 0$ (i.e., $x = 0$ w.p. 1). More generally, $\varrho_z \in \partial\Omega \Leftrightarrow r_z = 1$ which in turn by Theorem 1 holds if and only if $\varepsilon^2 = 1 \Leftrightarrow \lambda_2 = 0 \Leftrightarrow x = 0$ or $y = 0$ (w.p. 1), or $x = cy$ for some $c \in \mathbb{R} \setminus \{0\}$. Thus, $\varrho_z \in \partial\Omega \setminus \{\pm 1\} \Leftrightarrow x = cy$ for some $c \in \mathbb{R} \setminus \{0\}$ (i.e., $\rho = \pm 1$).

(c) For $0 < r_z < 1$: $\varrho_z = r_z\{\cos(\theta) + \jmath\sin(\theta)\} = \pm r_z \Leftrightarrow \theta \in \{0, \pi\} \Leftrightarrow \mathrm{Im}[\varrho_z] = 2\sigma_{xy}/\sigma_z^2 = 0 \Leftrightarrow \rho = 0$.

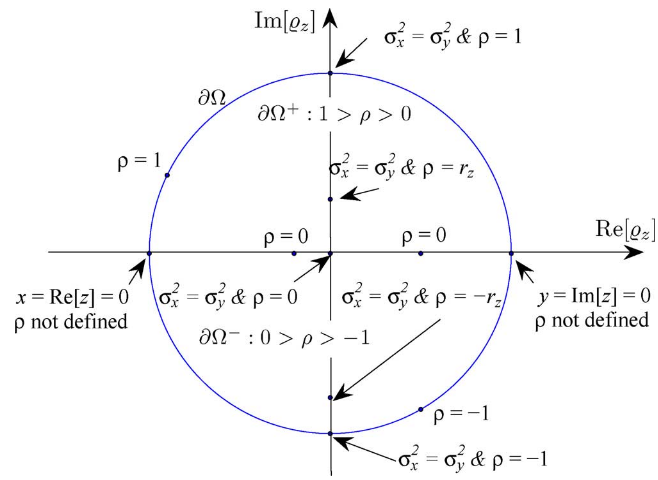

Fig. 2. Pictorial presentation of Theorem 2 with some exemplary points of ϱ_z. Recall that $\varrho_z \in \Omega$, i.e., ϱ_z lies inside or on the unit circle $\partial\Omega$.

(d) For $0 < r_z \leq 1$: $\varrho_z = r_z\{\cos(\theta) + \jmath\sin(\theta)\} = \pm\jmath r_z \Leftrightarrow \theta = \pm(\pi/2) \Leftrightarrow \mathrm{Re}[\varrho_z] = 0 \Leftrightarrow \sigma_x^2 = \sigma_y^2$. If $\sigma_x^2 = \sigma_y^2$, then $\rho = \sigma_{xy}/\sigma_x^2$ and $\mathrm{Im}[\varrho_z] = 2\sigma_{xy}/(\sigma_x^2 + \sigma_y^2) = \sigma_{xy}/\sigma_x^2$, i.e., $\rho = \mathrm{Im}[\varrho_z] = \pm r_z$.

(e) Observe that $\varrho_z \in \Omega^+$ if $\mathrm{Im}[\varrho_z] > 0$ (i.e., $\theta \in (0, \pi)$) and $r_z \neq 1$. Note that $\mathrm{Im}[\varrho_z] > 0 \Leftrightarrow \sigma_{xy} > 0 \Leftrightarrow \rho > 0$. The fact that $r_z \neq 1$ shows by (b)-part of the Theorem that $\rho < 1$. Proof for the case $\varrho_z \in \Omega^-$ proceeds similarly. ∎

Fig. 2 summarizes the findings of Theorem 2. In general, a scatter plot of r.va.'s distributed as z with $r_z = 1$ (resp. $r_z = 0$) looks the "least circular" (resp. "most circular") in the complex plane. Note that $r_z = 1$ (i.e., $\varrho_z \in \partial\Omega$) if z is purely real-valued such as BPSK modulated communication signal, or if the signal lies on a line in the scatter plot (also called constellation or I/Q diagram) as is the case for BPSK, ASK, AM, or PAM-modulated communications signals.

The next theorem shows the explicit connection between ϱ_z and ρ and derives simple bounds on them.

Theorem 3: Assume that ρ exists (i.e., x and y are nondegenerate with finite variances).

(a) Connection between ρ and circularity quotient $\varrho_z = r_z e^{\jmath\theta}$ of $z = x + \jmath y$ is

$$\begin{aligned}\rho &= \frac{\mathrm{Im}[\varrho_z]}{\sqrt{1 - \mathrm{Re}[\varrho_z]^2}} \\ &= \frac{r_z \sin(\theta)}{\sqrt{1 - r_z^2\cos^2(\theta)}} \\ &= \mathrm{sign}[\rho]\sqrt{\tan(\theta')\tan(\theta'')}\end{aligned}$$

where $\tan(\theta') = \mathrm{Im}[\varrho_z]/(1 - \mathrm{Re}[\varrho_z])$ and $\tan(\theta'') = \mathrm{Im}[\varrho_z]/(1 + \mathrm{Re}[\varrho_z])$.

(b) Assume that $\rho \neq 0$. Then $\mathrm{sign}[\theta] = \mathrm{sign}[\rho]$ and $\rho \leq r_z\mathrm{sign}[\rho]$ with equality if and only if $\rho = \pm 1$ (i.e., $\varrho_z \in \partial\Omega \setminus \{\pm 1\}$) or $\theta = \pm\pi/2$ (i.e., $\sigma_x^2 = \sigma_y^2$).

Proof: First we note that the assumption that ρ exists implies that $\varrho_z \neq \pm 1$ by Theorem 2(b).

(a) Using (3), we get

$$\sigma_x\sigma_y = \frac{1}{2}\sqrt{\sigma_z^4 - \mathrm{Re}[\tau_z]^2} = \frac{1}{2}\sigma_z^2\sqrt{1 - \mathrm{Re}\left[\tau_z/\sigma_z^2\right]^2}.$$

Since $\varrho_z = \tau_z/\sigma_z^2$ and $\sigma_{xy} = \mathrm{Im}[\tau_z]/2 = \sigma_z^2\mathrm{Im}[\varrho_z]/2$, we write the first stated form for $\rho = \sigma_{xy}/\sigma_x\sigma_y$. The

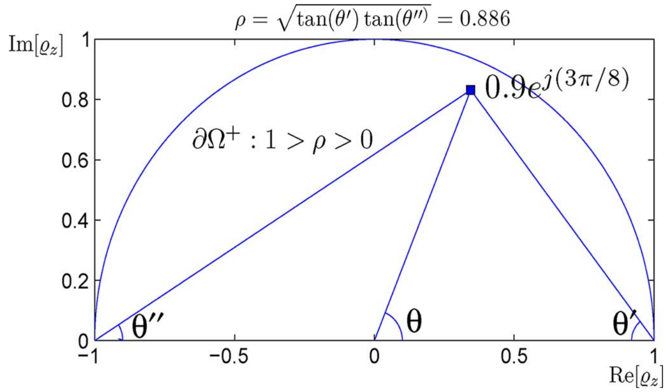

Fig. 3. Graphical illustration of the relation of ρ with $\varrho_z = r_z e^{j\theta}$ given by Theorem 3. In the example, $\varrho_z = 0.9e^{j(3\pi/8)}$, i.e., $r_z = 0.9$ and $\theta = 3\pi/8$.

second form follows since $\varrho_z = r_z e^{j\theta}$, i.e., $\mathrm{Re}[\varrho_z] = r_z\cos(\theta)$ and $\mathrm{Im}[\varrho_z] = r_z\sin(\theta)$. We can write the first form as

$$\rho = \mathrm{sign}\,\{\mathrm{Im}[\varrho_z]\}\sqrt{\frac{\mathrm{Im}[\varrho_z]}{1-\mathrm{Re}[\varrho_z]}\frac{\mathrm{Im}[\varrho_z]}{1+\mathrm{Re}[\varrho_z]}}.$$

This gives the last form for ρ since $\mathrm{sign}\{\mathrm{Im}[\varrho_z]\} = \mathrm{sign}[\sigma_{xy}] = \mathrm{sign}[\rho]$.

(b) Note that $\rho \neq 0 \Leftrightarrow \mathrm{Im}[\varrho_z] = r_z\sin(\theta) \neq 0 \Leftrightarrow \theta \neq 0, \pi$ and $r_z \neq 0$ (i.e., $\varrho_z \neq 0$). Note that $1 - r_z^2\cos^2(\theta) \geq \sin^2(\theta)$ since $r_z \in (0,1]$. This together with the second form for ρ indicate that $\rho \leq r_z\mathrm{sign}[\sin(\theta)]$. Then note that $\mathrm{sign}[\sin(\theta)] = \mathrm{sign}[\theta] = \mathrm{sign}[\rho]$ since $\theta \neq 0, \pi$. Hence, the equality $\rho = r_z\mathrm{sign}[\rho]$ is obtained if and only if $|\rho| = r_z$. Based on the second form for ρ, equality is obtained if and only if $|\sin(\theta)| = \sqrt{1 - r_z^2\cos^2(\theta)}$ which holds true if and only if $\varrho_z \in \partial\Omega \setminus \{\pm 1\}$ [i.e., $\rho = \pm 1$ by Theorem 2(b)], or $\theta = \pm\pi/2$ [i.e., $\sigma_x^2 = \sigma_y^2$ by Theorem 2(d)]. ∎

Fig. 3 elucidates the relationship of ρ with ϱ_z as stated in Theorem 3. In general, the larger the triangle formed by connecting the end points $1 + \jmath 0$ and $-1 + \jmath 0$ of the diameter of the circle with the point ϱ_z, the larger is $|\rho|$. Since $r_z = \varepsilon^2$ and $\theta = 2\alpha$ (cf. Theorem 1), the bound on ρ can also be written as $\rho \leq \mathrm{sign}[\alpha]\varepsilon^2$, that is, ρ is always smaller than the squared eccentricity multiplied by the sign of the orientation of the ellipse. The bound on ρ and the fact that $0 \leq r_z \leq 1$ provide the following upper and lower bounds for the circularity coefficient r_z: $\rho \leq r_z \leq 1$ when $\rho > 0$ and $0 < r_z \leq |\rho|$ when $\rho < 0$. See also [13] for bounds in the vector case. The bounds in [13] are however useful only in the vector case since the assumption about the knowledge of the eigenvalues λ_1 and λ_2 (and hence of ε) of the covariance matrix Σ, provide *exact* knowledge of r_z in the scalar case as $r_z = \varepsilon^2$ by Theorem 1.

IV. Generalized Likelihood Ratio Test (GLRT) of Circularity

Statistical hypothesis test of circularity of the sample $\{z_i = x_i + \jmath y_i, i = 1,\ldots,n\}$ is equivalent with the test of sphericity of the composite sample $\{v_i = (x_i, y_i)^T\}$. Hence, a test of sphericity of the composite sample is also a test of circularity. Naturally, this holds only for samples in $\mathbb{C}$. If $z_1,\ldots,z_n$ is a random sample from $CN(0,\sigma_z^2,\tau_z)$, i.e., $v_1,\ldots,v_n$ is a random sample from $N_2(0,\Sigma)$, then the GRLT decision statistic for testing $H_0 : \tau_z = 0$ (i.e., $\Sigma = \sigma^2 I$) against a general alternative $H_1 : \tau_z \neq 0$ $(\Sigma \neq \sigma^2 I)$ is [12]

$$l_n \triangleq \frac{\max_{\sigma^2}\prod_{i=1}^n f(x_i,y_i|\sigma^2 I)}{\max_{\Sigma}\prod_{i=1}^n f(x_i,y_i|\Sigma)} = \left[\frac{|\hat{\Sigma}|^{1/2}}{\frac{1}{2}\mathrm{tr}[\hat{\Sigma}]}\right]^n = \hat{q}^n$$

where $\hat{\Sigma} = (1/n)\sum_{i=1}^n v_i v_i^T$ is the sample covariance matrix and $\hat{q}$ is the sample version of (5), i.e., the ratio of the geometric mean and the mean of the eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of $\hat{\Sigma}$. Furthermore, if H_0 is true, then $-2n\log\hat{q} \to \chi_2^2$ in distribution [12]. Since $\hat{q} = \sqrt{1-\hat{\varepsilon^4}}$ and $\hat{r}_z = |\hat{\varrho}_z| = \hat{\varepsilon}^2$ by Theorem 1, we have the following result.

Theorem 4: $l_n = (1-\hat{r}_z^2)^{n/2}$ and, under H_0, $-n\log(1-\hat{r}_z^2) \to \chi_2^2$ in distribution.

The test that rejects H_0 whenever $-n\ln(1-\hat{r}_z^2)$ exceeds the quantile $\chi^2_{2,1-p}$ is thus GLRT with asymptotic level p (e.g., $p = 0.05$). See also [5] and[11] for GLRT of circularity in k-variate $(k \geq 1)$ case. However, the asymptotic distribution of GLRT was not derived in [5] and [11].

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