# UNIVERSITY OF TRENTO

## Department of Industrial Engineering

### Master Degree in Mechatronics Engineering


# Influence of Slip Versus Torque Control Formulation on Minimum Time Manoeuvres of a Motorcycle

**Academic Year 2018/2019**


Supervisor:
Prof. Francesco Biral.

Graduant:
Mattia Piazza



$17^{th}$ march 2019

# Acknowledgements

Ringrazio tutti

# Contents

# Introduction

The simulation of mechanical and mechatronics systems allow to test and validate the design in a safe and efficient environment without the need to build the physical object and measure the parameters. The simulation is based on digital technology with major benefits as cost and efficiency and the possibility of an easy reconfiguration and retesting of a system which is usually impossible or infeasible for real model in terms of cost and time. [1]

A large number of vehicle model are available in scientific literature some are fairly complex and other are simple. Depending on the application and the time constraints a proper model should be chosen. Simple models are faster and therefore suitable for real-time purposes while complex models are time-consuming and are used in the case where the model cannot be simplified or the goal of the study is to replicate in detail the behaviour of the analysed system.

The work of this thesis aims to derive three different models of racing motorcycle with increasing complexity taking into account the force exchange between tyre and ground using Pacejka's magic formula [2]. The models will then be used to calculate controls and trajectory to achieve the minimum lap time of a specific circuit.

In particular, the first model of the motorcycle will represent a vehicle with fixed suspensions meaning that the rear swingarm and the steering fork have respectively a fixed angle and a fixed length. The motorcycle is controlled with the steering torque and the longitudinal slip of front and rear wheel.

The second model has again fixed suspensions, however, the vehicle is controlled with steering torque, braking torque at the front wheel and braking/traction torque at the rear.

The third model takes into account the internal motion due to suspension deformation and the motorcycle is again controlled with torques.

All the model are derived with the multi-body approach and symbolic formulation. In order to achieve this goal, the model is defined using Maple, a software well known for its capability in symbolic computation. Moreover, the equations of motion are obtained using MBSymba which is a custom free library for Maple available online at `http://www.multibody.net`.

All three models are derived in a similar way as in the publication of Cossalter *et al* [3], without taking into account the lateral flexibility of the front torque and the torsional flexibility of the swingarm. The tyre forces are derived using the Pacejka's magic formula [2] which is an empirical formula obtained from the assumption of similarity.

The minimum time trajectory and control is computed formulating a custom optimal control problem using XOptima package for Maple and then solving with PINS (acronym for *PINS Is Not a Solver*). The first is a library developed to transform the symbolic model of the vehicle (DAE), constraints, and target functions in C++ code that can be used by

PINS. PINS is a software, free for academic purposes, developed at University of Trento by Prof. Bertolazzi, Prof. Biral and Prof. Bosetti that can solve optimal control problems (OCPs) with indirect method. As far as the author knows, there are not other optimal control solvers that exploit Pontryagin maximum principle and calculus of variations to solve the problem with the indirect method. [4]
The thesis is organized in the following way. In the first chapter, there is a brief overview of the state of the art in optimal control, motorcycle dynamic model and minimum time application. In the second chapter, there is a description of how the motorcycle models are derived and the optimal control problem. The third chapter confronts the results of the three models. The thesis is ended with the conclusion, references and appendix.

# Chapter 1

# State of art on minimum lap time application in racing motorcycle

## 1.1 State of the art on Optimal Control Problems

Optimal control problem, also known as dynamic optimisation, are minimisation problem where the variables and parameters change with time. Dynamic systems are characterized by the states and often are controlled by a convenient choice of inputs (controls).
Dynamic optimisation aims to compute those controls and states for a dynamic system over a time interval to minimise one or more performance indexes. In other words, the input is chosen to optimize (minimize) an objective function while complying to constraint equations.

### 1.1.1 Optimal Control Problems

Optimal control problems are challenging from the theoretical point of view and of practical interest. However due to dimensionality and complexity of system of equations the application in real problems and industrial environment is still not so widespread.
In general, OPC can be continuous or discrete, linear or non-linear, time-variant or time-invariant. However, in this thesis are addressed only optimal control problems that are continuous time-variant and highly non-linear. Those properties will be discussed in the following sections. In general, there are four main approaches to solve continuous-time OPC: state space approach, direct methods, indirect methods and differential dynamic programming.

#### State-space approaches

State-space approaches follow the principle of optimality for which each sub-arc of an optimal trajectory must be optimal. In literature, those are referred to as Hamilton-Jacobi-Bellman (HJB) equation. However, the problem needs numerical methods to be solved, moreover, a solution can be found only for small dimension problems due to *course of dimensionality*. There is no practical application of this method to solve highly non-linear problem as a dynamic optimisation of a motorcycle model.

#### Direct Method

Direct methods discretise the original optimal control problem into a non linear programming problem (NLP). In other words, the OPC is transformed in a discrete-time system that can be solved using numerical schemes and optimization techniques, namely Initial Value Solver (IVS) and Sequential Quadratic Programming (SQP) [5] The main advantage of direct methods is the possibility to use inequality constraints even in case of change in the constraints active set ( activation/deactivation) [6]
Direct methods are easier to implement compared to the other three categories and this is one of the reasons why they are by far the most widespread. In fact, almost 90% of the available optimal control software rely on direct method. [7, 8]

#### Indirect Method

Indirect methods exploits the necessary condition of optimality to derive a boundary value problem (BVP) in ordinary differential equations(ODE). Therefore the BVP can be solved numerically as a non linear problem. The indirect method allow to first optimize and then discretise meaning that the problem con be firstly written in continuous time and discretised later using different discretisation techniques. The class of indirect methods exploits the well known calculus of variations and the Euler-Lagrange differential equations, and the so-called Pontryagin Maximum Principle. [4]
The numerical solution can be computed either by shooting techniques (single/multiple shooting) or by collocation. The major drawbacks of indirect methods are that the problem could be difficult to solve or unstable due to the nature of the underlying differential equations (non linearity and instability) and the changes in control structure (active constraints in specific arcs). Moreover, in some arcs, singularity arises therefore the DAE index increase leading to the necessity of specialized solution techniques. [6]

### 1.1.2 Minimum time Optimal Control Problem

### 1.1.3 PINS

## 1.2 Development in motorcycle dynamic

# Chapter 2

# Motorcycle model

There are multiple way to derive describe the behaviour of the motorcycle and to derive the equation of motion. Some use the lagrangian approach to use a minimum set of coordinates [2, 9, 10]. Other derive the equation of motion using Newton-Euler equations. In this thesis the dynamic model of the motorcycle has been derived using a multi-body approach and assuming the ISO convention for the orientation of the $z$-axis (upward). To this end multiple reference frames, points, bodies, forces and torques are defined in the following sections. The model was derived symbolically using Maple and MBSymba that deals with rotation and transformation matrices using homogeneous coordinates. In particular, working on the model, the author chose to derive the kinematics using a combination of global and recursive approach in order to use a minimum set of coordinates while containing the size of the derived equations.

## 2.1 Kinematic model

### 2.1.1 Reference frames

A common choice is to start by defining a reference frame in movement with respect to the ground, or fixed, one. This reference frame have in general three linear and three angular velocities. However, for the purpose of this thesis, we consider only planar roads. This means that we need to define the movement of a frame in a plane, therefore only three degrees of freedom are needed (three velocities). The moving reference frame ia addressed as $RF_1$ and has velocities $u(t)$, $v(t)$ and $\Omega(t)$. This set of velocities, also called quasi-coordinates, are suitable to be used later in the definition of curvilinear coordinate. $RF_1$ has the $x$-axis aligned with the direction of motion of the motorcycle.
The frame $RF_\phi$ is the reference frame attached to a plane rotated of an angle $\phi(t)$ commonly addressed as rolling angle around the moving $x$-axis and it is obtained recursively multiplying $RF_1$ for a rotation matrix.

$$RF_\phi = RF_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi(t)) & -\sin(\phi(t)) & 0 \\ 0 & \sin(\phi(t)) & \cos(\phi(t)) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

Then a reference frame attached to the joint between the swingarm and the rear frame is defined with a translation in the vertical direction of the rolled frame of a certain height

$h(t)$ and a rotation around the rolled $y$-axis of an angle $\theta(t)$ plus the caster angle $\epsilon$. The new frame will be from here addressed as $RF_{Rear}$.

$$RF_{Rear} = RF_{\phi} \begin{bmatrix} \cos(\theta(t)+\epsilon) & 0 & -\sin(\theta(t)+\epsilon) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta(t)+\epsilon) & 0 & \cos(\theta(t)+\epsilon) & h(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

From $RF_{Rear}$ frame one can define the reference frame attached to the swingarm and the one attached to the steering assembly. The first is obtained with a rotation around the $y$-direction of the previous reference frame of a relative angle $eta(t)$ and a translation of the length of the swingarm. The advantage of choosing this relative angle is that there are already define relationship between this rotation an the force of the suspension. The new reference frame is addressed as $RF_{\eta}$ and will coincide with the centre of the rear wheel.

$$RF_{\eta} = RF_{Rear} \begin{bmatrix} \cos(\eta(t)) & 0 & \sin(\eta(t)) & -\cos(\eta(t))\, L_{swa} \\ 0 & 1 & 0 & 0 \\ -\sin(\eta(t)) & 0 & \cos(\eta(t)) & \sin(\eta(t))\, L_{swa} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

The second reference frame, the one attached to the steering assembly has already the $z$-axis with the same direction of the rotation axis of the steer. Therefore, $RF_{\delta}$ can be obtained with a translation of a fixed quantity in the $x$ direction and a rotation of an angle $\delta(t)$ around the rotation axis $z$. This steering angle is small for racing motorcycle and it is always smaller than 10 degrees therefore can be linearised from here since the equation of motion are derived using Newton-Euler approach instead of Lagrange.

$$RF_{\delta} = RF_{Rear} \begin{bmatrix} 1 & -\delta(t) & 0 & L_b \\ \delta(t) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.4}$$

The reference frame attached to the centre of the front wheel is defined as $RF_{susp}$ and it is obtained with a translation in the negative vertical direction of a fixed quantity $s_{fs}$ plus a time-varying $s_f(t)$ which is the deformation of the suspension and a translation in the $x$ direction of $x_{off}$, an offset always present in the suspension fork.

$$RF_{susp} = RF_{\delta} \begin{bmatrix} 1 & 0 & 0 & x_{\mathit{off}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -s_{fs} + s_f(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.5}$$

In order to have a simplified model one can introduce two new reference frames one for the front and one for the rear wheel starting from $RF_1$. $RF_{FW}$ is defined with a translation of the components $x_f(t)$, $y_f(t)$ and $z_f(t)$ and a rotation of an angle $\delta_f(t)$ around the vertical direction and then one around the new longitudinal direction of angle $\phi_f(t)$. Once again

$\delta_f(t)$, which is the steering angle projected to the horizontal plane, is small and can be linearised here.

$$RF_{FW} = RF_1 \begin{bmatrix} 1 & -\delta_f(t)\cos(\phi_f(t)) & \delta_f(t)\sin(\phi_f(t)) & x_f(t) \\ \delta_f(t) & \cos(\phi_f(t)) & -\sin(\phi_f(t)) & y_f(t) \\ 0 & \sin(\phi_f(t)) & \cos(\phi_f(t)) & z_f(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6}$$

The second reference fram $RF_{RW}$ is defined with a translation of the components $x_r(t)$, $y_r(t)$ and $z_r(t)$ and a rotation of an angle $\phi_r(t)$ around the $x$-axis. However since there is no rotation in other planes $\phi_r(t) = \phi(t)$.

$$RF_{RW} = RF_1 \begin{bmatrix} 1 & 0 & 0 & -x_r(t) \\ 0 & \cos(\phi(t)) & -\sin(\phi(t)) & y_r(t) \\ 0 & \sin(\phi(t)) & \cos(\phi(t)) & z_r(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

Those last two reference frames are attached to the wheels centre. One should also define the reference frame that is spinning by multiplying with a rotation matrix of an angle respectively $\theta_r$ and $\theta_f$.

$$RF_{FWspin} = RF_{FW} \begin{bmatrix} \cos(\theta_f(t)) & 0 & \sin(\theta_f(t)) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_f(t)) & 0 & \cos(\theta_f(t)) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

$$RF_{RWspin} = RF_{RW} \begin{bmatrix} \cos(\theta_r(t)) & 0 & \sin(\theta_r(t)) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_r(t)) & 0 & \cos(\theta_r(t)) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.9}$$

Those angles will not appear directly in the equation of motion. However, their first and second derivative will. Later on $\theta_r$ and $\theta_f$ will be substituted as variable by their derivative, the angular velocities $\omega_r$ and $\omega_f$.

One can chose to model the interaction of tyre and asphalt as a pure contact and therefore define two constraints equations, one for each wheel. However, the goal of this thesis is to compute the optimal control and those constraints are not linear and analytically unsolvable. This leads to a system of equation that is a Differential Algebraic Equation (DAE). This can be modelled in two ways. The first is by imposing a penalty in the cost function that minimize those constraints, while the second is deriving the constraints, find an ODE and incorporate the algebraic constrains with a Baumgarte stabilization [11]. However, those two methods leads to an enlarged problem that should be solved by the optimisation.

In this thesis, the chosen solution is to treat the constraints as soft. As did in literature [10] the contact is imposed using a force which is proportional to the penetration and the penetration velocity as shown in the next section. For this reason four points are defined in order to get the slip velocities and the penetration of the wheels with the ground. All of those points are defined starting from the reference frame centred in the wheels $RF_{FW}$ and $RF_{RW}$. The points used for the computation of the slips are:

$$P_r = \begin{bmatrix} 0 \\ 0 \\ -rr + rtr - \frac{rtr}{\cos(\phi(t))} \\ 1 \end{bmatrix} \quad \text{in } RF_{RW} \tag{2.10}$$

$$P_f = \begin{bmatrix} 0 \\ 0 \\ -rf + rtf - \frac{rtf}{\cos\left(\phi_f(t)\right)} \\ 1 \end{bmatrix} \quad \text{in } RF_{FW} \tag{2.11}$$

While the contact points are defined assuming that the shape of the tyre is a torus.

$$C_r = \begin{bmatrix} 0 \\ -\sin(\phi(t))\, rtr \\ -rr + rtr - rtr\, \cos(\phi(t)) \\ 1 \end{bmatrix} \quad \text{in } RF_{RW} \tag{2.12}$$

$$C_f = \begin{bmatrix} 0 \\ -\sin(\phi_f(t))\, rtf \\ -rf + rtf - \cos(\phi_f(t))\, rtf \\ 1 \end{bmatrix} \quad \text{in } RF_{FW} \tag{2.13}$$

Where $rr$, $rf$, $rtr$, $rtf$ are respectively the radius of the rear and front wheel and the radii of the section.
DISEGNI DEI PUNTI

### 2.1.2 Kinematic solution

Not all the variable introduced are state variable nor are needed for the final purpose of this thesis. Those have been introduced just to simplify the formulation. However, they can be solved now by imposing some constraints. $x_f(t)$, $y_f(t)$ and $z_f(t)$ can be solved by imposing that the point of the origin of $RF_{susp}$ is equal to the origin of $RF_{FW}$. This yield 3 algebraic equation in 3 variable.

$$\begin{aligned}
x_f(t) &= (L_b + x_{off})\cos(\theta(t)+\epsilon) + \sin(\theta(t)+\epsilon)(s_{fs} - s_f(t)) \\
y_f(t) &= \sin(\phi(t))(s_{fs} - s_f(t))\cos(\theta(t)+\epsilon) - \sin(\phi(t))(L_b + x_{off})\sin(\theta(t)+\epsilon) + \dots \\
&\qquad \dots + \delta(t)\cos(\phi(t))\,x_{off} - \sin(\phi(t))\,h(t) \\
z_f(t) &= -\cos(\phi(t))(s_{fs} - s_f(t))\cos(\theta(t)+\epsilon) + \cos(\phi(t))(L_b + x_{off})\sin(\theta(t)+\epsilon) + \dots \\
&\qquad \dots + \delta(t)\sin(\phi(t))\,x_{off} + \cos(\phi(t))\,h(t)
\end{aligned} \tag{2.14}$$

The same can be said for the rear wheel. The origin of $RF_\eta$ is the same point as the origin of $RF_{RW}$ yielding the following.

$$\begin{aligned}
x_r(t) &= (\cos(\theta(t)+\epsilon)\cos(\eta(t)) + \sin(\theta(t)+\epsilon)\sin(\eta(t)))\,L_{swa} \\
y_r(t) &= -\sin(\phi(t))\,(L_{swa}\,\sin(\eta(t))\cos(\theta(t)+\epsilon) - L_{swa}\,\cos(\eta(t))\sin(\theta(t)+\epsilon) + h(t)) \\
z_r(t) &= \cos(\phi(t))\,(L_{swa}\,\sin(\eta(t))\cos(\theta(t)+\epsilon) - L_{swa}\,\cos(\eta(t))\sin(\theta(t)+\epsilon) + h(t))
\end{aligned} \tag{2.15}$$

The rotation angles $\delta_f(t)$ and $\phi_f(t)$ can be solved as a function of the other states. The constrain equation can be multiple and each formulation are equal. One can impose orthogonality between reference frames unit vectors or can impose the equality of such vectors. Both ways lead to the following solution.

$$\begin{aligned}\phi_f\left(t\right) &= -\arcsin\left(\cos\left(\phi\left(t\right)\right)\ \sin\left(\theta\left(t\right)+\epsilon\right)\ \delta\left(t\right)-\sin\left(\phi\left(t\right)\right)\right)\\ \delta_f\left(t\right) &= \frac{\cos\left(\theta\left(t\right)+\epsilon\right)\ \delta\left(t\right)}{\sin\left(\phi\left(t\right)\right)\ \sin\left(\theta\left(t\right)+\epsilon\right)\ \delta\left(t\right)+\cos\left(\phi\left(t\right)\right)}\end{aligned} \tag{2.16}$$

This can be further simplified keeping in mind that we are considering small angles for $delta(t)$.

$$\begin{aligned}\phi_f\left(t\right) &= -\delta\left(t\right)\sin\left(\theta\left(t\right)+\epsilon\right)+\phi\left(t\right)\\ \delta_f\left(t\right) &= \frac{\cos\left(\theta\left(t\right)+\epsilon\right)\delta\left(t\right)}{\cos\left(\phi\left(t\right)\right)}\end{aligned} \tag{2.17}$$

The values obtained will be later substituted in the equation of motion.

### 2.1.3 Tyre-Ground penetration

As previously introduced in this thesis, the contact with ground is modelled as a soft constraint using a force which will be a function of the penetration and penetration velocity. The tyre is in this case in equivalent to a spring-damper system. [9,10] The penetration can be obtained by evaluating the vector joining the origin of the $RF_1$ supposed on ground and the two points $C_f$ and $C_r$ at the surface of the torus. The $z$ component of those vectors give a measure of how much the tyre is deformed.

$$\begin{aligned}p_r(t) &= -\text{comp\_Z}(O_{RF1}C_r, RF_1)\\ p_f(t) &= -\text{comp\_Z}(O_{RF1}C_f, RF_1)\end{aligned} \tag{2.18}$$

The minus sign is present because a positive penetration will generate a positive contact force. comp_Z indicate the component of the vector in the $z$ direction. The results are not reported here because too long to display.

### 2.1.4 Longitudinal and lateral slips

Longitudinal and lateral slips are defined following the definition of practical slip [2, 12]. The lateral slips are the arctangent of the ratio between longitudinal and lateral velocity of the wheel.

$$\begin{aligned}\alpha_r(t) &= -\arctan\left(\frac{\text{comp\_Y}(VP_r, RF_1)}{\text{comp\_X}(VP_r, RF_1)}\right)\\ \alpha_f(t) &= -\arctan\left(\frac{\text{comp\_Y}(VP_f, RF_1\cdot R_{\delta_f})}{\text{comp\_X}(VP_f, RF_1\cdot R_{\delta_f})}\right)\end{aligned} \tag{2.19}$$

The longitudinal slip, instead, is defines as the difference between longitudinal velocity and velocity of the point on the wheel divided by the longitudinal velocity.

$$\begin{aligned}\lambda_r(t) &= -\frac{(\text{comp\_X}, \text{RF}_1(VP_r) - \omega_r(t)rr)}{\text{comp\_X}(VP_r, RF_1)}\\ \lambda_f(t) &= -\frac{(\text{comp\_X}(VP_f, RF_1\cdot R_{\delta_f}) - \omega_f(t)rf)}{\text{comp\_X}(VP_f, RF_1\cdot R_{\delta_f})}\end{aligned} \tag{2.20}$$

$\omega_r(t)$ and $\omega_f(t)$ are the angular velocities of the wheels also addressed as the time derivative of angles $\theta_r(t)$ and $\theta_f(t)$. $VP_r$ and $VP_r$ are the time derivative of $P_r$ and $P_r$. It is important to notice that for the front wheel one should project the velocity vector in the reference frame rotated of the angle $\delta_f(t)$ with respect to the reference frame $RF_1$. The results are not reported here because too long to display ($RF_1 \cdot R_{\delta_f}$).

## 2.2 Dynamic Model

In the previous section all variables describing the motion have been introduced. One can globally consider how many degrees of freedom the motorcycle will have in the space. The vehicle as a body will have 6 DoF. The first 3 are translations identified in the quasi-coordinate $u(t)$, $v(t)$ (longitudinal and lateral velocity) and the vertical translation $h(t)$. The other DoF are the rotation around 3 axis. The first is around the $z$ direction and identified by the quasi-coordinate $\Omega(t)$ (yaw rate) while the others are angle $\phi(t)$ (roll) and $\theta(t)$(pitch).
In addition to those "external" degrees of freedom, the motorcycle is described by a set of "internal" variables. The word internal is used because the variables describe a motion between parts of the motorcycle. Fist of all the degree of freedom of the steer ($\delta(t)$). Then there is the motion of the front suspension $s_f(t)$ and the one of the rear suspension ($\eta(t)$). Finally the two DoF of the spinning wheels, $\omega_r(t)$ and $\omega_f(t)$.
From the previous description about DoF it is clear that the motorcycle model have 11 degrees of freedom, therefore 11 equation of motion are needed.

### 2.2.1 Motorcycle rigid body

One techniques to write equation of motion in an efficient way is to define a body that describe the whole motorcycle as a rigid body and then add only the dynamic contribute of the internal motion of the other bodies.
The motorcycle as a rigid body is composed by the following bodies:

- the rear frame (main body)
- the driver
- the steering assembly (fork)
- the unsprung mass at the end of the front suspension
- the swingarm
- the front wheel
- the rear wheel

In order to describe the motorcycle all the internal degrees of freedom should be fixed. This means that the following substitution should be made for the next calculations.

$$\theta\left(t\right)=\theta_{00},\ h\left(t\right)=h_{00},\ \delta\left(t\right)=0,\ \eta\left(t\right)=\eta_{00},\ s_f\left(t\right)=s_{f_{00}},\ \theta_f\left(t\right)=\theta_{f_{00}},\ \theta_r\left(t\right)=\theta_{r_{00}}$$

#### Bodies

Before computing the centre of mass of the complete rigid body we should define the centre of gravity of each body.
All the bodies at play are defined starting from the convenient reference frames defined in the precious section. The data and convention in of length and physical values of the motorcycle are take primary from FastBike a fortran code for real time simulation of motorcycles. [13, 14]
The rear frame (main body) is linked to the reference frame $RF_{Rear}$. The centre of gravity of this body will be in a point $G_{Rear}$ that has only $x$ and $z$ components. This comes directly from the assumption that the vehicle is symmetrical and the reference frame lays

on the symmetric plane of the body. The same is true for the body of the rider. The condition on the position of the rider wil be relaxed later. In fact the rider is not static with respect to the motorcycle but can move, lean forward and laterally.

$$G_{Rear} = \begin{bmatrix} x_{Rear} \\ 0 \\ z_{Rear} \end{bmatrix} \text{ in } RF_{rear} \tag{2.21}$$

$$G_{rdr} = \begin{bmatrix} x_{rdr} \\ 0 \\ z_{rdr} \end{bmatrix} \text{ in } RF_{rear} \tag{2.22}$$

The body representing the steering assembly is defined starting from $RF_\delta$.

$$G_\delta = \begin{bmatrix} x_\delta \\ 0 \\ z_\delta \end{bmatrix} \text{ in } RF_\delta \tag{2.23}$$

where $z_{delta}$ in this case will be negative since the reference frame is define with the ISO convention.
The body of the swingarm in define wth the following centre of gravity starting from $RF_\eta$.

$$G_{Swing} = \begin{bmatrix} x_{Swing} \\ 0 \\ z_{Swing} \end{bmatrix} \text{ in } RF_\eta \tag{2.24}$$

The two body of the wheels have their CoM in the origin of the attached spinning reference frame.

$$G_{FW} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } RF_{FW spin} \tag{2.25}$$

$$G_{RW} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } RF_{RW spin} \tag{2.26}$$

The last body is the unsprung front suspension. However, the mass of this element is small compared to the other and it can be eventually integrated in the mass of the front wheel. The body of the unsprung suspension has as a virtual centre of mass the centre of the front wheel, wont have mass (mass= 0) and has no inertia. This is a modelling expedient to derive the equation of motions and to transmit the reaction forces.
From all the previously defined centre of gravity one can define the six bodies at play with their inertial properties, mass and moment of inertia.
In the following sections masses and moment of inertia belonging to bodies will appear. The notation used in this thesis follow the subsequent rule. Masses and inertia are indicated with symbols $m$ and $Ix$, $Iy$, $Iz$, $Cxy$, $Cxz$, $Cyz$ in general. When addressed to a specific one a suffix is added to represent the body or the reference frame. For instance the mass of the steering assembly will be $m_\delta$.
Drawings of CoM of each body
TO BE IMPLEMENTED

#### Centre of Mass of the motorcycle

The CoM of the motorcycle can be simply computed as the weighted average sum of the masses of all the parts. This mean yields a vector of 3 components that can be projected in the rolled reference frame $RF_\phi$. Once substituted the relationship of the previous paragraph for the freezed DoF, the values of the vector are the coordinate of the CoM of the motorcycle in the rolled reference frame.

$$G_{moto} = \begin{bmatrix} XG \\ YG \\ ZG \end{bmatrix} \tag{2.27}$$

However, the definition of $YG$ is simple and evaluated is equal to zero. This is due to the assumption that the motorcycle is always symmetric with respect to the rolled plane.

#### Moment of Inertia of the motorcycle

The moment of inertia of the whole motorcycle, can be computed in a similar way as the CoM. The fist thing to derive is the angular velocity of the rolled reference frame $RF_\phi$. This will not contain angle $\theta$ because is considered as freezed.
The three components of the angular velocity are

$$\begin{aligned} \omega_x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \\ \omega_y(t) &= \Omega(t)\sin(\phi(t)) \\ \omega_z(t) &= \Omega(t)\cos(\phi(t)) \end{aligned} \tag{2.28}$$

Proceeding with the equation derivation the angular momentum of the whole motorcycle ic computed. The angular momentum is additive, therefore it can be calculated as the sum of all the angular momentums. Those should be calculated using as a pole the CoM of the motorcycle. The obtained vector is projected in the rolled reference frame to get rid off almost all the contribute of $\phi$ and evaluated considering the freezeed DoF.
At this point the previous relationship 2.28 can be exploited and substituted. Therefore the angular momentum can be used to generate the inertia matrix collecting $\omega_x$, $\omega_y$ and $\omega_z$.

$$\mathbf{A_M} = I_{tot}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} + \mathbf{res} \tag{2.29}$$

where $I_{tot}$ is the matrix of inertia of the whole motorcycle, $\mathbf{A_M}$ is the vector of the angular momentum and **res** is the vector of residual from the matrix generation. This should be checked to be equal to zero and it is. With the previously computed data a body can be created. This have the centre of gravity in the point of equation 2.27, mass equal to the sum of all masses ($M_{tot}$) and the moment of inertia $I_{tot}$.

### 2.2.2 Dummy bodies

The moment of inertia and the masses of the different parts of the motorcycle are already inside the definition of the rigid body of the previous section. Therefore in the derivation of the equation of motion we should only consider the effect of the motion of the parts. In fact in the rigid body all the pats are frezed and considered as static. However, the dynamic component plays a great role in the motion.

One solution is to take into account only those components is to introduce a set of dummy bodies also known as anti-bodies. This bodies are defined for each moving component of the motorcycle. The dummy body has as a centre of gravity the body they are referred to, but evaluated with the internal DoF freezed. The anti-bodies, as the name suggest, has inertial properties that are the one of the real body with changed sign. This means that all dummy bodies have negative mass and negative inertia. The concept of negative mass and negative inertia have no meaning in physical and real life. However, is a modelling trick to simplify the resulting equations of motion.
Here is reported a list of dummy bodies that have to be defined.

- anti rear frame
- anti rider
- anti steering assembly
- anti swingarm
- anti front wheel
- anti rear wheel

There is no need to create the antibody for the unsprung front suspension since it does not have mass nor inertia. If the model consider also mass and inertia of this body then an antibody must be created to be consistent.

**Forces**

First of all the force of gravity is acting on all bodies and the acceleration of gravity is set as a vector in the negative $z$ direction due to convention choices.
The other external forces acting on the motorcycle are the aerodynamic drag and the force exchanged by the tyre with the ground.
The drag is modelled with a simple law that scale with the squared of the longitudinal velocity. It has only component in the $x$ direction in the rolled reference frame. The force act on the motorcycle rigid body.

$$F_a = \begin{bmatrix} -C_a u(t)^2 \\ 0 \\ 0 \end{bmatrix} \text{in} RF_\phi \tag{2.30}$$

Where $C_a$ is this case is a constant, but in theory and reality it is not. This coefficient depend on multiple thermodynamic factors and more important for this application the resisting cross section of the motorcycle and the rider. The motorcycle profile is constant, but the pilot is moving leaning forward and opening the knee at curve entrance. Moreover, the drag force has an application point which is constant in this model, but in reality change with changing configuration (position) of the driver.

$$P_a = \begin{bmatrix} x_a \\ 0 \\ z_a \end{bmatrix} \text{in} RF_{Rear} \tag{2.31}$$

The forces of the tyre are expressed in a reference frame with the tyre itself. For the rear wheel it coincide with $RF_1$, while for the front it is necessary to rotate of an angle $\delta_f$.

The force vector on the rear wheel is composed of $Fxr(t)$ the longitudinal force, $Fyr(t)$ the lateral force and $Fzr(t)$ the vertical force.

$$F_{RW} = \begin{bmatrix} Fxr(t) \\ Fyr(t) \\ Fzr(t) \end{bmatrix} \text{in} RF_1 \tag{2.32}$$

and it is supposed as applied in the point $P_r$ and it is acting on the rear wheel. This is not true however some overturning moment will be introduced in the next section.
The force vector on the rear wheel is composed of $Fxf(t)$ the longitudinal force, $Fyf(t)$ the lateral force and $Fzf(t)$ the vertical force.

$$F_{FW} = \begin{bmatrix} Fxf(t) \\ Fyf(t) \\ Fzf(t) \end{bmatrix} \text{in} RF_1 \cdot R_{\delta_f} \tag{2.33}$$

and it is supposed as applied in the point $P_f$ and it is acting on the front wheel. This is not true however some overturning moment will be introduced in the next section.
$Fxr(t)$, $Fyr(t)$, $Fxf(t)$ and $Fyf(t)$ will be defined following the Magic Formula of Pacejka [2].
The vertical forces are defined as a function of the penetration (equation 2.18) and penetration velocity. It is modelled assuming that the tyre behave as a spring-damper (second-order) system.

$$\begin{aligned} Fzf\,(t) &= Kp_f\, p_f\,(t) + Cp_f \frac{\mathrm{d}}{\mathrm{d}t} p_f\,(t) \\ Fzr\,(t) &= Kp_r\, p_r\,(t) + Cp_r \frac{\mathrm{d}}{\mathrm{d}t} p_r\,(t) \end{aligned} \tag{2.34}$$

There is another force present in the model and is the one of the front suspension. It is model as a spring-damper with linear proportional and damping factor. A more complicated model can be used in future applications.

$$F_s = \begin{bmatrix} 0 \\ 0 \\ s_f\,(t)\, k_{fs} + \left(\frac{\mathrm{d}}{\mathrm{d}t} s_f\,(t)\right) c_{fs} \end{bmatrix} \text{in} RF_\delta \tag{2.35}$$

For modelling purposes it is defined as acting on the origin of $RF_\delta$. However, since it is an internal force, the reacting body must be specified. In this case it is the unsprung suspension.

### Torques

The torques acting in this dynamic model are both external and internal. The external torques are the one acting on the wheels. Those are present for two reasons. The first is that we are considering a point of application which is not the real point of application of the forces. The contact is in fact on a patch where the distribution of forces is not known. The second reason is that the model of the magic formula [2] takes into account the effect of trail and camber calculating forces and moments with respect to the point $P_r$ and $P_f$ previously defined.
The torques vector acting on the front wheel is:

$$M_{RW} = \begin{bmatrix} Mxr(t) \\ 0 \\ Mzr(t) \end{bmatrix} \text{in} RF_1 \tag{2.36}$$

while the momentum vector on the front wheel is:

$$M_{FW} = \begin{bmatrix} Mxf(t) \\ 0 \\ Mzf(t) \end{bmatrix} \text{in} RF_1 \cdot R_{\delta_f} \tag{2.37}$$

Then there are other toques that are traction and braking torques. Those are acting in the $y$ direction of the reference frame attached to the wheels. Front wheel can only brake. The traction is acting on the rear wheel and reacting on the swingarm.

$$TB_{RW} = \begin{bmatrix} 0 \\ Myr(t) \\ 0 \end{bmatrix} \text{in} RF_{RW} \tag{2.38}$$

The braking torque is acting on the front wheel and reacting on the unsprung suspension.

$$B_{FW} = \begin{bmatrix} 0 \\ Myf(t) \\ 0 \end{bmatrix} \text{in} RF_{FW} \tag{2.39}$$

The internal momentums are instead the torque of the steering damper, the torque applied by the rider and the torque of the rear suspension. The first is modelled as a viscous term.

$$T_d = \begin{bmatrix} 0 \\ 0 \\ -C_\delta \frac{\mathrm{d}}{\mathrm{d}t}\delta(t) \end{bmatrix} \text{in} RF_\delta \tag{2.40}$$

It is acting on the steering frame and reacting on the motorcycle.
The rider torque has only component in the $z$ direction of the $RF_\delta$.

$$T_r = \begin{bmatrix} 0 \\ 0 \\ \tau(t) \end{bmatrix} \text{in} RF_\delta \tag{2.41}$$

It is acting on the steering assembly and reacting on the motorcycle rigid body. The last internal torque is the rear suspension torque. It is actually a force of a spring-damper system applied with a certain arm. However, for modelling purposes can be written in terms of a torque with a torsional stiffness and a rotational viscosity.

$$M_{RSusp} = \begin{bmatrix} 0 \\ \eta(t)\, a_1\, k_{rs} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\right) a_1\, c_{rs} \\ 0 \end{bmatrix} \text{in} RF_{Rear} \tag{2.42}$$

It is acting on the swingarm and reacting on the motorcycle rigid body.

## 2.3 Equations of motion

## Chapter 3

# Optimal Control Problem

## Chapter 4

# Analysis of results

# Conclusions

# Bibliography

[1] A. Maria, “Introduction to modeling and simulation,” in *Winter simulation conference*, vol. 29, pp. 7–13, 1997.

[2] H. Pacejka, *Tyre and Vehicle Dynamics*. Automotive engineering, Butterworth-Heinemann, 2006.

[3] V. Cossalter, R. Lot, and M. Massaro, “The influence of frame compliance and rider mobility on the scooter stability,” *Vehicle System Dynamics*, vol. 45, no. 4, pp. 313–326, 2007.

[4] E. Bertolazzi, F. Biral, and M. Da Lio, “Symbolic-numeric efficient solution of optimal control problems for multibody systems,” *Journal of computational and applied mathematics*, vol. 185, no. 2, pp. 404–421, 2006.

[5] E. Bertolazzi, F. Biral, and M. Da Lio, “Symbolic–numeric indirect method for solving optimal control problems for large multibody systems,” *Multibody System Dynamics*, vol. 13, no. 2, pp. 233–252, 2005.

[6] F. Biral, E. Bertolazzi, and P. Bosetti, “Notes on numerical methods for solving optimal control problems,” *IEEJ Journal of Industry Applications*, vol. 5, no. 2, pp. 154–166, 2016.

[7] A. V. Rao, “A survey of numerical methods for optimal control,” *Advances in the Astronautical Sciences*, vol. 135, no. 1, pp. 497–528, 2009.

[8] H. S. Rodrigues, M. T. T. Monteiro, and D. F. Torres, “Optimal control and numerical software: an overview,” *arXiv preprint arXiv:1401.7279*, 2014.

[9] R. Sharp, S. Evangelou, and D. J. Limebeer, “Advances in the modelling of motorcycle dynamics,” *Multibody system dynamics*, vol. 12, no. 3, pp. 251–283, 2004.

[10] L. Leonelli and D. Limebeer, “Optimal control of a road racing motorcycle on a three-dimensional closed track,” *Vehicle System Dynamics*, pp. 1–25, 2019.

[11] J. Baumgarte, “Stabilization of constraints and integrals of motion in dynamical systems,” *Computer methods in applied mechanics and engineering*, vol. 1, no. 1, pp. 1–16, 1972.

[12] R. Lot, “A motorcycle tire model for dynamic simulations: Theoretical and experimental aspects,” *Meccanica*, vol. 39, no. 3, pp. 207–220, 2004.

[13] V. Cossalter and R. Lot, “A motorcycle multi-body model for real time simulations based on the natural coordinates approach,” *Vehicle system dynamics*, vol. 37, no. 6, pp. 423–447, 2002.

[14] V. Cossalter, R. Lot, and F. Maggio, "A multibody code for motorcycle handling and stability analysis with validation and examples of application," tech. rep., SAE Technical Paper, 2003.

# Appendix