Non-Riemannian effects in non-perturbative quantum gravity

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Some notes, which we hope to grow into a paper through watering and sunlight.

Introduction. –

Models to try. – Presumably you will want to replace this with your findings from some weeks previous. A fairly general model to consider would be

$$\begin{aligned} L_{\text{G}} = &-\frac{1}{2}\phi^2\mathcal{R} + \frac{1}{2}\nu\mathcal{D}^*{}_i\phi\mathcal{D}^{*i}\phi - \lambda\phi^4 \\ &+ \frac{1}{2}\xi\mathcal{H}_{ij}\mathcal{H}^{ij} + \phi^2\Big(\beta_1\mathcal{T}^*{}_{ijk}\mathcal{T}^{*ijk} \\ &+ \beta_2\mathcal{T}^*{}_{ijk}\mathcal{T}^{*jik} + \beta_3\mathcal{T}^*{}_i\mathcal{T}^{*i}\Big), \end{aligned} \tag{1}$$

where $\mathcal{R}^i{}_{jkl}$ is the Riemann-Cartan field strength, $\mathcal{H}^{ij}$ is the Maxwell strength for the Weyl gauge field $\mathcal{B}^i$, the dilaton or compensator is ϕ, and $\mathcal{T}^*{}_{ijk}$ is the Weyl torsion. This model has the advantage of Weyl gauge invariance, and it is 'conservative' in the sense that it connects continuously to the Einstein–Cartan model. The fields ϕ and $\mathcal{B}^i$ are allowed to be dynamical. In the Einstein gauge, we would presumably get free Einstein–Cartan theory with a Proca-like Weyl field. Otherwise, the compensator has a mass induced by the torsion, but the torsion is not yet dynamical and itself enters by a collection of pure mass terms – we would probably want to change that as below.

Since a large portion of the available structure is represented within (1), it would be good to see how that whole theory looks when it is separated into second-order formalism, and reduced to two dimensions. The key question to ask is then how Weyl invariance relates to conformal invariance (if at all), and whether the signature matters.

In order to have dynamical torsion, we would expect to need quadratic curvature terms. To this end, it would be good to have a full understanding of the structure of $\mathcal{R}^i{}_{jkl}$, separated in two Euclidean dimensions. Probably it is better to do this before considering quadratic invariants and their degeneracies.

What about tuned scale invariance? It is possible that we could obtain this using *quartic* torsion invariants whose couplings are tuned against the Einstein constant, and thereby not need a Weyl gauge theory or compensator at all.

In summary there are two things to get done at this point:

1. Concise statement of the general Weyl and/or Poincaré gauge theory in two Euclidean dimensions, at least encompassing (1) but if possible also up to *quadratic* Riemann-Cartan invariants and *quartic* torsion invariants, including an understanding of how many free parameters we have at our disposal in lower dimensions.
2. Understanding of how Weyl invariance carries over to two Euclidean dimensions, and any relation to conformal invariance, or whether we would absolutely need tuned (Ising-like) couplings.

Once again, the short-term objective is to obtain a 2D, Euclidean theory which looks like Einstein gravity coupled to a CFT, and which descends from a reasonably conservative but non-Riemannian model of pure gravity coupled to nothing at all.

Riemann–Cartan setup. – Following your finding that pure non-Riemannian extensions do not behave like interesting CFTs in lower dimensional models, the desire to connect with Liouville theory suggests that matter coupling might be the best route forwards.

By now you will have verified $c = 1/2$ for the free fermion models, or at least have shown that there is a key problem with our assumptions in setting these models up on the lattice (or perhaps you do not measure the flat-space c for reasons to do with the gravitational dressing. Something like this happens with Onsager I believe, so let me know what you found and we can try to compare with literature).

The next step before going over into CDT will be to see whether non-perturbative lattice methods can be used to 'detect' the effective interactions that emerge in one of the most conservative non-Riemannian models: the Einstein–Cartan model in four dimensions. For decades, people have speculated that such interactions could remove the various singularities that blight GR, but in four dimensions these higher vertices are suppressed by the Planck scale and nobody has ever detected them. The two dimensional lattice is the closest we will be able to come in the short term of the next week or so!

So now I'll sketch out what I think is the minimal model (minimal as in minimal nontrivial, not a finite collection of Virsasoro irreps) of the Euclidean system on the 2-sphere, since it connects with work I am doing now in four dimensions with some very cool papers Claire's supervisor produced, and which I mentioned back when we did the last 'seminar' (need to get on top of these agian..). You should check yourself that you are happy with the setup and precise couplings, since these will be vital to the cancellation if it can be made to work.

First I think we need to be happy embracing the second-order geometric interpretation of Einstein–Cartan theory. It seems sensible to go over from the gauge picture in the previous section to the geometric one, since triangulations are attempting to describe a curved spacetime. In a curved and torsionful spacetime of general dimenison we say that there is a tetrad

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and spin structure

$$\nabla\!\!\!/\Psi \equiv \gamma^i e_i^{\ \mu}\nabla_\mu\Psi, \tag{2}$$

so that Ψ is (perhaps) a spinor and the covariant derivative is

$$\nabla_\mu\Psi \equiv \left(\partial_\mu + \frac{1}{2}\omega^{ij}_{\ \ \mu}\Sigma_{ij}\right)\Psi, \tag{3}$$

i.e. spinorial indices are implicit. Now we can also break the spin connection down into the torsionful and torsionless parts

$$\omega^{ij}_{\ \ \mu} \equiv \Delta^{ij}_{\ \ \mu} + K^{ij}_{\ \ \mu}, \tag{4}$$

where the Ricci rotation coefficients and contorsion tensor are defined as in [1]. So we can also think of the curved spacetime described by triangulations as using the covariant derivative

$$\mathring{\nabla\!\!\!/}\Psi \equiv \gamma^i e_i^{\ \mu}\mathring{\nabla}_\mu\Psi, \tag{5}$$

where the torsion is removed

$$\mathring{\nabla}_\mu\Psi \equiv \left(\partial_\mu + \frac{1}{2}\Delta^{ij}_{\ \ \mu}\Sigma_{ij}\right)\Psi. \tag{6}$$

Aside from the matter fields, the gravitational action will derive from the Riemann–Cartan scalar $R \equiv R^\lambda_{\ \lambda}$ and $R_{\mu\nu} \equiv R^\lambda_{\ \mu\lambda\nu}$ for $R_{\mu\nu\lambda}^{\ \ \ \ \sigma} \equiv 2\partial_{[\nu}\Gamma^\sigma_{\mu]\lambda} + 2\Gamma^\sigma_{[\nu|\kappa}\Gamma^\kappa_{|\mu]\lambda}$, and analogously for the Riemann tensor $\mathring{R}_{\mu\nu\lambda}^{\ \ \ \ \sigma}$ in terms of $\mathring{\Gamma}^\sigma_{\mu\nu}$. The connections are respectively torsonful $\Gamma^\mu_{\lambda\nu}$ and Levi–Civita $\mathring{\Gamma}^\mu_{\lambda\nu}$, according to $\Gamma^\mu_{\lambda\nu} \equiv \mathring{\Gamma}^\mu_{\lambda\nu} - \frac{1}{2}\left(T^\mu_{\ \lambda\nu} - T_{n\ \ \lambda}^{\ \mu} + T_{\lambda\nu}^{\ \ \ \mu}\right)$. Here $\mathring{\Gamma}^\mu_{\lambda\nu}$ derives from $\Delta^{ij}_{\ \ \mu}$ with an extra tetrad gradient, and the torsion terms are equal to the contorsion with tetrads used to change indices, all according to [1] again.

Two Euclidean dimensions. – Now we need to connect with the physics of your existing triangulation codes. We first take the matter sector. The two dimensional gamma matrices are given by the Pauli matrices $\gamma^i \equiv \sigma_i$, where the third Pauli matrix acts chirally. We can label the time and space coordinates $x^1 = x$ and $x^2 = y$, with z and $\bar{z}$ defined as usual. The Majorana–Weyl components of the spinor are

$$\Psi \equiv \begin{pmatrix}\psi \\ \bar{\psi}\end{pmatrix}. \tag{7}$$

We take the string theory normalisation of the free fermion matter Lagrangian in a curved spacetime with torsion

$$L_\text{M} = \frac{1}{4\pi}\bar{\Psi}\nabla\!\!\!/\Psi. \tag{8}$$

Now for the gravitational sector. In two dimensions the torsion derives from a vector, regardless of the signature of the spacetime. The particular configuration I'd be inclined to use would be that of Katanaev, since you were using it the other week for your computations, whereby the spin connection is equated quite naturally with the SO(2) gauge boson $\omega^{ij}_{\ \ \mu} \equiv \epsilon^{ij}A_\mu$, where in holonomic indices $\epsilon_{\nu\sigma} \equiv \sqrt{-g}\varepsilon_{\nu\sigma}$ is the volume form. Something that would be quite interesting to understand is how this setup is distinct from the U(1) $\simeq$ SO(2) theory: it is nice that the Lorentz group becomes compact as you go down into the two-sphere, and the implications for Yang–Mills gravity could be quite interesting. What if you could recover Euclidean electrodynamics as a complexification of torsion? Does that idea hold water? Perhaps you would like to investigate. Moving on with Katanaev, I imagine this would translate to

$$T^\mu_{\ \nu\sigma} \equiv g_{\mu\nu}T_\sigma - g_{\mu\sigma}T_\nu, \tag{9}$$

where we can work with some field T_ν. We take for the gravitational lagrangian the usual Einstein–Hilbert term. Something that never quite sat right with me was the result from the end of the day on Friday 22nd that the Einstein–Cartan action differed from the Riemannian counterpart by a nontrivial term, i.e. the torsion 'mass' $T_\mu T^\mu$. As a general rule, these topological identities don't care much about the connection (I guess the concept of a 'curvature tensor' is more fundamental than it seems). The connection can invoke new invariants (e.g. the Nieh–Yan number), but the old ones are usually preserved (e.g. the Gauss-Bonnet term), and the dimensionality restrictions are preserved, too[1].

Indeed, a check of that result we had immediately gives

$$\begin{aligned} R &\equiv \mathring{R} + \frac{1}{4}T_{\mu\nu\sigma}T^{\mu\nu\sigma} + \frac{1}{2}T_{\mu\nu\sigma}T^{\nu\mu\sigma} \\ &\quad - T^\mu_{\ \mu\sigma}T^{\nu\ \ \sigma}_{\ \nu} - 2\mathring{\nabla}_\mu T^{\nu\ \ \mu}_{\ \nu} \\ &= \mathring{R} - 2\mathring{\nabla}_\mu T^\mu. \end{aligned} \tag{10}$$

So, this would seem to be unsuitable classically. If that is right (please check!) it seems the extended action in (1) will be needed to some extent after all, so minimally let's try bundling the $\{\beta_i\}$ into some new parameter μ

$$L_\text{G} = R + \mu T^\mu_{\ \mu\sigma}T^{\nu\ \ \sigma}_{\ \nu}. \tag{11}$$

Hopefully this effective Proca roton mass (though dimensionless) in (11) will give us something similar to the higher-dimensional contact interaction. It is at least consistent with the Wilsonian expansion, and puts us back on the higher-order operator track that feels more natural if you imagine these things as gauge theories.

Something else for you to check is whether the total torsion divergence in (10) will have an effect on the quantum fluctuations. After all, the Riemannian Einstein–Hilbert term is nontrivial on a lattice (higer-dimensional broccoli ;)), so please do explore the effect of this!

Effective theory. – Now I'd be interested to determine whether the torsion contact interaction survives in two Euclieann dimensions. Combining Eqs. (8) and (11) I seem to get up to surface

[1] there might be exceptions to this that I don't know about, this is mostly a gut feeling. However, I did hear a very interesting conversation with Cumrun Vafa in recent days where he was talking about the classification of C–Y manifolds based on some characteristics, and this must be independent of the contents of the manifold – maybe we could circle back to the idea if you are interested.

terms

$$L_{\mathrm{T}} = \mathring{R} + \mu T^{\mu} T_{\mu} + \frac{1}{4\pi} \bar{\Psi} \mathring{\not{\nabla}} \Psi + \frac{1}{8\pi} T^{\mu} e_{i}{}^{\mu} \bar{\Psi} \gamma^{i} \Psi. \quad (12)$$

As expected, the spin tensor will then just be the vector current. The algebraic equation for torsion then looks like it becomes

$$T^{\mu} = \frac{1}{16\pi\mu} e_{i}{}^{\mu} \bar{\Psi} \gamma^{i} \Psi, \quad (13)$$

and so we can in principle substitute this back into (12) to give an effective action in $e_{i}{}^{\mu}$, or equally in $g^{\mu\nu} \equiv \eta^{ij} e_{i}{}^{\mu} e_{j}{}^{\nu}$. Following this line of thinking, I'd be more interested to discover what happens when you make simple changes to your current code to *directly* sample from the gauge-link theory

$$L_{\mathrm{T}} = \mathring{R} + \mu T^{\mu} T_{\mu} + \frac{1}{4\pi} \bar{\Psi} \not{\nabla} \Psi + \alpha \left(\bar{\Psi} \gamma_{i} \Psi \right) \left(\bar{\Psi} \gamma^{i} \Psi \right). \quad (14)$$

Do we, for example, recover in the limit $\alpha \mapsto 1/(256\pi^{2}\mu)$ the critical behaviour expected of the theory

$$L_{\mathrm{T}} = \mathring{R} + \frac{1}{4\pi} \bar{\Psi} \mathring{\not{\nabla}} \Psi, \quad (15)$$

i.e. the theory you have already studied at the beginning of this week? (Again, assuming there is nothing fundamentally wrong with putting fermions on the lattice in this way.) Would be good to verify that the reasoning above is watertight, and that the numerical coefficients match with your own results. Some other things to consider during this process: since α is dimensionless, could it be that the effective theory, or something like it, is *also* a CFT? I would be inclined to explore this in complexified coordinates, should be a snappy calculation. If not, are there other fermionic theories which, when torsion is integrated out, are automatically CFTs (perhaps deploy the γ^{5} analogue to explore this)? Very curious to explore any other possibilities in this space as you find them!

EDT implementation. – Perhaps some beautiful `TikZ`?

Other fermion models. – Are there more options in lower dimensions?

CDT implementation. – More beautiful `TikZ`?

Novel CDT pheomena. – This section to attract citations.

Concluding remarks. –

[1] M. Blagojević, *Gravitation and Gauge Symmetries*, Series in high energy physics, cosmology, and gravitation (Institute of Physics Publishing, Bristol, UK, 2002).