UNIVERSITY OF TRENTO

Department of Industrial Engineering

Master Degree in Mechatronics Engineering

Influence of Slip Versus Torque Control Formulation on Minimum Time Manoeuvres of a Motorcycle

Academic Year 2018/2019

Supervisor:
Prof. Francesco Biral.

Graduant:
Mattia Piazza

17^{th} march 2019

Acknowledgements

I would first like to thank my thesis advisor Professor Francesco Biral for his dedication and the huge amount of time and the late evenings that he spent to assist my work.
I would also like to thank Professor Enrico Bertolazzi, his office door was always open whenever I ran into a trouble spot or had a question about optimal control or about PINS.
I must express my very profound gratitude to my parents for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.
Most importantly, I wish to thank my loving and supportive girlfriend, Sara. She provides me unending inspiration and never let me down.
Finally, I would like to thanks all my friends and flatmates for all the fun and support given in the past years.

Mattia Piazza

Contents

List of Figures

List of Tables

Introduction

The simulation of mechanical and mechatronics systems allow to test and validate the design in a safe and efficient environment without the need to build the physical object and measure the parameters. The simulation is based on digital technology with major benefits as cost and efficiency and the possibility of an easy reconfiguration and retesting of a system which is usually impossible or infeasible for real model in terms of cost and time. [1]

A large number of vehicle model are available in the scientific literature with different levels of complexity. Depending on the application a proper model must be chosen. Simple models are faster and therefore suitable for real-time purposes while complex models are time-consuming and are used in the case where the model cannot be simplified or the goal of the study is to replicate in detail the behaviour of the analysed system.

In literature a lot of studies concerning optimal control and in particular minimum lap time problems. However, most of them concern only four wheels vehicles. For some reasons, motorcycle manoeuvre had a minor interest in the research. This is due to the fact that the motorcycle model is highly non-linear and computational demanding to solve. Moreover, it is not always possible to reduce the system of dynamic equation in explicit form and most of the optimal control solver cannot deal with implicit forms.

Most of the optimal control problems for motorcycles in literature are solved using relatively simple dynamics and direct methods [2, 3] or using the assumption of quasi-steady-state behaviour. [4, 5]

In some papers, the problem is solved using indirect approach [6].

As far as the author knows the publication of Leonelli and Limber [3] is the only one that takes into account tyre ground interaction while having a complicated dynamic model. However, it is not specified which version of the Magic formula is being used, moreover, the motorcycle suspensions are considered as fixed.

The work of this thesis aims to derive different models of a racing motorcycle with increasing complexity taking into account the force exchange between tyre and ground using Pacejka's magic formula [7]. The dynamic model has eleven degrees of freedom considering both moving and fixed suspensions and gyroscopic effect coming from wheels acceleration.

The models will then be used to calculate controls to achieve minimum time manoeuvre in different scenarios controlling the motorcycle with longitudinal slip or torque.

All the models are derived with the multi-body approach and symbolic formulation. To achieve this goal, the model is defined using Maple, a software well known for its capability in symbolic computation. Moreover, the equations of motion are obtained using MBSymba which is a custom library for Maple available online at http://www.multibody.net.

All three models are derived in a similar way as in the publication of Cossalter *et al* [8], without taking into account the lateral flexibility of the front torque and the torsional flexibility of the swingarm. The tyre forces are derived using the Pacejka's magic formula [7] which is an empirical formula obtained from the assumption of similarity.

The minimum time trajectory and control is computed formulating a custom optimal control problem using XOptima package for Maple and then solving with PINS (the acronym for *PINS Is Not a Solver*). The first is a library developed to transform the symbolic model of the vehicle (DAE), constraints, and target functions in C++ code that can be used by PINS. PINS is a software, free for academic purposes, developed at the University of Trento by Prof. Bertolazzi, Prof. Biral and Prof. Bosetti that can solve optimal control problems (OCPs) with the indirect method. As far as the author knows, there are not other optimal control solvers that exploit Pontryagin maximum principle and calculus of variations to solve the problem with the indirect method. [6]

The thesis is organized in the following way. In chapter 1, there is a brief overview of the state of the art in optimal control techniques and motorcycle dynamic model for minimum time application. Chapter 2 describe the derivation of kinematic and dynamic model of the motorcycle. In Chapter 3 tyre ground interaction is modelled following the magic formula of H. Pacejka [9] reporting formulas and tyre data used. Chapter 4 report static condition solution and steady-state derived for the motorcycle. Those results are then used in chapter 5 as a suitable initial condition to solve the optimal control problems. Chapter 6 presents and confront the results of the OCP. Finally in the conclusions here is a wrap-up of all the obtained results with possible future developments.

Chapter 1

State of the Art

1.1 Optimal Control Problems

Optimal control problem, also known as dynamic optimisation, are minimisation problem where the variables and parameters change with time. Dynamic systems are characterized by the states and often are controlled by a convenient choice of inputs (controls).
Dynamic optimisation aims to compute those controls and states for a dynamic system over a time interval to minimise one or more performance indexes. In other words, the input is chosen to optimize (minimize) an objective function while complying to constraint equations.
Optimal control problems are challenging from the theoretical point of view and of practical interest. However, due to dimensionality and complexity of the system of equations the application in real problems and the industrial environment is still not so widespread.
In general, OPC can be continuous or discrete, linear or non-linear, time-variant or time-invariant. However, in this thesis are addressed only optimal control problems that are continuous time-variant and highly non-linear. Those properties will be discussed in the following sections.
In general, there are four main approaches to solve continuous-time OPC: state space approach, direct methods, indirect methods and differential dynamic programming.

1.1.1 State-space Approaches

State-space approaches follow the principle of optimality for which each sub-arc of an optimal trajectory must be optimal. In literature, those are referred to as the Hamilton-Jacobi-Bellman (HJB) equation. However, the problem needs numerical methods to be solved, moreover, a solution can be found only for small dimension problems due to *course of dimensionality.* There is no practical application of this method to solve highly non-linear problem as a dynamic optimisation of a motorcycle model.

1.1.2 Direct Method

Direct methods discretise the original optimal control problem into a non-linear programming problem (NLP). In other words, the OPC is transformed into a discrete-time system that can be solved using numerical schemes and optimization techniques, namely Initial Value Solver (IVS) and Sequential Quadratic Programming (SQP) [10] The main advantage of direct methods is the possibility to use inequality constraints even in case of changes in the constraints active set (activation/deactivation) [11]

Direct methods are easier to implement compared to the other three categories and this is one of the reasons why they are by far the most widespread. In fact, almost 90% of the available optimal control software rely on the direct method. [12,13] One most common software is GPOPS-II [14], a tool developed by Patterson and Rao. Other worth citing software are ACADO Toolkit [15] and CasADi [16].

Most of the previously cited tool make use of IPOPT for the solution of the non-linear system of equations. [17]

1.1.3 Indirect Methods

The indirect method exploits the necessary condition of optimality to derive a boundary value problem (BVP) in ordinary differential equations(ODE). Therefore the BVP can be solved numerically as a non-linear problem. The indirect method allows to first optimize and then discretise meaning that the problem con be firstly written in continuous time and discretised later using different discretisation techniques.

The class of indirect methods exploits the well-known calculus of variations and the Euler-Lagrange differential equations, and the so-called Pontryagin Maximum Principle. [6]

The numerical solution can be computed either by shooting techniques (single/multiple shooting) or by collocation. The major drawbacks of indirect methods are that the problem could be difficult to solve or unstable due to the nature of the underlying differential equations (non-linearity and instability) and the changes in control structure (active constraints in specific arcs). Moreover, in some arcs, singularity arises therefore the DAE index increase leading to the necessity of specialized solution techniques. [11] Biral *et al.* [11] develop PINS, a software that transforms the boundary value problem in a non-linear system of equation using finite difference. The system is then solved with a dumped Newton method. PINS (PINS Is Not a Solver) is a software under copyright, but free for academic purposes.

1.1.4 Differential Dynamic Programming

Differential Dynamic Programming (DDP) is a technique developed to overcome the limitation of the dynamic programming approach. The simple approach was developed firstly by Bellman and Dreyfus and it is based on Bellman's principle of optimality. [18]

The DDP is a second-order shooting method where the cost function is approximated around a

reference with a quadratic function. [19]
This approach is well known and dates back to 1966. A remarkable result has been obtained by Huang *et al.* [20] where they use DDP approach to compute the trajectory of an autonomous vehicle with simple kinematics. However, as far as the author knows, it has never been used to solve the optimal control problem with complex dynamic models. Furthermore, there is no trace, in literature, of OCP concerning motorcycles solved using DDP.

1.2 Optimal Control with Motorcycle Dynamics

Several optimal control problem for vehicle dynamics have been proposed in the literature, starting from simple dynamics model increasing to complicated ones and with different applications. Some concern safe manoeuvre minimising a certain functional while others minimize the time to complete a manoeuvre or a circuit (minimum time OC problems). Dal Bianco *et al.* [21] offers a deep and detailed analysis regarding the state-of-the-art in optimal control problems for minimum time application.
In general, optimal control problems can be addressed in multiple ways. One method is the quasi-steady-state approach. In this case, the race line is precomputed and it is an input of the problem. The optimal control is therefore computed keeping the vehicle on the surface of a precomputed maximum performance envelope (G-G surface). The QSS (Quasi-Steady-State) approaches neglect all transient dynamics except for the longitudinal. The lateral acceleration is computed using the curvature of the trajectory and the tangential speed while the vehicle can only accelerate or decelerate in the longitudinal direction. This approach has the advantage of small computational times but it lacks accuracy due to the neglected transient dynamic.
Another method to approach the problem is the so-called transient optimal control. As the name suggests, the dynamic in no more neglected and the differential equations are the constraints of the minimization problem. This formulation allows to describe the full motion of the motorcycle and optimise both controls and trajectory. As addressed in the previous section, for complicated non-linear models the solution techniques are mainly three: differential dynamic programming (DDP), indirect optimal control (IOC) and direct optimal control (DOC).
However, for some reason, the academic research on optimal control is being focused on cars and four-wheeled vehicles leaving motorcycle almost uncovered. This is probably due to the high complexity of the model and the instability.
One of the first publication in the field of optimal control for motorcycle dynamics is from Sharp [2]. In his work, a dynamic model with 5 degrees of freedom is proposed assuming rolling without slipping and saturating the possible applied force. Other publications propose more complicated model considering also driver motion and gearbox effect [5, 22, 23]. However, those do not investigate the effect of tyre forces using any tire model. Cossalter *et al.* [22] propose a model where the forces are bounded with an adherence ellipse.
Bertolazzi *et al.* [10] presents the results for an optimal control problem with a simple model of

a motorcycle obtained with the multi-body approach and it is one of the few OCP solved with indirect methods in literature. [11]

Cossalter *et al.* propose a further complicated model with the purpose of safety manoeuvre and investigation of rider position effect. [24, 25]. It is worth to highlight the recent work of Leonelli and Limebeerb [3]. In their publication, they solve an optimal control problem for minimum lap time of a racing motorcycle using a model with 7 degrees of freedom. As well as other they discard the dynamic effect due to suspension motion. However, their work stands out for the use of the Magic Formula [9] in the modelling of exchanged forces. Another advance in their research is the use of three-dimensional curvilinear coordinate in combination with a complex model. This publication solves the OCP with the direct method using GPOPS-II [14], which is a direct pseudo-spectral method based on Legendre–Gauss–Radau collocation and Radau's integration formula.

Chapter 2

Motorcycle model

There are multiple way to derive describe the behaviour of the motorcycle and to derive the equation of motion. Some use the lagrangian approach to use a minimum set of coordinates [3,7,26]. Other derive the equation of motion using Newton-Euler equations.
In this thesis the dynamic model of the motorcycle has been derived using a multi-body approach and assuming the ISO convention for the orientation of the z-axis (upward). To this end multiple reference frames, points, bodies, forces and torques are defined in the following sections. The model was derived symbolically using Maple and MBSymba that deals with rotation and transformation matrices using homogeneous coordinates. In particular, working on the model, the author chose to derive the kinematics using a combination of global and recursive approach in order to use a minimum set of coordinates while containing the size of the derived equations.

2.1 Kinematic model

2.1.1 Reference frames

A common choice is to start by defining a reference frame in movement with respect to the ground, or fixed, one. This reference frame have in general three linear and three angular velocities. However, for the purpose of this thesis, we consider only planar roads. This means that we need to define the movement of a frame in a plane, therefore only three degrees of freedom are needed (three velocities). The moving reference frame ia addressed as RF_1 and has velocities $u(t)$, $v(t)$ and $\Omega(t)$ as can be appreciated from figure 2.1.1. This set of velocities, also called quasi-coordinates, are suitable to be used later in the definition of curvilinear coordinate. RF_1 has the x-axis aligned with the direction of motion of the motorcycle.
The frame RF_ϕ is the reference frame attached to a plane rotated of an angle $\phi(t)$ commonly addressed as rolling angle around the moving x-axis and it is obtained recursively multiplying

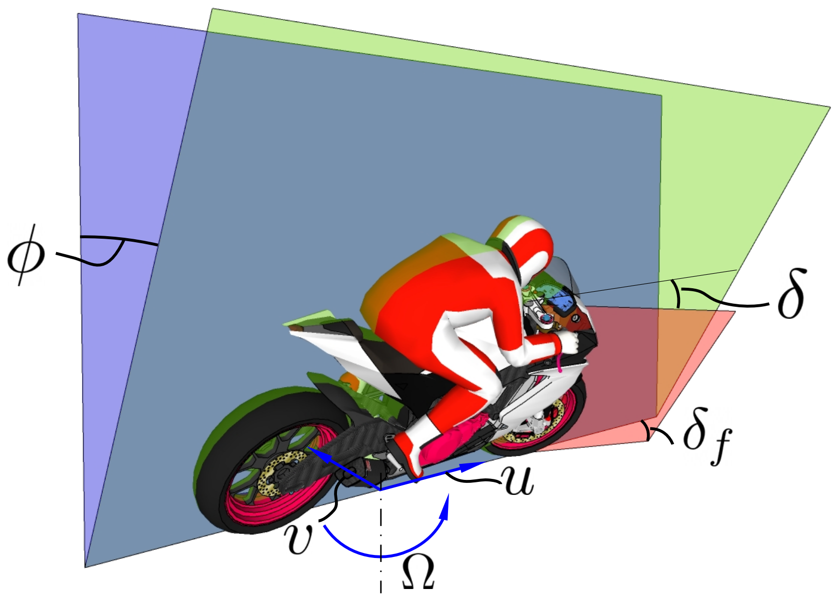


Figure 2.1.1: Motorcycle representation with some of the Degrees of Freedom

RF_1 for a rotation matrix.

$$RF_{\phi} = RF_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi(t)) & -\sin(\phi(t)) & 0 \\ 0 & \sin(\phi(t)) & \cos(\phi(t)) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

Then a reference frame attached to the joint between the swingarm and the rear frame is defined with a translation in the vertical direction of the rolled frame of a certain height $h(t)$ and a rotation around the rolled y-axis of an angle $\theta(t)$ plus the caster angle ϵ (Figure 2.1.2). The new frame will be from here addressed as RF_{Rear}.

$$RF_{Rear} = RF_{\phi} \begin{bmatrix} \cos(\theta(t)+\epsilon) & 0 & -\sin(\theta(t)+\epsilon) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta(t)+\epsilon) & 0 & \cos(\theta(t)+\epsilon) & h(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

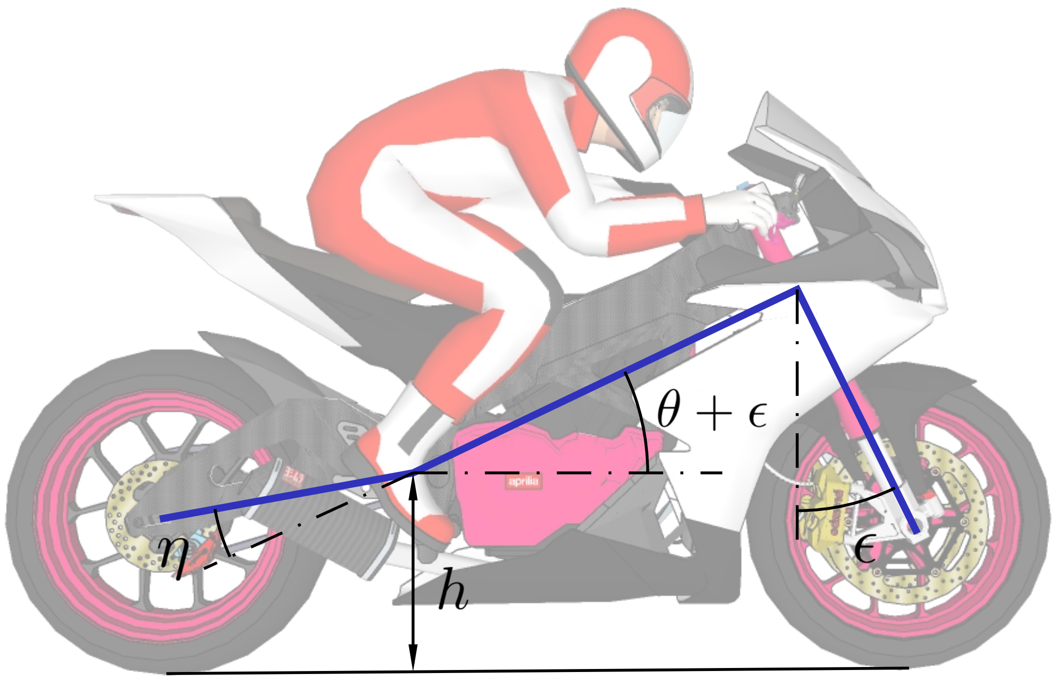


Figure 2.1.2: Motorcycle representation with some of the Degrees of Freedom

From RF_{Rear} frame one can define the reference frame attached to the swingarm and the one attached to the steering assembly. The first is obtained with a rotation around the y-direction of the previous reference frame of a relative angle $eta(t)$ and a translation of the length of the swingarm. The advantage of choosing this relative angle is that there are already define relationship between this rotation an the force of the suspension. The new reference frame is addressed as RF_η and will coincide with the centre of the rear wheel.

$$RF_\eta = RF_{Rear} \begin{bmatrix} \cos(\eta(t)) & 0 & \sin(\eta(t)) & -\cos(\eta(t)) L_{swa} \\ 0 & 1 & 0 & 0 \\ -\sin(\eta(t)) & 0 & \cos(\eta(t)) & \sin(\eta(t)) L_{swa} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

The second reference frame, the one attached to the steering assembly has already the z-axis with the same direction of the rotation axis of the steer. Therefore, RF_δ can be obtained with a translation of a fixed quantity in the x direction and a rotation of an angle $\delta(t)$ around the rotation axis z. This steering angle is small for racing motorcycle and it is always smaller than 10 degrees therefore can be linearised from here since the equation of motion are derived using

Newton-Euler approach instead of Lagrange.

$$RF_{\delta} = RF_{Rear} \begin{bmatrix} 1 & -\delta(t) & 0 & L_b \\ \delta(t) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.4}$$

The reference frame attached to the centre of the front wheel is defined as RF_{susp} and it is obtained with a translation in the negative vertical direction of a fixed quantity s_{fs} plus a time-varying $s_f(t)$ which is the deformation of the suspension and a translation in the x direction of x_{off}, an offset always present in the suspension fork.

$$RF_{susp} = RF_{\delta} \begin{bmatrix} 1 & 0 & 0 & x_{\mathit{off}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -s_{fs} + s_f(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.5}$$

In order to have a simplified model one can introduce two new reference frames one for the front and one for the rear wheel starting from RF_1. RF_{FW} is defined with a translation of the components $x_f(t)$, $y_f(t)$ and $z_f(t)$ and a rotation of an angle $\delta_f(t)$ around the vertical direction and then one around the new longitudinal direction of angle $\phi_f(t)$. Once again $\delta_f(t)$, which is the steering angle projected to the horizontal plane, is small and can be linearised here.

$$RF_{FW} = RF_1 \begin{bmatrix} 1 & -\delta_f(t)\cos(\phi_f(t)) & \delta_f(t)\sin(\phi_f(t)) & x_f(t) \\ \delta_f(t) & \cos(\phi_f(t)) & -\sin(\phi_f(t)) & y_f(t) \\ 0 & \sin(\phi_f(t)) & \cos(\phi_f(t)) & z_f(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6}$$

The second reference fram RF_{RW} is defined with a translation of the components $x_r(t)$, $y_r(t)$ and $z_r(t)$ and a rotation of an angle $\phi_r(t)$ around the x-axis. However since there is no rotation in other planes $\phi_r(t) = \phi(t)$.

$$RF_{RW} = RF_1 \begin{bmatrix} 1 & 0 & 0 & -x_r(t) \\ 0 & \cos(\phi(t)) & -\sin(\phi(t)) & y_r(t) \\ 0 & \sin(\phi(t)) & \cos(\phi(t)) & z_r(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

Those last two reference frames are attached to the wheels centre. One should also define the reference frame that is spinning by multiplying with a rotation matrix of an angle respectively θ_r and θ_f.

$$RF_{FW\,spin} = RF_{FW} \begin{bmatrix} \cos\left(\theta_f\left(t\right)\right) & 0 & \sin\left(\theta_f\left(t\right)\right) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\left(\theta_f\left(t\right)\right) & 0 & \cos\left(\theta_f\left(t\right)\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

$$RF_{RW\,spin} = RF_{RW} \begin{bmatrix} \cos\left(\theta_r\left(t\right)\right) & 0 & \sin\left(\theta_r\left(t\right)\right) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\left(\theta_r\left(t\right)\right) & 0 & \cos\left(\theta_r\left(t\right)\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.9}$$

Those angles will not appear directly in the equation of motion. However, their first and second derivative will. Later on θ_r and θ_f will be substituted as variable by their derivative, the angular velocities ω_r and ω_f.

One can chose to model the interaction of tyre and asphalt as a pure contact and therefore define two constraints equations, one for each wheel. However, the goal of this thesis is to compute the optimal control and those constraints are not linear and analytically unsolvable. This leads to a system of equation that is a Differential Algebraic Equation (DAE). This can be modelled in two ways. The first is by imposing a penalty in the cost function that minimize those constraints, while the second is deriving the constraints, find an ODE and incorporate the algebraic constrains with a Baumgarte stabilization [27]. However, those two methods leads to an enlarged problem that should be solved by the optimisation.

In this thesis, the chosen solution is to treat the constraints as soft. As did in literature [3] the contact is imposed using a force which is proportional to the penetration and the penetration velocity as shown in the next section. For this reason four points are defined in order to get the slip velocities and the penetration of the wheels with the ground. All of those points are defined starting from the reference frame centred in the wheels RF_{FW} and RF_{RW}. The points used for the computation of the slips are:

$$P_r = \begin{bmatrix} 0 \\ 0 \\ -rr + rtr - \frac{rtr}{\cos(\phi(t))} \\ 1 \end{bmatrix} \quad \text{in } RF_{RW} \tag{2.10}$$

$$P_f = \begin{bmatrix} 0 \\ 0 \\ -rf + rtf - \frac{rtf}{\cos\left(\phi_f(t)\right)} \\ 1 \end{bmatrix} \quad \text{in } RF_{FW} \tag{2.11}$$

While the contact points are defined assuming that the shape of the tyre is a torus.

$$C_r = \begin{bmatrix} 0 \\ -\sin\left(\phi\left(t\right)\right) rtr \\ -rr + rtr - rtr\ \cos\left(\phi\left(t\right)\right) \\ 1 \end{bmatrix} \quad \text{in } RF_{RW} \tag{2.12}$$

$$C_f = \begin{bmatrix} 0 \\ -\sin\left(\phi_f\left(t\right)\right) rtf \\ -rf + rtf - \cos\left(\phi_f\left(t\right)\right) rtf \\ 1 \end{bmatrix} \quad \text{in } RF_{FW} \tag{2.13}$$

Where rr, rf, rtr, rtf are respectively the radius of the rear and front wheel and the radii of the section.

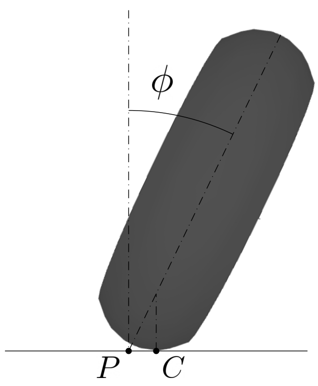


Figure 2.1.3: Contact point and pont of application of the forces

2.1.2 Kinematic solution

Not all the variable introduced are state variable nor are needed for the final purpose of this thesis. Those have been introduced just to simplify the formulation. However, they can be solved now by imposing some constraints. $x_f(t)$, $y_f(t)$ and $z_f(t)$ can be solved by imposing that the point of the origin of RF_{susp} is equal to the origin of RF_{FW}. This yield 3 algebraic equation in 3 variable.

$$
\begin{aligned}
x_f(t) &= (L_b + x_{off})\cos(\theta(t)+\epsilon) + \sin(\theta(t)+\epsilon)(s_{fs} - s_f(t)) \\
y_f(t) &= \sin(\phi(t))(s_{fs} - s_f(t))\cos(\theta(t)+\epsilon) - \sin(\phi(t))(L_b + x_{off})\sin(\theta(t)+\epsilon) + \ldots \\
&\qquad \cdots + \delta(t)\cos(\phi(t))\,x_{off} - \sin(\phi(t))\,h(t) \\
z_f(t) &= -\cos(\phi(t))(s_{fs} - s_f(t))\cos(\theta(t)+\epsilon) + \cos(\phi(t))(L_b + x_{off})\sin(\theta(t)+\epsilon) + \ldots \\
&\qquad \cdots + \delta(t)\sin(\phi(t))\,x_{off} + \cos(\phi(t))\,h(t)
\end{aligned}
\tag{2.14}
$$

The same can be said for the rear wheel. The origin of RF_η is the same point as the origin of RF_{RW} yielding the following.

$$
\begin{aligned}
x_r(t) &= (\cos(\theta(t)+\epsilon)\cos(\eta(t)) + \sin(\theta(t)+\epsilon)\sin(\eta(t)))\,L_{swa} \\
y_r(t) &= -\sin(\phi(t))(L_{swa}\sin(\eta(t))\cos(\theta(t)+\epsilon) - L_{swa}\cos(\eta(t))\sin(\theta(t)+\epsilon) + h(t)) \\
z_r(t) &= \cos(\phi(t))(L_{swa}\sin(\eta(t))\cos(\theta(t)+\epsilon) - L_{swa}\cos(\eta(t))\sin(\theta(t)+\epsilon) + h(t))
\end{aligned}
\tag{2.15}
$$

The rotation angles $\delta_f(t)$ and $\phi_f(t)$ can be solved as a function of the other states. The constrain equation can be multiple and each formulation are equal. One can impose orthogonality between reference frames unit vectors or can impose the equality of such vectors. Both ways lead to the following solution.

$$
\begin{aligned}
\phi_f(t) &= -\arcsin(\cos(\phi(t))\ \sin(\theta(t)+\epsilon)\ \delta(t) - \sin(\phi(t))) \\
\delta_f(t) &= \frac{\cos(\theta(t)+\epsilon)\ \delta(t)}{\sin(\phi(t))\ \sin(\theta(t)+\epsilon)\ \delta(t) + \cos(\phi(t))}
\end{aligned}
\tag{2.16}
$$

This can be further simplified keeping in mind that we are considering small angles for $\delta(t)$.

$$
\begin{aligned}
\phi_f(t) &= -\delta(t)\sin(\theta(t)+\epsilon) + \phi(t) \\
\delta_f(t) &= \frac{\cos(\theta(t)+\epsilon)\,\delta(t)}{\cos(\phi(t))}
\end{aligned}
\tag{2.17}
$$

The values obtained will be later substituted in the equation of motion.

2.1.3 Tyre-Ground penetration

As previously introduced in this thesis, the contact with ground is modelled as a soft constraint using a force which will be a function of the penetration and penetration velocity. The tyre is in this case in equivalent to a spring-damper system. [3, 26] The penetration can be obtained by evaluating the vector joining the origin of the RF_1 supposed on ground and the two points C_f and C_r at the surface of the torus. The z component of those vectors give a measure of how much the tyre is deformed.

$$
\begin{aligned}
p_r(t) &= -\text{comp\_Z}(O_{RF1}C_r, RF_1) \\
p_f(t) &= -\text{comp\_Z}(O_{RF1}C_f, RF_1)
\end{aligned}
\tag{2.18}
$$

The minus sign is present because a positive penetration will generate a positive contact force. comp_Z indicate the component of the vector in the z direction. The results are not reported here because too long to display.

2.1.4 Longitudinal and lateral slips

Longitudinal and lateral slips are defined following the definition of practical slip [7, 28]. The lateral slips are the arctangent of the ratio between longitudinal and lateral velocity of the wheel.

$$\begin{aligned}\alpha_r(t) &= -\arctan\left(\frac{\text{comp\_Y}(VP_r, RF_1)}{\text{comp\_X}(VP_r, RF_1)}\right)\\ \alpha_f(t) &= -\arctan\left(\frac{\text{comp\_Y}(VP_f, RF_1\cdot R_{\delta_f})}{\text{comp\_X}(VP_f, RF_1\cdot R_{\delta_f})}\right)\end{aligned} \tag{2.19}$$

The longitudinal slip, instead, is defines as the difference between longitudinal velocity and velocity of the point on the wheel divided by the longitudinal velocity.

$$\begin{aligned}\lambda_r(t) &= -\frac{(\text{comp\_X}, \text{RF}_1(VP_r) - \omega_r(t)rr)}{\text{comp\_X}(VP_r, RF_1)}\\ \lambda_f(t) &= -\frac{(\text{comp\_X}(VP_f, RF_1\cdot R_{\delta_f}) - \omega_f(t)rf)}{\text{comp\_X}(VP_f, RF_1\cdot R_{\delta_f})}\end{aligned} \tag{2.20}$$

$\omega_r(t)$ and $\omega_f(t)$ are the angular velocities of the wheels also addressed as the time derivative of angles $\theta_r(t)$ and $\theta_f(t)$. VP_r and VP_r are the time derivative of P_r and P_r. It is important to notice that for the front wheel one should project the velocity vector in the reference frame rotated of the angle $\delta_f(t)$ with respect to the reference frame RF_1. The results are not reported here because too long to display ($RF_1 \cdot R_{\delta_f}$).

2.2 Dynamic Model

In the previous section all variables describing the motion have been introduced. One can globally consider how many degrees of freedom the motorcycle will have in the space. The vehicle as a body will have 6 DoF. The first 3 are translations identified in the quasi-coordinate $u(t)$, $v(t)$ (longitudinal and lateral velocity) and the vertical translation $h(t)$. The other DoF are the rotation around 3 axis. The first is around the z direction and identified by the quasi-coordinate $\Omega(t)$ (yaw rate) while the others are angle $\phi(t)$ (roll) and $\theta(t)$(pitch).
In addition to those "external" degrees of freedom, the motorcycle is described by a set of "internal" variables. The word internal is used because the variables describe a motion between parts of the motorcycle. Fist of all the degree of freedom of the steer ($\delta(t)$). Then there is the motion of the front suspension $s_f(t)$ and the one of the rear suspension ($\eta(t)$). Finally the two DoF of the spinning wheels, $\omega_r(t)$ and $\omega_f(t)$.
From the previous description about DoF it is clear that the motorcycle model have 11 degrees of freedom, therefore 11 equation of motion are needed.

2.2.1 Motorcycle rigid body

One techniques to write equation of motion in an efficient way is to define a body that describe the whole motorcycle as a rigid body and then add only the dynamic contribute of the internal

motion of the other bodies.
The motorcycle as a rigid body is composed by the following bodies:

- the rear frame (main body)
- the driver
- the steering assembly (fork)
- the unsprung mass at the end of the front suspension
- the swingarm
- the front wheel
- the rear wheel

In order to describe the motorcycle all the internal degrees of freedom should be fixed. This means that the following substitution should be made for the next calculations.

$$\theta(t) = \theta_{00},\ h(t) = h_{00},\ \delta(t) = 0,\ \eta(t) = \eta_{00},\ s_f(t) = s_{f_{00}},\ \theta_f(t) = \theta_{f_{00}},\ \theta_r(t) = \theta_{r_{00}}$$

Bodies

Before computing the centre of mass of the complete rigid body we should define the centre of gravity of each body.
All the bodies at play are defined starting from the convenient reference frames defined in the precious section. The data and convention in of length and physical values of the motorcycle are take primary from FastBike a fortran code for real time simulation of motorcycles. [29, 30]
The rear frame (main body) is linked to the reference frame RF_{Rear}. The centre of gravity of this body will be in a point G_{Rear} that has only x and z components. This comes directly from the assumption that the vehicle is symmetrical and the reference frame lays on the symmetric plane of the body. The same is true for the body of the rider. The condition on the position of the rider wil be relaxed later. In fact the rider is not static with respect to the motorcycle but can move, lean forward and laterally.

$$G_{Rear} = \begin{bmatrix} x_{Rear} \\ 0 \\ z_{Rear} \end{bmatrix} \text{ in } RF_{rear} \tag{2.21}$$

$$G_{rdr} = \begin{bmatrix} x_{rdr} \\ 0 \\ z_{rdr} \end{bmatrix} \text{ in } RF_{rear} \tag{2.22}$$

The body representing the steering assembly is defined starting from RF_{δ}.

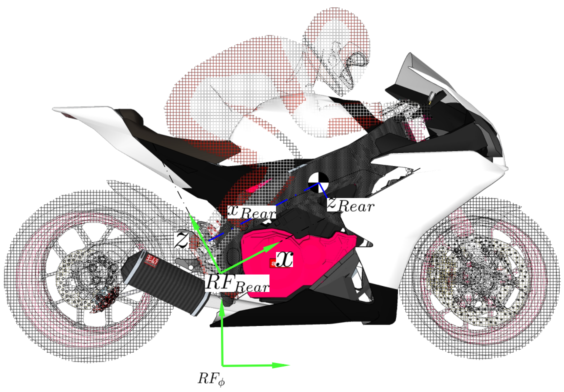


Figure 2.2.1: CoM of the rear frame

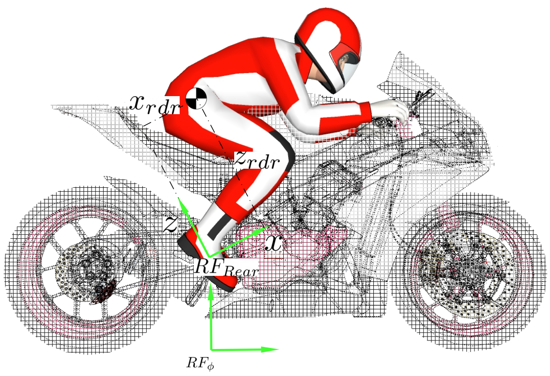


Figure 2.2.2: CoM of the rider

$$G_{\delta} = \begin{bmatrix} x_{\delta} \\ 0 \\ z_{\delta} \end{bmatrix} \text{ in } RF_{\delta} \tag{2.23}$$

where z_{delta} in this case will be negative since the reference frame is define with the ISO convention.

The body of the swingarm in define wth the following centre of gravity starting from RF_{η}.

$$G_{Swing} = \begin{bmatrix} x_{Swing} \\ 0 \\ z_{Swing} \end{bmatrix} \text{ in } RF_{\eta} \tag{2.24}$$

The two body of the wheels have their CoM in the origin of the attached spinning reference frame.

$$G_{FW} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } RF_{FWspin} \tag{2.25}$$

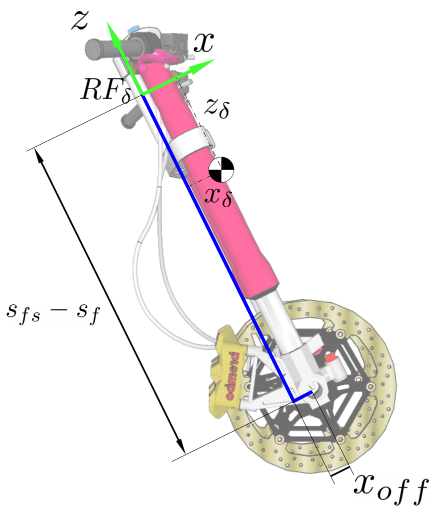


Figure 2.2.3: CoM of the steering assembly

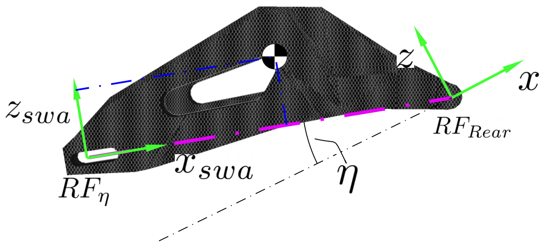


Figure 2.2.4: CoM of the swingarm

$$G_{RW} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } RF_{RW\,spin} \tag{2.26}$$

The last body is the unsprung front suspension. However, the mass of this element is small compared to the other and it can be eventually integrated in the mass of the front wheel. The body of the unsprung suspension has as a virtual centre of mass the centre of the front wheel, wont have mass (mass= 0) and has no inertia. This is a modelling expedient to derive the equation of motions and to transmit the reaction forces.

From all the previously defined centre of gravity one can define the six bodies at play with their inertial properties, mass and moment of inertia.

In the following sections masses and moment of inertia belonging to bodies will appear. The notation used in this thesis follow the subsequent rule. Masses and inertia are indicated with symbols m and Ix, Iy, Iz, Cxy, Cxz, Cyz in general. When addressed to a specific one a suffix is added to represent the body or the reference frame. For instance the mass of the steering assembly will be m_δ.

Centre of Mass of the motorcycle

The CoM of the motorcycle can be simply computed as the weighted average sum of the masses of all the parts. This mean yields a vector of 3 components that can be projected in the rolled reference frame RF_ϕ. Once substituted the relationship of the previous paragraph for the freezed DoF, the values of the vector are the coordinate of the CoM of the motorcycle in the rolled reference frame.

$$G_{moto} = \begin{bmatrix} XG \\ YG \\ ZG \end{bmatrix} \text{ in } RF_1 \tag{2.27}$$

However, the definition of YG is simple and evaluated is equal to zero. This is due to the assumption that the motorcycle is always symmetric with respect to the rolled plane.

Moment of Inertia of the motorcycle

The moment of inertia of the whole motorcycle, can be computed in a similar way as the CoM. The fist thing to derive is the angular velocity of the rolled reference frame RF_ϕ. This will not contain angle θ because is considered as freezed.
The three components of the angular velocity are

$$\begin{aligned} \omega_x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \\ \omega_y(t) &= \Omega(t)\sin(\phi(t)) \\ \omega_z(t) &= \Omega(t)\cos(\phi(t)) \end{aligned} \tag{2.28}$$

Proceeding with the equation derivation the angular momentum of the whole motorcycle ic computed. The angular momentum is additive, therefore it can be calculated as the sum of all the angular momentums. Those should be calculated using as a pole the CoM of the motorcycle. The obtained vector is projected in the rolled reference frame to get rid off almost all the contribute of ϕ and evaluated considering the freezeed DoF.
At this point the previous relationship 2.28 can be exploited and substituted. Therefore the angular momentum can be used to generate the inertia matrix collecting ω_x, ω_y and ω_z.

$$\mathbf{A_M} = I_{tot} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} + \mathbf{res} \tag{2.29}$$

where I_{tot} is the matrix of inertia of the whole motorcycle, $\mathbf{A_M}$ is the vector of the angular momentum and **res** is the vector of residual from the matrix generation. This should be checked to be equal to zero and it is. With the previously computed data a body can be created. This have the centre of gravity in the point of equation 2.27, mass equal to the sum of all masses (M_{tot}) and the moment of inertia I_{tot}.

2.2.2 Dummy bodies

The moment of inertia and the masses of the different parts of the motorcycle are already inside the definition of the rigid body of the previous section. Therefore in the derivation of the equation of motion we should only consider the effect of the motion of the parts. In fact in the rigid body all the pats are frezed and considered as static. However, the dynamic component plays a great role in the motion.
One solution is to take into account only those components is to introduce a set of dummy bodies also known as anti-bodies. This bodies are defined for each moving component of the motorcycle. The dummy body has as a centre of gravity the body they are referred to, but evaluated with the internal DoF freezed. The anti-bodies, as the name suggest, has inertial properties that are the one of the real body with changed sign. This means that all dummy bodies have negative mass and negative inertia. The concept of negative mass and negative inertia have no meaning in physical and real life. However, is a modelling trick to simplify the resulting equations of motion.
Here is reported a list of dummy bodies that have to be defined.

- anti rear frame
- anti rider
- anti steering assembly
- anti swingarm
- anti front wheel
- anti rear wheel

There is no need to create the antibody for the unsprung front suspension since it does not have mass nor inertia. If the model consider also mass and inertia of this body then an antibody must be created to be consistent.

2.2.3 Forces

First of all the force of gravity is acting on all bodies and the acceleration of gravity is set as a vector in the negative z direction due to convention choices.
The other external forces acting on the motorcycle are the aerodynamic drag and the force exchanged by the tyre with the ground.

The drag is modelled with a simple law that scale with the squared of the longitudinal velocity. It has only component in the x direction in the rolled reference frame. The force act on the motorcycle rigid body.

$$F_a = \begin{bmatrix} -C_a u(t)^2 \\ 0 \\ 0 \end{bmatrix} \text{ in } RF_\phi \tag{2.30}$$

Where C_a is this case is a constant, but in theory and reality it is not. This coefficient depend on multiple thermodynamic factors and more important for this application the resisting cross section of the motorcycle and the rider. The motorcycle profile is constant, but the pilot is moving leaning forward and opening the knee at curve entrance. Moreover, the drag force has an application point which is constant in this model, but in reality change with changing configuration (position) of the driver.

$$P_a = \begin{bmatrix} x_a \\ 0 \\ z_a \end{bmatrix} \text{ in } RF_{Rear} \tag{2.31}$$

The forces of the tyre are expressed in a reference frame with the tyre itself. For the rear wheel it coincide with RF_1, while for the front it is necessary to rotate of an angle δ_f. The force vector on the rear wheel is composed of $Fxr(t)$ the longitudinal force, $Fyr(t)$ the lateral force and $Fzr(t)$ the vertical force.

$$F_{RW} = \begin{bmatrix} Fxr(t) \\ Fyr(t) \\ Fzr(t) \end{bmatrix} \text{ in } RF_1 \tag{2.32}$$

and it is supposed as applied in the point P_r and it is acting on the rear wheel. This is not true however some overturning moment will be introduced in the next section.
The force vector on the rear wheel is composed of $Fxf(t)$ the longitudinal force, $Fyf(t)$ the lateral force and $Fzf(t)$ the vertical force.

$$F_{FW} = \begin{bmatrix} Fxf(t) \\ Fyf(t) \\ Fzf(t) \end{bmatrix} \text{ in } RF_1 \cdot R_{\delta_f} \tag{2.33}$$

and it is supposed as applied in the point P_f and it is acting on the front wheel. This is not true however some overturning moment will be introduced in the next section.
$Fxr(t)$, $Fyr(t)$, $Fxf(t)$ and $Fyf(t)$ will be defined following the Magic Formula of Pacejka [7]. The vertical forces are defined as a function of the penetration (equation 2.18) and penetration velocity. It is modelled assuming that the tyre behave as a spring-damper (second-order) system.

$$
\begin{aligned}
Fzf\,(t) &= Kp_f\, p_f\,(t) + Cp_f\,\frac{\mathrm{d}}{\mathrm{d}t} p_f\,(t) \\
Fzr\,(t) &= Kp_r\, p_r\,(t) + Cp_r\,\frac{\mathrm{d}}{\mathrm{d}t} p_r\,(t)
\end{aligned}
\tag{2.34}
$$

There is another force present in the model and is the one of the front suspension. It is model as a spring-damper with linear proportional and damping factor. A more complicated model can be used in future applications.

$$
F_s = \begin{bmatrix} 0 \\ 0 \\ s_f\,(t)\,k_{fs} + \left(\frac{\mathrm{d}}{\mathrm{d}t} s_f\,(t)\right) c_{fs} \end{bmatrix} \text{in } RF_\delta \tag{2.35}
$$

For modelling purposes it is defined as acting on the origin of RF_δ. However, since it is an internal force, the reacting body must be specified. In this case it is the unsprung suspension.

2.2.4 Torques

The torques acting in this dynamic model are both external and internal. The external torques are the one acting on the wheels. Those are present for two reasons. The first is that we are considering a point of application which is not the real point of application of the forces. The contact is in fact on a patch where the distribution of forces is not known. The second reason is that the model of the magic formula [7] takes into account the effect of trail and camber calculating forces and moments with respect to the point P_r and P_f previously defined.
The torques vector acting on the front wheel is:

$$
M_{RW} = \begin{bmatrix} Mxr(t) \\ 0 \\ Mzr(t) \end{bmatrix} \text{in } RF_1 \tag{2.36}
$$

while the momentum vector on the front wheel is:

$$
M_{FW} = \begin{bmatrix} Mxf(t) \\ 0 \\ Mzf(t) \end{bmatrix} \text{in } RF_1 \cdot R_{\delta_f} \tag{2.37}
$$

Then there are other toques that are traction and braking torques. Those are acting in the y direction of the reference frame attached to the wheels. Front wheel can only brake. The traction is acting on the rear wheel and reacting on the swingarm.

$$
TB_{RW} = \begin{bmatrix} 0 \\ Myr(t) \\ 0 \end{bmatrix} \text{in } RF_{RW} \tag{2.38}
$$

The braking torque is acting on the front wheel and reacting on the unsprung suspension.

$$B_{FW} = \begin{bmatrix} 0 \\ Myf(t) \\ 0 \end{bmatrix} \text{in } RF_{FW} \tag{2.39}$$

The internal momentums are instead the torque of the steering damper, the torque applied by the rider and the torque of the rear suspension. The first is modelled as a viscous term.

$$T_d = \begin{bmatrix} 0 \\ 0 \\ -C_\delta \frac{\mathrm{d}}{\mathrm{d}t}\delta(t) \end{bmatrix} \text{in } RF_\delta \tag{2.40}$$

It is acting on the steering frame and reacting on the motorcycle.
The rider torque has only component in the z direction of the RF_δ.

$$T_r = \begin{bmatrix} 0 \\ 0 \\ \tau(t) \end{bmatrix} \text{in } RF_\delta \tag{2.41}$$

It is acting on the steering assembly and reacting on the motorcycle rigid body. The last internal torque is the rear suspension torque. It is actually a force of a spring-damper system applied with a certain arm. However, for modelling purposes can be written in terms of a torque with a torsional stiffness and a rotational viscosity.

$$M_{RSusp} = \begin{bmatrix} 0 \\ \eta(t)\, a_1\, k_{rs} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\right) a_1\, c_{rs} \\ 0 \end{bmatrix} \text{in } RF_{Rear} \tag{2.42}$$

It is acting on the swingarm and reacting on the motorcycle rigid body.

2.3 Equations of motion

The MBSymba library [31] allow to derive the newton euler equations of motion. As previously highlighted, the equation of motion needed are 11. The first six are derived from the Newton and Euler equation of the whole system that is composed of all bodies, all anti-bodies and all forces at play. Th equation are projected in RF_1 and use the origin of RF_1 as a pole for the momentum equilibrium.
The equation are huge and it is pointless to show them if not in a simplified case. For instance with the internal degrees of freedom freezed and some other set to zero. Specifically the imposed values are:

$$\phi(t) = 0, \eta(t) = \eta_0 0, \Omega(t) = 0, \delta(t) = 0, \theta(t) = \theta_0 0, s_f(t) = s_{f00} \tag{2.43}$$

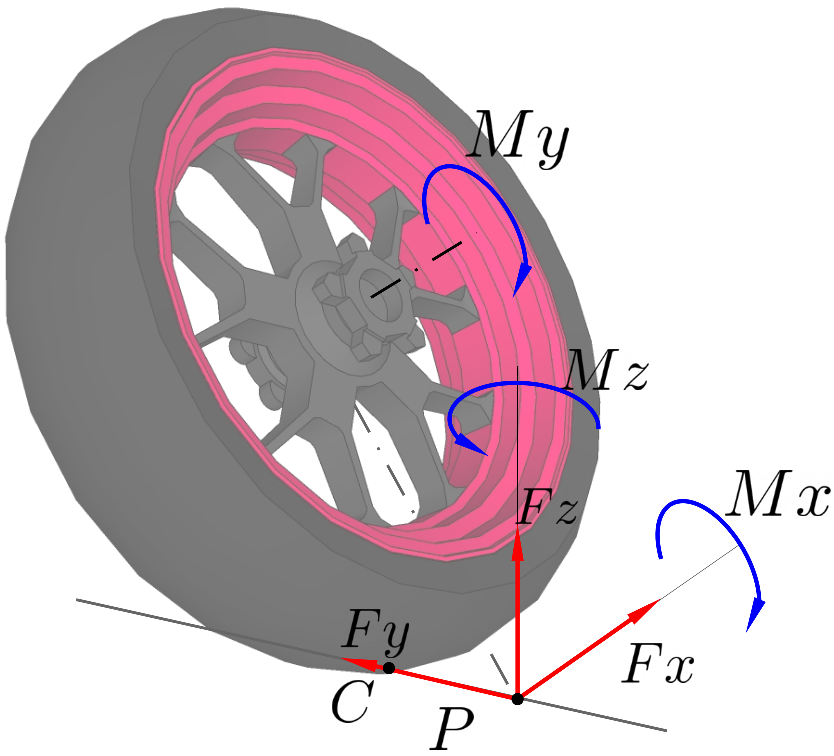


Figure 2.2.5: Forces scheme

The DoF set to zero are the roll angle of the motorcycle, the steering angle and the yaw rate. The motorcycle is in up-straight static condition. This yields a simplified version of the Newton equation such as:

$$\begin{aligned}
&(u(t))^2\, Ca + M_{tot}\, \tfrac{\mathrm{d}}{\mathrm{d}t} u(t) - Fxf(t) - Fxr(t) = 0\\
&M_{tot}\, \tfrac{\mathrm{d}}{\mathrm{d}t} v(t) - Fyf(t) - Fyr(t) = 0\\
&(m_m + m_{rdr} + m_\delta + m_{swa} + m_{wf} + m_{wr})\, \frac{\mathrm{d}^2}{\mathrm{d}t^2} h(t) + M_{tot}\, g - Fzf(t) - Fzr(t) = 0
\end{aligned} \tag{2.44}$$

The simplification performed transform the complex formulas of the dynamic of the motorcycle in something equal to the single track model of car.
The Euler equation are too complex to show even with such greater simplifications.
The equation of motion that should be derived from the steering dynamic is only one. It is the Euler equation around the z axis of RF_δ and it is projected in this reference frame. As well as in the other cases the equation is long an complex. However the simplified version (with relationship in 2.43) can be displayed.

$$(m_{wf}\, x_{off} + m_\delta\, x_\delta)\, \frac{\mathrm{d}}{\mathrm{d}t} v(t) + (rf\, \sin(\theta_0 + \epsilon) - x_{off})\, Fyf(t) - Mzf(t) \cos(\theta_0 + \epsilon) + Mxf(t) \sin(\theta_0 + \epsilon) - \tau(t) = 0 \tag{2.45}$$

The Euler equation of the rotation of the system composed by rear swingarm and rear wheel is derived using as pole the joint between rear frame and swingarm and it is projected in RF_{Rear}.

As for the previous the equation simplified is reported here.

$$
\begin{aligned}
&(((L_{swa} - x_{Swing})\, m_{swa} + L_{swa}\, m_{wr}) \cos(-\eta_0 + \theta_0 + \epsilon) + z_{Swing}\, m_{swa}\, \sin(-\eta_0 + \theta_0 + \epsilon)) \frac{\mathrm{d}^2}{\mathrm{d}t^2} h(t) \ldots \\
&\cdots + (z_{Swing}\, m_{swa}\, \cos(-\eta_0 + \theta_0 + \epsilon) + (m_{swa}\, (-L_{swa} + x_{Swing}) - L_{swa}\, m_{wr}) \sin(-\eta_0 + \theta_0 + \epsilon)) \frac{\mathrm{d}}{\mathrm{d}t} u(t) + \ldots \\
&\cdots + (-L_{swa}\, Fzr(t) - (m_{swa}\, (-L_{swa} + x_{Swing}) - L_{swa}\, m_{wr})\, g) \cos(-\eta_0 + \theta_0 + \epsilon) + \ldots \\
&\cdots + (z_{Swing}\, m_{swa}\, g + L_{swa}\, Fxr(t)) \sin(-\eta_0 + \theta_0 + \epsilon) + \eta_0\, a_1\, k_{rs} + Fxr(t)\, rr + Iy_{wr} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \theta_r(t) = 0
\end{aligned}
\tag{2.46}
$$

The equation of motion describing the dynamic of front suspension is the z component of the Newton equations of the system unsprung suspension plus front wheel. It is projected in RF_δ As for the previous the equation simplified is reported here.

$$
\cos(\theta_0 + \epsilon) \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} h(t) \right) m_{wf} + (g m_{wf} - Fzf(t)) \cos(\theta_0 + \epsilon) + \left(- \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t) \right) m_{wf} + Fxf(t) \right) \sin(\theta_0 + \epsilon) + s_{f_0}\, k_{fs} = 0 \tag{2.47}
$$

The only two equation left out are the one describing the rotational dynamic of the rear and front wheels. Those are derived considering only the wheels and the forces applied on those. The Euler equations are projected respectively in RF_{FW} and RF_{RW}. The simplified version for the front is the following

$$
Iy_{wf} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \theta_f(t) + Fxf(t)\, rf - Myf(t) \tag{2.48}
$$

While for the rear we have

$$
Iy_{wr} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \theta_r(t) + Fxr(t)\, rr - Myr(t) \tag{2.49}
$$

2.4 Reduction to Single Track model

As highlighted in the previous section, the complete equation of motion are complex and cannot be shown. However, one way to validate the model is to reduce the problem to simple case to observe the terms in the equation. As a proof of concept one can isolate the the two equation of motion of the whole motorcycle concerning the pitch and the vertical translation (θ and h).
DISEGNO
From the equilibrium of momentum is clear that from those equation one can solve for the vertical forces. Those will have a complex formulation that simplified in the case of the motorcycle in vertical position ($\phi(t) = 0$), no steering ($\delta(t) = 0$), lateral velocity null ($v(t) = 0$), zero yaw rate ($\Omega(t) = 0$) and internal degrees of freedom freezed ($\eta(t) = \eta_0 0$, $\theta(t) = \theta_0 0$, $s_f(t) = s_{f00}$). The equations are still complicated, but a lot of terms can be collected yielding the following

expression.

$$
\begin{aligned}
Fzr\left(t\right) &= +\frac{M_{tot}\,ax\left(t\right)ZG}{L}+\frac{Ca\,\left(u\left(t\right)\right)^{2}ZA}{L}+\frac{Iy_{wf}\,\frac{\mathrm{d}^{2}}{\mathrm{d}t^{2}}\theta_{f}\left(t\right)}{L}+\frac{Iy_{wr}\,\frac{\mathrm{d}^{2}}{\mathrm{d}t^{2}}\theta_{r}\left(t\right)}{L}+\frac{M_{tot}\,gLf}{L}\\
Fzf\left(t\right) &= -\frac{M_{tot}\,ax\left(t\right)ZG}{L}-\frac{Ca\,\left(u\left(t\right)\right)^{2}ZA}{L}-\frac{Iy_{wf}\,\frac{\mathrm{d}^{2}}{\mathrm{d}t^{2}}\theta_{f}\left(t\right)}{L}-\frac{Iy_{wr}\,\frac{\mathrm{d}^{2}}{\mathrm{d}t^{2}}\theta_{r}\left(t\right)}{L}+\frac{M_{tot}\,gLr}{L}
\end{aligned}
\tag{2.50}
$$

In the previous equations $ax(t)$ is actually the longitudinal acceleration that in general is equal to $\frac{\mathrm{d}}{\mathrm{d}t}u(t)-\Omega(t)v(t)$. However, both $\omega(t)$ and $v(t)$ are considered null, therefore $ax(t)=\dfrac{\mathrm{d}}{\mathrm{d}t}u(t)$. L is the total length defined as $L=L_r+L_f$ where L_r and L_f are rear axis length and front axis length. Those are measure the distance of the contact point from the CoM of the motorcycle. ZA is the height of the pressure point where the drag force is applied redefined in the reference frame RF_1.

The solution for the vertical forces in equation 2.50 shows clearly the dependency on static load distribution between front and rear wheel due to the position of the centre of mass $\dfrac{M_{tot}\,gLf}{L}$. The other therms depends on load transfer due to drag, to acceleration and wheels angular acceleration.

Chapter 3

Magic Formula Pacejka

The model used in this thesis for the forces is the same proposed for motorcycles by Hans Pacejka. [9] The data for the tyre are taken from the example chapter 12 of "Vehicle and Tyre Dynamics" and other publications. [2, 26]

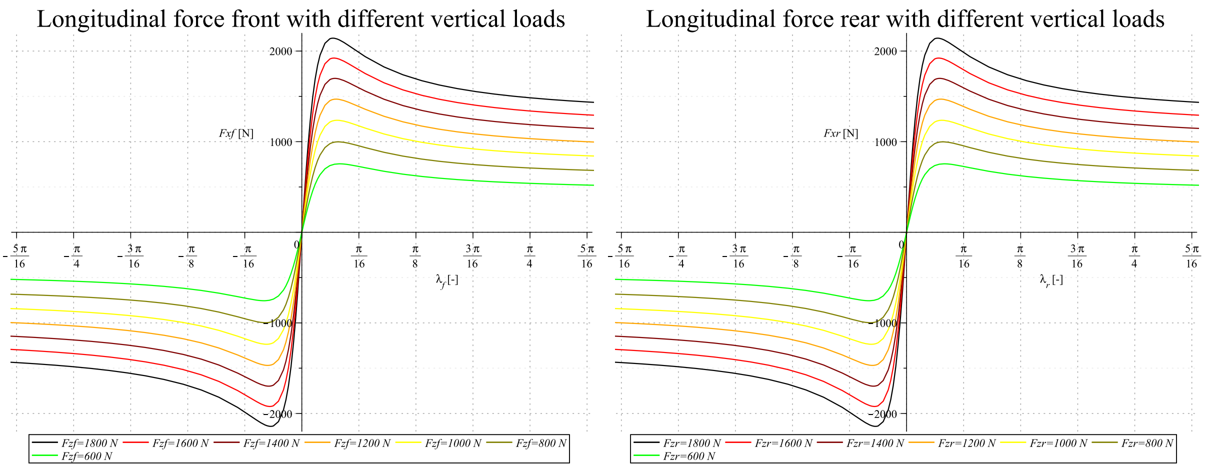
3.1 Longitudinal force

The longitudinal forces are described by the following relationship where λ represent the longitudinal slip defined in the previous chapters. Usually longitudinal slip is small or at list constrained to be small to have good performances.

$$\begin{aligned} Fxf &= Dx_f \sin\Big(Cx_f \arctan\Big(Bx_f \lambda_{x_f} + SHx_f\Big)\Big) + SVx_f \\ Fxr &= Dx_r \sin(Cx_r \arctan(Bx_r \lambda_{x_r} + SHx_r)) + SVx_r \end{aligned} \tag{3.1}$$

Here, the time dependency is being dropped to have a lighter notation. This formula comes from the book "Tire and Vehicle Dynamics" 3^{rd} edition [9] which is more compact with respect to the one of the older version [7]. All parameters inside are function of other variables with the following definition.

$$\left[\begin{array}{c} dfz_i = \dfrac{Fzi\,(t) - Fz0}{Fz0} \\ \lambda_{x_i} = \lambda_i\,(t) + SHx_i \\ Cx_i = \lambda_{C_{x_i}}\ pCx1_i \\ Dx_i = Fzi\,(t)\ \mu_{x_i} \\ \mu_{x_i} = (dfz_i\ pDx2_i + pDx1_i)\ \lambda_{\mu_{x_i}} \\ Ex_i = \lambda_{E_{x_i}}\ (1 - pEx4_i)\ \Big(dfz_i^2\ pEx3_i + dfz_i\ pEx2_i + pEx1_i\Big) \\ Kx_{\lambda_i} = \lambda_{K_{x_i}}\ e^{dfz_i\ pKx3_i}\ (dfz_i\ pKx2_i + pKx1_i)\ Fzf\,(t) \\ Bx_i = \dfrac{Kx_{\lambda_i}}{Dx_i\ Cx_i + \epsilon_{x_i}} \\ SHx_i = 0 \\ SVx_i = 0 \end{array}\right] \tag{3.2}$$



(a) (b)

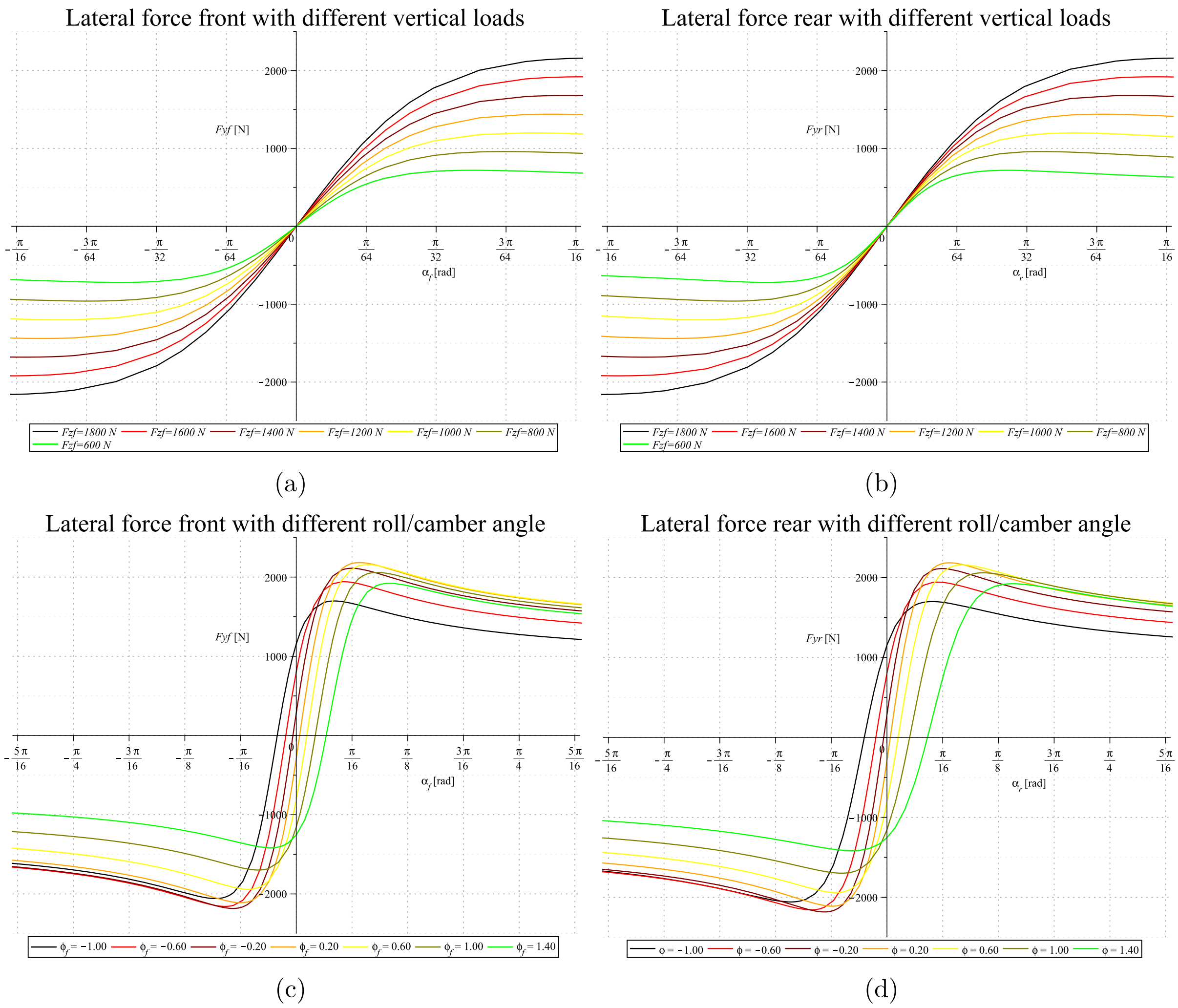
Figure 3.1.1: Graphs of the longitudinal force for front and rear wheel

where i is everywhere in place of r or f, rear and front wheel.
As one can appreciate from figure 3.1.1a and 3.1.1b the longitudinal force scale linearly with the vertical load. This is the expected behaviour in traction and braking conditions. In order to get the minimum time performance the optimal control should be able to reach the peak either positive or negative while on traction or on braking manoeuvre.

3.2 Lateral force

The lateral force depends on the lateral slip, namely α_f and α_r. They are both function of time even if here the time dependency is dropped out to simplify the notation. In straight motion there is no lateral velocity therefore the lateral slip is null along with the lateral forces. Lateral forces also depends on camber angle that in the case of motorcycle coincide with the roll angle. The rear wheel has a camber angle that is exactly ϕ while the front wheel is affected also by the steering angle and therefore his camber is equal to ϕ_f.

$$\begin{aligned} Fyf &= D_f \sin\Big(C_f \arctan\Big(B_f(\alpha_{Feq_f} + SH_f)\Big)\Big) + SV_f \\ Fyr &= D_r \sin(C_r \arctan(B_r(\alpha_{Feq_r} + SH_r))) + SV_r \end{aligned} \tag{3.3}$$



(a) (b) (c) (d)

Figure 3.2.1: Graphs of the lateral force for front and rear wheel

All coefficient are defined hereafter.

$$\left[\begin{array}{c} Fz_i = Fzi\,(t) \\ CF_{\alpha_i} = \dfrac{CF_{\alpha 0_i}}{d5_i\ \gamma_i^2+1} \\ CF_{\alpha 0_i} = d1_i\ Fz0_i + d2_i\ (Fz_i - Fz0_i) \\ CF_{\gamma_i} = d3_i\ Fz_i \\ CM_{\alpha_i} = e1_i\ Fz_i \\ CM_{\gamma_i} = e2_i\ Fz_i \\ CMx_{\gamma_i} = e3_i\ Fz_i \\ Fx_i = Fxi\,(t) \\ C_i = d8 \\ K_i = CF_{\alpha_i} \\ D0_i = \dfrac{d4_i\ Fz_i}{d7_i\ \gamma_i^2+1} \\ D_i = \sqrt{D0_i^2} \\ B_i = \dfrac{K_i}{C_i\ D0_i} \\ SHf_i = \dfrac{CF_{\gamma_i}\ \gamma_i}{CF_{\alpha_i}} \\ SV_i = \dfrac{d6_i\ Fz_i\ \gamma_i\ D_i}{D0_i} \\ SH_i = SHf_i - \dfrac{SV_i}{CF_{\alpha_i}} \\ \alpha_{Feq_i} = \dfrac{D0_i\ (\alpha_i - SHf_i)}{D_i} - SHf_i \end{array}\right] \tag{3.4}$$

where i is everywhere in place of r or f, rear and front wheel.
In the previous definitions γ_i is actually the camber angle ϕ or ϕ_f depending on the addressed wheel. The angle $alpha_i$ is the side slip angle and is used to calculate the equivalent side slip α_{Feq_i}.
In figure 3.2.1a and 3.2.1b the effect of vertical load Fz on lateral force is show for both wheels. As expected from the model definition, the lateral force scale with the applied load. Figure 3.2.1c and 3.2.1d, instead, represent the effect of the different chamber angles on lateral loads. It is clear that this angle plays a major role in the range between -60 and 60 degrees.

3.3 Self-aligning moment

The self-aligning moment depends on the lateral slips and chamber angle. As well as fot the other forces previously defined, the time dependency is dropped out to simplify the notation.

$$\begin{aligned} Mzf &= -Fy_{\alpha_f}\, t_{\alpha_f} + Mzr_f - \tan(\gamma_f)\, Fx_f\, rtf \\ Mzr &= -Fy_{\alpha_r}\, t_{\alpha_r} + Mzr_r - \tan(\gamma_r)\, Fx_r\, rtr \end{aligned} \tag{3.5}$$

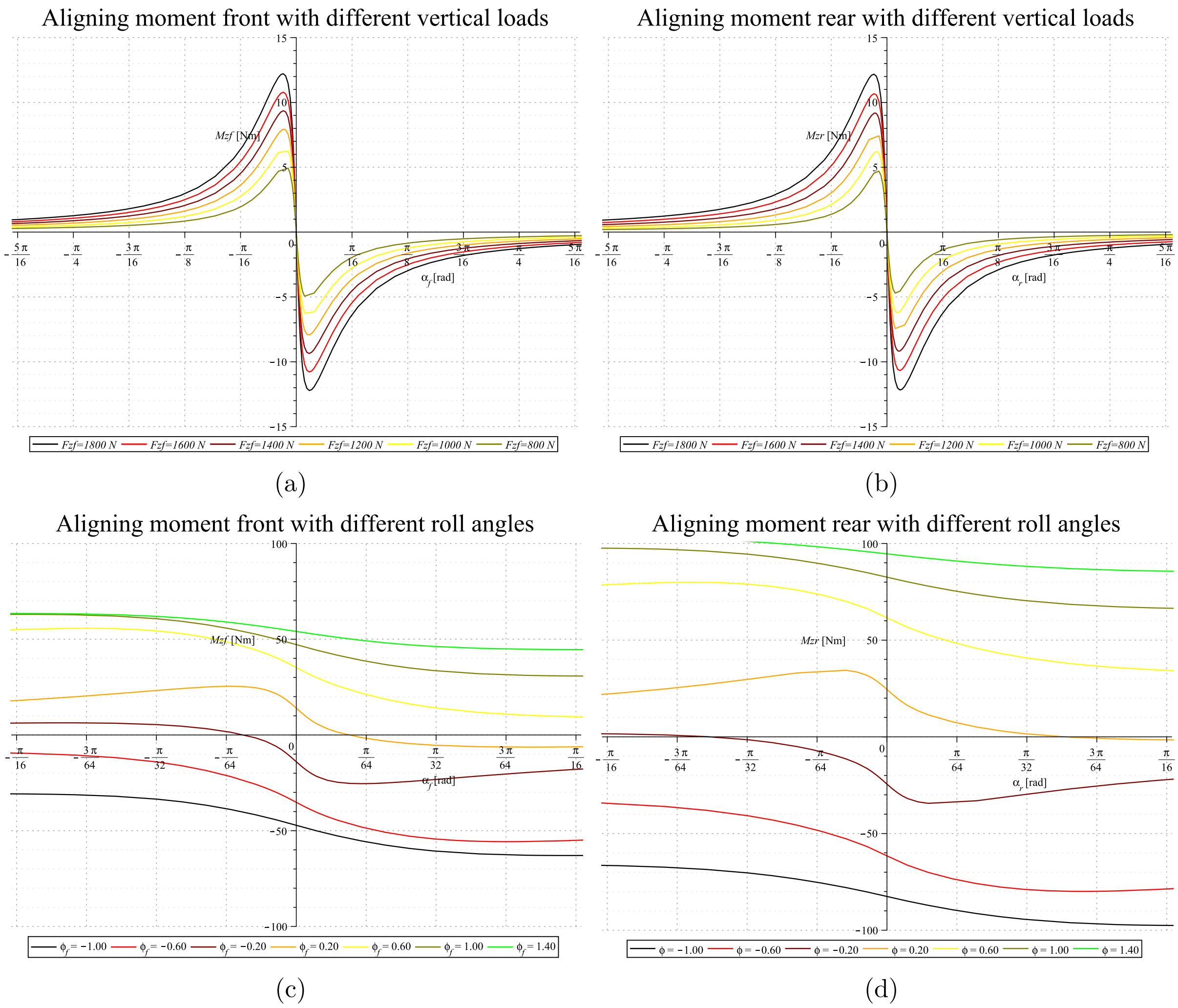


Figure 3.3.1: Graphs of the self-aligning moments for front and rear wheel

The self-aligning moments depend on three components. The first is the product of the lateral force by a trail (t). In fact the lateral force is not acting on the point P defined in chapter 2, this was a modelling expedient. However, there is no such thing as a precise contact point. The force exchange between tyre and ground occurs in a contact patch with an unknown distribution. In fact, this distribution depends on the side slip.

The second component of the self-aligning moment depends on the combined effect of chamber and side slip angle. The third therm is an additional torque present because the longitudinal force is not applied in the actual contact point C but in P. [9]

All coefficient are defined hereafter.

$$\left[\begin{aligned} t_{\alpha 0_i} &= \frac{CM_{\alpha_i}}{CF_{\alpha 0_i}} \\ \alpha_{eq0_i} &= \frac{\alpha_i \ D0_i}{D_i} \\ Fy_{\alpha_i} &= \sin\left(\arctan\left(\alpha_{eq0_i} \ B_i\right) \ C_i\right) \ D_i \\ B_{t_i} &= e7 \\ C_{t_i} &= e8 \\ B_{r_i} &= \frac{e9}{\gamma_i^2 \ e4 + 1} \\ C_{r_i} &= \frac{e10}{\gamma_i^2 \ e5 + 1} \\ t_{\alpha_i} &= \frac{\cos\left(\arctan\left(\alpha_{eq0_i} \ B_{t_i}\right) \ C_{r_i}\right) \ t_{\alpha 0_i}}{\gamma_i^2 \ e5 + 1} \\ Mzr0_i &= \frac{\arctan\left(\gamma_i \ e6\right) \ CM_{\gamma_i}}{e6} \\ Mzr_i &= \cos\left(\arctan\left(\alpha_{eq0_i} \ B_{r_i}\right) \ C_{r_i}\right) \ Mzr0_i \end{aligned}\right] \tag{3.6}$$

In figure 3.3.1a and 3.3.1b the effect of vertical load Fz on self aligning moment is show for both wheels. As expected from the model definition, the peak of the moment scale with the applied load. Figure 3.3.1c and 3.3.1d, instead, represent the effect of the different chamber angles on the aligning moment. It is clear that this angle plays a major role in the range between -60 and 60 degrees.

3.4 Overturning moment

The overturning moment is added to complete the modelling. In fact, as previously highlighted, the force are applied in P both for modelling purposes and for consistence with parameters measurements from literature [2, 9]. The overturning moment appears from the translation of the vertical force from C to P. Under the assumption of circular cross section of the tyre (torus) the overturning moment is defined as

$$\begin{aligned} Mxf\left(t\right) &= -Fzf\left(t\right)\tan\left(\phi_f\left(t\right)\right) rtf \\ Mxr\left(t\right) &= -Fzr\left(t\right)\tan\left(\phi\left(t\right)\right) rtr \end{aligned} \tag{3.7}$$

where rtf and rtr are the radii of the toroidal section. In straight condition both chamber angles ϕ and ϕ_f are null and therefore there is no overturning moment.

Chapter 4

Static conditions and Steady State

4.1 Static condition

The evaluation of static condition of the motorcycle is a necessary to compute the inertial values and the coordinate of the CoM of the whole rigid body. Up to this point XG, YG and ZG are function of the freeze DoF (which correspond to the value of the variable in static condition). Moreover, the only inertial value known is the total mass that can be easily calculated as a sum of all the other masses. The rest of inertial unknowns are, as well as the CoM coordinates, function of the freezed DoF.

4.1.1 Problem formulation

In static condition all time derivative are zero which is equivalent to impose a constant value to all variables. This will drive to zero most of the equation. Moreover, static conditions means no velocity, therefore all the quasi-coordinate (u, v, Ω) are set to zero. The equation defining the slips are discarded due to singularity in static condition. In fact, whenever the longitudinal velocity is equal to zero the values of the longitudinal and lateral slips are not defined. Since our interest is in finding the static values of the variables, one should consider an equilibrium condition. Therefore both ϕ and δ are set to zero. The problem will be still undetermined and can not be solved analytically. Furthermore, it can be solved as a non linear problem with a quadratic minimisation.

The minimisation problem is solved using a specific tool in Maple from the Optimisation package. The objective function to be minimised in the sum of the square of all the ordinary differential equations with the addition of the algebraic definition of some quantities (CoM,Inertia,x_r, *etc.*). Actually the ODE system has no more differential parts since we are considering static condition. Therefore the system is composed of algebraic equations. The values of some angles are bounded with value consistent with physical system. The algebraic system, in fact, is non-linear and multiple local minimum can be found. For this reason one should restrict the allowable region for the NLP. Furthermore, the convergence of the solution is highly dependent on initial point

or initial guess.
The results of the minimisation problem are reported in the following section.

4.1.2 Solution

The solution of the minimisation problem yields first of all the vertical forces in table 4.1. The overall force distribution is fairly balanced between front and rear axle.

Vertical Loads	
Fzf_{00}	1320.980 [N]
Fzr_{00}	1325.438 [N]

Table 4.1: Vertical load distribution in static condition

The position of the two wheels defined in chapter 2 with the variables x_r, y_r, z_r, x_f, y_f and z_f are reported here. The results are not really interesting, however it is a check that everything works as it is supposed to. As highlighted in table 4.2 the y coordinate of both wheels is at zero, the z coordinate represent exactly the radii of front and rear wheel. The x coordinate gives the position of the contact point that in this case overlaps the point of application of the forces.

Wheel Coordinates			
	Front [m]		**Rear** [m]
x_{f00}	0.8594	x_{r00}	0.5138
y_{f00}	-6.449×10^{-308}	y_{r00}	-1.905×10^{-308}
z_{f00}	0.2788	z_{r00}	0.3068

Table 4.2: Wheels centre coordinates

For obvious reasons and trivially from their definition in chapter 2 the angles of the front wheel in static case ere both zero.

$$\delta_{f00} = 0$$
$$\phi_{f00} = 0$$

As defined in chapter 2 the motorcycle CoM is define with the 3 coordinate XG, YG and ZG. YG is zero for the assumption of symmetry while the other two are computed in the NLP and yields the following values.(Table 4.3)

CoM position		
XG [m]	YG [m]	ZG [m]
0.1717	0	0.6661

Table 4.3: CoM position with respect to RF_ϕ

The most interesting result are the values of the freezed DoF in table 4.4. The computed values are the one expected from the definition of the variables in the model derivation. In particular front ar rear suspension are in compression state. As one can notice the relative angle of the swing arm (η) is very small due to the rigidity of the rear suspension. On the other hand, the front suspension is more compliant and the deflection in greater. This deformation reflects directly on the pitch angle $t\theta$. Since the rear deformation is almost null it is trivial to show that there is a rigid rotation around the y axis of the motorcycle.

DoF freezed			
η_{00} [rad]	h_{00} [m]	$s_{f_{00}}$ [m]	θ_{00} [rad]
0.0288	0.4558	0.0584	−0.1169

Table 4.4: Freezed DoF values in static condition

The last parameters computed are the inertial ones. As highlighted in chapter 2 the motorcycle is consider as symmetric and therefore most of the off diagonal moment of inertia are zero. In particular, the only moment of inertia needed are IX, IY, IZ and CXZ

Inertial Values				
M_{tot} [kg]	IX [kgm^2]	IY [kgm^2]	IZ [kgm^2]	CXZ [kgm^2]
269.85	31.121	67.723	38.681	1.772

Table 4.5: Inertial properties of the motorcycle rigid body

Those values are fundamental in the next steps of this study. Furthermore, for now on they will be considered as data of the problem.

4.2 Steady state

The computation of steady state values is done to achieve a reference state to use as a initial guess for the integration and for the optimal control problem.

4.2.1 Problem formulation

The SS (Steady˙State) is calculated using a similar formulation as in the static case.
All time derivative are zero therefore all states are constant. In this case, the quasi-coordinate (u, v, Ω) are free and the equation defining the slips are taken into account.
The substitution of the constant values yield a system o f algebraic equations 11 comes from the ODE, 2 from the definition of the vertical forces, 8 from the definition of kinematic parameters (δ_f, x_f, *etc.*), 8 from the definition of the tyre forces and 4 from the definition of slips. This make a total of 33 non linear algebraic equations while the variable at play are 36. In fact, 3 of them can be imposed as controls. From the statement of the problem the controls are the couple applied to front and rear wheel and the torque applied to the steer. However, in this case, one can conveniently choose which variable to chose as control.
The author choose to impose the torque applied to the front wheel with value zero (not braking).

This means that the front wheel is left free to run along with the vehicle. The other two imposed values are the rolling angle and the velocity. For the sake of model verification, here the author choose the following values.

$$\begin{aligned} u(t) &= u = 10 \text{ [m/s]} \\ \phi(t) &= \phi = 0 \text{ [rad]} \end{aligned} \tag{4.1}$$

This means that the motorcycle is going straight with a slow speed of 36 km/h. It is important to highlight that any other choice is possible providing feasible values. Moreover, any other two controls can be chosen.
The problem will be still non linear and can not be solved analytically. However, it can be solved as a non linear problem with a quadratic minimisation of the algebraic equation under some constraints. Those will be the same as for the static case.
As highlighted in the previous sections the solution is highly sensitive to initial conditions. A good choice for the first step will be to use as a guess the values computed in the static state. Not all variable are computed and those can be chosen, such as the slips and some forces.
The results of the minimisation problem are reported in the following section.

4.2.2 Solution

All the parameters have been computed, even the redundant one such as x_r, y_r, δ_f *etc.*. However those are not of great interest and are reported in appendix.
The important ones are the forces and the values of the states of the dynamical system.
First of all the vertical loads can be confronted with the one obtained in the static case.

Vertical Loads	
Fzf_{ss}	1291.135 [N]
Fzr_{ss}	1355.284 [N]

Table 4.6: Vertical load distribution in steady-state condition

As highlighted from table 4.6 with respect to table 4.1 there is a slight unload of the front wheel and therefore the load is transferred to the rear axis. It is small due to the small velocity of the vehicle.
As expected and reported in table 4.7 the angular velocity and the lateral velocity are almost zero. They are not zero ($10^{-12} \approx 0$) in the table because the solution through the minimization yields result up to a certain precision.

Quasi-coordinates	
v_{ss}	-7.977×10^{-12} [m/s]
Ω_{ss}	2.901×10^{-12} [rad/s]

Table 4.7: Quasi-coordinate in steady-state condition

The other states assumes the following values reported in table 4.8.

States in SS				
η_{ss} [rad]	h_{ss} [m]	$s_{f_{ss}}$ [m]	θ_{ss} [rad]	δ_{ss} [rad]
0.0288	0.456	0.057	-0.115	1.041×10^{-10}

Table 4.8: States value in steady-state

The computed values are very similar to the static ones. This was expected since the velocity is small and there is no internal dynamic. The value of the steering angle δ_{ss} is very small and comparable to zero for the same reason as for yaw rate. This is due to the specific condition tracked here. In fact in straight running one can expect zero torque applied by the driver. This is confirmed by the results.

$$\tau_{ss} = -1.363 \times 10^{-10}\ \mathrm{N\,m} \tag{4.2}$$

The angular velocity of the two wheel is trivial. In steady-state there is almost no slip therefore the assumption of free rolling can be accepted. The minimisation problem yields

$$\begin{aligned} \omega_{f_{ss}} &= 34.247\ \mathrm{rad/s} \\ \omega_{r_{ss}} &= 31.573\ \mathrm{rad/s} \end{aligned} \tag{4.3}$$

This was expected since the radius of the wheel is around 0.3 m.
The minimisation also yields the slips.

Slips in SS			
Longitudinal [$-$]		**Lateral** [rad]	
$\lambda_{f_{ss}}$	-2.162×10^{-14}	$\alpha_{f_{ss}}$	-1.426×10^{-12}
$\lambda_{r_{ss}}$	0.877×10^{-3}	$\alpha_{r_{ss}}$	1.451×10^{-13}

Table 4.9: Slips in steady-state

In table 4.9 it is clear that all lateral slips are zero along with the front longitudinal one. The longitudinal slip at the rear wheel is greater because in order to maintain a certain speed a couple must be applied at the rear. Therefore, the trust force needed derive from the Magic Formula and as a consequence some slip should be present.
As previously stated, to keep a constant velocity a torque should be applied (air drag effect).

$$Myr_{ss} = 9.510\ \mathrm{N\,m} \tag{4.4}$$

Force and torques applied at wheels are reported in the following table 4.10 and 4.11.
In this steady state condition we are considering a motion with a dynamic somehow constrained in the vertical plane. This means that all moments and forces exerting ”out of the plane” should be zero. This is in fact confirmed in table 4.10 and 4.11.
The only force different from zero is the longitudinal force of the rear wheel. As expected this

Forces and Torques at front wheels			
Force [N]		**Torque** [N m]	
Fxf_{ss}	-6.999×10^{-10}	Mxf_{ss}	2.578×10^{-9}
Fyf_{ss}	4.353×10^{-9}	Mzf_{ss}	-9.160×10^{-10}

Table 4.10: Forces and torques in steady-state at front wheel

Forces and Torques at rear wheel			
Force [N]		**Torque** [N m]	
Fxr_{ss}	30.00	Mxr_{ss}	-2.029×10^{-307}
Fyr_{ss}	3.399×10^{-9}	Mzr_{ss}	-7.867×10^{-11}

Table 4.11: Forces and torques in steady-state at rear wheel

is the driving force applied to keep the velocity of the vehicle constant.
The computed steady-state will became a useful starting point for both the integration of the dynamic and the solution of the optimal control.

Chapter 5

Optimal Control Problem

Once the model is fully formulated the OCP can be build around it. The focus of this thesis is to compute minimum time manoeuvres, therefore the OCP must be constructed following some specific steps depending on the method chosen. In this work the OCP is solved using indirect method through the software PINS [6]. In the following section the whole process is explained in details.

5.1 General OCP

In general the OCP problem is formulated in this way.

$$
\begin{aligned}
\underset{x(\cdot),u(\cdot)}{minimize} \quad & \mathcal{J}(x(t), u(t), t) \\
\text{subject to} \quad & \mathcal{F}(x'(t)x(t), u(t), t) = 0, \quad t \in [a, b] \quad \text{(Dynamic System)} \\
& b(x(a), x(b)) = 0, \quad \text{(Boundary Conditions)} \\
& h(x(t), u(t), t) = 0, \quad \text{(Equality constraints)} \\
& g(x(t), u(t), t) >= 0, \quad \text{(Inequality constraints)} \\
& x(t) \in \mathbb{R}^{n_s},\ u(t) \in \mathbb{R}^{n_c}
\end{aligned}
\tag{5.1}
$$

where $\mathcal{J}x(\cdot)$ is the cost function that in general is written like the following.

$$
\mathcal{J}(x(t), u(t), t) = \mathcal{M}(x(a), x(b)) + \int_a^b \mathcal{L}(x(t), u(t), t)\mathrm{d}t \tag{5.2}
$$

$\mathcal{F}(x'(t)x(t), u(t), t) = 0$ is the general expression for the system of ODE that can also be written in the form

$$
A(x(t), t)x'(t) = f(x(t), u(t), t) \tag{5.3}
$$

in which all derivatives of the states appear linearly. Consequently, one can collect the mass matrix $A(x(t), t)$ which is not always invertible. The system of ODE is implicit in this case. This make the direct method hard to use if not impossible.

$x(t)$ is the vector of states of dimension n_s and $u(t)$ is the vector of the controls of size n_c. The aim of this study is to compute minimum time manoeuvre. Therefore the extremes of integration became 0 and T instead of a and b. Moreover, the model constructed do not need any equality constraints. The lagrange function in this case is simple because of our interest in the minimum time problem.

$$\mathcal{L}(x(t), u(t), t) = 1 \tag{5.4}$$

This combined with neglecting the Mayer therm $\mathcal{M}(\cdot)$ (most of the time in unnecessary) yields a cost function such as:

$$\mathcal{J}(x(t), u(t), t) = \mathcal{M}(x(a), x(b)) + \int_0^T \mathrm{d}t = T \tag{5.5}$$

This lead to a simplified version of the problem

$$\begin{aligned} \underset{x(\cdot),u(\cdot)}{minimize} \quad & T \\ \text{subject to} \quad & \mathcal{F}(x'(t)x(t), u(t), t) = 0, \quad t \in [0, T] \quad \text{(Dynamic System)} \\ & b(x(0), x(T)) = 0, \quad \text{(Boundary Conditions)} \\ & g(x(t), u(t), t) >= 0, \quad \text{(Inequality constraints)} \\ & x(t) \in \mathbb{R}^{n_s},\ u(t) \in \mathbb{R}^{n_c} \end{aligned} \tag{5.6}$$

T is a parameter and the problem cannot be solved in this therm. However there are several techniques to obtain a solvable OCP. The transformation will be discussed in the following sections.

5.2 Curvilinear Coordinates

The curvilinear coordinate are a convenient set of coordinate to compute OCPs. This is a well known technique in literature. In this thesis the curvilinear coordinate used consider only a motion on plane (2D). However it can be extended to the three-dimensional space as did by Leonelli and Limebeer [3].

The curvilinear coordinates ($s(t)$, $n(t)$,$\xi(t)$) can be mapped to quasi states can be obtained defining a reference frame mobile. This is translated with respect to ground of coordinate $xm(t)$ and $ym(t)$ and rotated around the vertical direction z of an angle $\theta(t)$ (Figure 5.2.1).

$$\begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) & 0 & xm(t) \\ \sin(\theta(t)) & \cos(\theta(t)) & 0 & ym(t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.7}$$

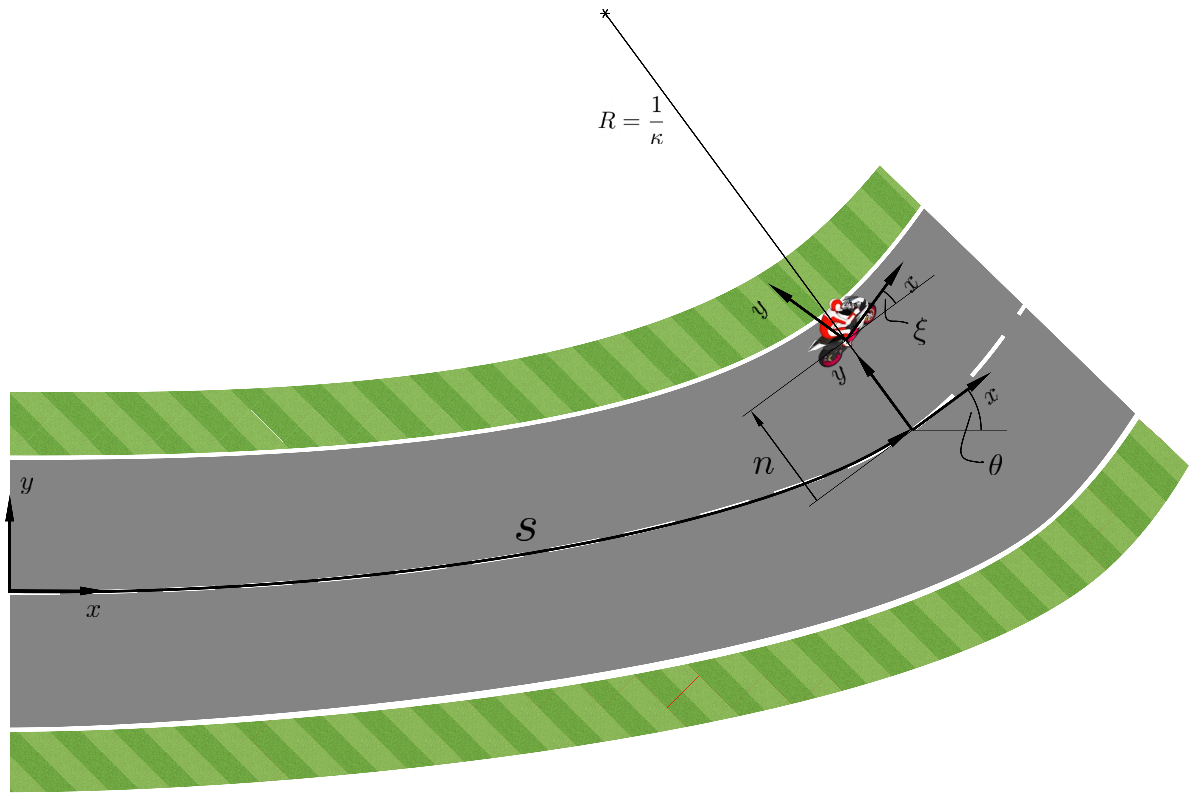


Figure 5.2.1: Curvilinear coordinates

θ is different from the pitch angle of the previous chapter and will be dropped in the next few passages.

From the reference frame previously defined one can define a moving reference frame with longitudinal velocity $\frac{\mathrm{d}}{\mathrm{d}t}s(t)$ and angular velocity $\frac{\mathrm{d}}{\mathrm{d}t}s(t)\kappa(t)$. The last term represent the linear velocity divided by the radius of curvature. In fact κ is the curvature of the curvilinear coordinate. κ is not a direct function of time. It depends on the curvilinear coordinate s, $\kappa(t) = \kappa(s) = \kappa(s(t))$. The velocity of this reference frame are

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}s\left(t\right) - \cos\left(\theta\left(t\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}xm\left(t\right) - \sin\left(\theta\left(t\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}ym\left(t\right) \\ \sin\left(\theta\left(t\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}xm\left(t\right) - \cos\left(\theta\left(t\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}ym\left(t\right) \\ \kappa\left(t\right)\frac{\mathrm{d}}{\mathrm{d}t}s\left(t\right) - \frac{\mathrm{d}}{\mathrm{d}t}\theta\left(t\right) \end{bmatrix} \tag{5.8}$$

From the moving reference frame above mentioned a translation and a rotation are performed. Specifically a lateral translation of $n(t)$ and a rotation of $\xi(t)$. The origin of this reference frame is in fact the position of the motorcycle in the space.

$$\begin{bmatrix} \cos\left(\xi\left(t\right)\right) & -\sin\left(\xi\left(t\right)\right) & 0 & 0 \\ \sin\left(\xi\left(t\right)\right) & \cos\left(\xi\left(t\right)\right) & 0 & n(t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.9}$$

Thus, the velocity of this point can be mapped to the velocity of the vehicle $u(t)$, $v(t)$, $\Omega(t)$. This can be obtained by defining a vector in the reference frame in equation 5.9 with components $u(t)$ and $v(t)$ and the angular velocity as

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) - \frac{\mathrm{d}}{\mathrm{d}t}\theta(t) - \frac{\mathrm{d}}{\mathrm{d}t}\xi(t) \tag{5.10}$$

with $\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \Omega(t)$ and $\theta(t) = \theta(s(t))$. Moreover the time derivative of θ can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(s(t)) = \frac{\mathrm{d}}{\mathrm{d}t}s(t)\frac{\mathrm{d}}{\mathrm{d}s}\theta(s) = \frac{\mathrm{d}}{\mathrm{d}t}s(t)\kappa(s(t)) \tag{5.11}$$

The terms $xm(t)$, $ym(t)$ and $\theta(t)$ are no more in the expression and the final equations for the curvilinear coordinates are

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}t}s\left(t\right)\right)\left(\kappa\left(s\left(t\right)\right)n\left(t\right)-1\right)+u\left(t\right)\cos\left(\xi\left(t\right)\right)-v\left(t\right)\sin\left(\xi\left(t\right)\right) \\ -\frac{\mathrm{d}}{\mathrm{d}t}n\left(t\right)+u\left(t\right)\sin\left(\xi\left(t\right)\right)+v\left(t\right)\cos\left(\xi\left(t\right)\right) \\ \Omega\left(t\right)-\kappa\left(s\left(t\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}s\left(t\right)-\frac{\mathrm{d}}{\mathrm{d}t}\xi\left(t\right) \end{cases} \tag{5.12}$$

Collecting all the differential parts

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}s\left(t\right) = -\frac{u\left(t\right)\cos\left(\xi\left(t\right)\right)-v\left(t\right)\sin\left(\xi\left(t\right)\right)}{\kappa\left(s\left(t\right)\right)n\left(t\right)-1} \\ \frac{\mathrm{d}}{\mathrm{d}t}n\left(t\right) = u\left(t\right)\sin\left(\xi\left(t\right)\right)+v\left(t\right)\cos\left(\xi\left(t\right)\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\xi\left(t\right) = \frac{\kappa\left(s\left(t\right)\right)u\left(t\right)\cos\left(\xi\left(t\right)\right)-\kappa\left(s\left(t\right)\right)v\left(t\right)\sin\left(\xi\left(t\right)\right)+\kappa\left(s\left(t\right)\right)n\left(t\right)\Omega\left(t\right)-\Omega\left(t\right)}{\kappa\left(s\left(t\right)\right)n\left(t\right)-1} \end{cases} \tag{5.13}$$

The last two equation will be integrated in the system of ODE for the OCP while the first will have a central role in the following section.

5.3 Coordinate change

There are multiple ways to transform equation 5.6 in a problem that can be solved.
The most common one is to make a coordinate change passing from time domain to another convenient domain. In the first section (5.3.1) is explained a change of variable from the time to the unitary interval $[0, 1]$. However, for the purpose of this study, it is convenient to perform a change from the time domain to the space one. It is similar to the latter, but it involves the

definition of curvilinear coordinates. It is explained in the second subsection 5.3.2.

5.3.1 Time to unitary space transformation

$$t = \zeta T(\zeta) \tag{5.14}$$

with $\zeta \in [0,1]$ $T(\zeta)$ is now a new state and need his own ODE. Since it is a constant his time derivative will be zero.

$$T'(\zeta) = 0 \tag{5.15}$$

with its own initial condition

$$T(0) = 0 \tag{5.16}$$

The coordinate change transform the cost function

$$\mathcal{J}(x(t), u(t), t) = \int_0^T \mathrm{d}t = \int_0^1 T(\zeta)\mathrm{d}\zeta = \mathcal{J}(x(\zeta), u(\zeta), \zeta) \tag{5.17}$$

The system of ODE is affected by the coordinate change as well. In fact, the differential part take into account the Jacobian of the changed coordinate. The states are now function of ζ which is a function of time t thus

$$\frac{\mathrm{d}}{\mathrm{d}t}x(\zeta(t)) = \frac{\mathrm{d}}{\mathrm{d}\zeta}x(\zeta(t))\frac{\mathrm{d}\zeta}{\mathrm{d}t} = x'(\zeta)\frac{1}{T(\zeta)} \tag{5.18}$$

This means that the system of ODE became

$$A(x(\zeta), \zeta)x'(\zeta) = T(\zeta)f(x(\zeta), u(\zeta), \zeta) \tag{5.19}$$

5.3.2 Time-Space Coordinate Change

The idea of the time-space coordinate change is to find the relationship between the time variable t and the space coordinate s. Thus the differential ds can be be expressed as

$$\mathrm{d}s = \frac{\mathrm{d}s}{\mathrm{d}t}dt = \dot{s}dt \tag{5.20}$$

Where $\dot{s}$ is exactly the time differential of the curvilinear coordinate in the equation 5.13.

$$\dot{s} = \frac{u(t)\cos(\xi(t)) - v(t)\sin(\xi(t))}{1 - \kappa(s(t))n(t)} \tag{5.21}$$

Therefore the cost function with the substitution of the new variable an reassigning ζ as the space variable (only for notation consistence).

$$\mathcal{J}(x(\zeta), u(\zeta), \zeta) = \int_{\zeta_i}^{\zeta_f} \frac{1}{\dot{s}}\mathrm{d}\zeta = \int_{\zeta_i}^{\zeta_f} \frac{1 - \kappa(\zeta)n(\zeta)}{u(\zeta)\cos(\xi(\zeta)) - v(\zeta)\sin(\xi(\zeta))}\mathrm{d}\zeta \tag{5.22}$$

It is clear that the change of variable affect all the differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}t}x(s(t)) = \frac{\mathrm{d}}{\mathrm{d}s}x(s(t))\frac{\mathrm{d}s}{\mathrm{d}t} = x'(s)\dot{s} \tag{5.23}$$

That with the substitution of ζ became

$$\frac{\mathrm{d}}{\mathrm{d}t}x(\zeta(t)) = x'(\zeta)\frac{u(\zeta)\cos(\xi(\zeta)) - v(\zeta)\sin(\xi(\zeta))}{1 - \kappa(\zeta)n(\zeta)} \tag{5.24}$$

5.4 Additional dynamics

5.4.1 Torque

5.4.2 Slip

5.5 Constraints

5.5.1 Feasible Values

5.5.2 Track

5.5.3 Convergence and Unrealistic Manoeuvre

5.6 Minimum Time

Chapter 6

Analysis of results

Conclusions

Bibliography

[1] A. Maria, "Introduction to modeling and simulation," in *Winter simulation conference*, vol. 29, pp. 7–13, 1997.

[2] R. Sharp, "A method for predicting minimum-time capability of a motorcycle on a racing circuit," *Journal of Dynamic Systems, Measurement, and Control*, vol. 136, no. 4, p. 041007, 2014.

[3] L. Leonelli and D. Limebeer, "Optimal control of a road racing motorcycle on a three-dimensional closed track," *Vehicle System Dynamics*, pp. 1–25, 2019.

[4] V. Cossalter, M. Da Lio, R. Lot, and L. Fabbri, "A general method for the evaluation of vehicle manoeuvrability with special emphasis on motorcycles," *Vehicle system dynamics*, vol. 31, no. 2, pp. 113–135, 1999.

[5] B. Simon, C. Vittore, M. Matteo, and P. Martino, "Application of the "optimal maneuver method" for enhancing racing motorcycle performance," *SAE International Journal of Passenger Cars-Mechanical Systems*, vol. 1, no. 2008-01-2965, pp. 1311–1318, 2008.

[6] E. Bertolazzi, F. Biral, and M. Da Lio, "Symbolic-numeric efficient solution of optimal control problems for multibody systems," *Journal of computational and applied mathematics*, vol. 185, no. 2, pp. 404–421, 2006.

[7] H. Pacejka, *Tyre and Vehicle Dynamics*. Automotive engineering, Butterworth-Heinemann, 2006.

[8] V. Cossalter, R. Lot, and M. Massaro, "The influence of frame compliance and rider mobility on the scooter stability," *Vehicle System Dynamics*, vol. 45, no. 4, pp. 313–326, 2007.

[9] H. Pacejka and I. Besselink, *Tire and Vehicle Dynamics*. Elsevier Science, 2012.

[10] E. Bertolazzi, F. Biral, and M. Da Lio, "Symbolic–numeric indirect method for solving optimal control problems for large multibody systems," *Multibody System Dynamics*, vol. 13, no. 2, pp. 233–252, 2005.

[11] F. Biral, E. Bertolazzi, and P. Bosetti, "Notes on numerical methods for solving optimal control problems," *IEEJ Journal of Industry Applications*, vol. 5, no. 2, pp. 154–166, 2016.

[12] A. V. Rao, "A survey of numerical methods for optimal control," *Advances in the Astronautical Sciences*, vol. 135, no. 1, pp. 497–528, 2009.

[13] H. S. Rodrigues, M. T. T. Monteiro, and D. F. Torres, "Optimal control and numerical software: an overview," *arXiv preprint arXiv:1401.7279*, 2014.

[14] M. A. Patterson and A. V. Rao, "Gpops-ii: A matlab software for solving multiple-phase optimal control problems using hp-adaptive gaussian quadrature collocation methods and sparse nonlinear programming," *ACM Transactions on Mathematical Software (TOMS)*, vol. 41, no. 1, pp. 1–37, 2014.

[15] B. Houska, H. Ferreau, and M. Diehl, "ACADO Toolkit – An Open Source Framework for Automatic Control and Dynamic Optimization," *Optimal Control Applications and Methods*, vol. 32, no. 3, pp. 298–312, 2011.

[16] J. A. E. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, "CasADi – A software framework for nonlinear optimization and optimal control," *Mathematical Programming Computation*, vol. 11, no. 1, pp. 1–36, 2019.

[17] A. Wächter and L. T. Biegler, "On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming," *Mathematical programming*, vol. 106, no. 1, pp. 25–57, 2006.

[18] R. E. Bellman and S. E. Dreyfus, *Applied dynamic programming*. Princeton university press, 2015.

[19] D. Mayne, "A second-order gradient method for determining optimal trajectories of nonlinear discrete-time systems," *International Journal of Control*, vol. 3, no. 1, pp. 85–95, 1966.

[20] W. Huang, X. Wu, Q. Zhang, N. Wu, and Z. Song, "Trajectory optimization of autonomous driving by differential dynamic programming," in *2014 13th International Conference on Control Automation Robotics & Vision (ICARCV)*, pp. 1758–1763, IEEE, 2014.

[21] N. Dal Bianco, E. Bertolazzi, F. Biral, and M. Massaro, "Comparison of direct and indirect methods for minimum lap time optimal control problems," *Vehicle System Dynamics*, vol. 57, no. 5, pp. 665–696, 2019.

[22] V. Cossalter, M. Peretto, and S. Bobbo, "Investigation of the influences of tyre—road friction and engine power on motorcycle racing performance by means of the optimal manoeuvre method," *Proceedings of the Institution of Mechanical Engineers, Part D: Journal of Automobile Engineering*, vol. 224, no. 4, pp. 503–519, 2010.

[23] R. Lot, M. Massaro, and R. Sartori, "Advanced motorcycle virtual rider," *Vehicle System Dynamics*, vol. 46, no. S1, pp. 215–224, 2008.

[24] V. Cossalter, R. Lot, and D. Tavernini, "Optimization of the centre of mass position of a racing motorcycle in dry and wet track by means of the "optimal maneuver method"," in *2013 IEEE International Conference on Mechatronics (ICM)*, pp. 412–417, IEEE, 2013.

[25] M. Massaro and R. Lot, "A virtual rider for two-wheeled vehicles," in *49th IEEE Conference on Decision and Control (CDC)*, pp. 5586–5591, IEEE, 2010.

[26] R. Sharp, S. Evangelou, and D. J. Limebeer, "Advances in the modelling of motorcycle dynamics," *Multibody system dynamics*, vol. 12, no. 3, pp. 251–283, 2004.

[27] J. Baumgarte, "Stabilization of constraints and integrals of motion in dynamical systems," *Computer methods in applied mechanics and engineering*, vol. 1, no. 1, pp. 1–16, 1972.

[28] R. Lot, "A motorcycle tire model for dynamic simulations: Theoretical and experimental aspects," *Meccanica*, vol. 39, no. 3, pp. 207–220, 2004.

[29] V. Cossalter and R. Lot, "A motorcycle multi-body model for real time simulations based on the natural coordinates approach," *Vehicle system dynamics*, vol. 37, no. 6, pp. 423–447, 2002.

[30] V. Cossalter, R. Lot, and F. Maggio, "A multibody code for motorcycle handling and stability analysis with validation and examples of application," tech. rep., SAE Technical Paper, 2003.

[31] "multibody.net." http://www.multibody.net/. (Accessed on 02/05/2020).

Appendix

Data and Parameters

Motorcycle Data

Tyre data

Numeric Solution

Static conditions

Steady-State conditions