GECaM equations: algorithmic implementation

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We consider the GECaM equations for a linearly-coupled system of Langevin equations with additive noise, i.e. a system of linearly interacting Ornstein-Uhlenbeck processes.

$$\frac{d}{dt}\mu_{i\setminus j}(t) = -\lambda_i \mu_{i\setminus j}(t) + \sum_{k\in\partial i\setminus j} J_{ik}\mu_{k\setminus i}(t) + \sum_{k\in\partial i\setminus j}\int_0^t ds J_{ik} R_{k\setminus i}(t,s) J_{ki}\mu_{i\setminus j}(s), \tag{1}$$

$$\frac{\partial}{\partial t}R_{i\setminus j}(t,t') = -\lambda_i R_{i\setminus j}(t,t') + \sum_{k\in\partial i\setminus j}\int_{t'}^t ds J_{ik} R_{k\setminus i}(t,s) J_{ki} R_{i\setminus j}(s,t') + \delta(t-t'), \tag{2}$$

$$\begin{aligned}\frac{\partial}{\partial t}C_{i\setminus j}(t,t') = {} & -\lambda_i C_{i\setminus j}(t,t') + \sum_{k\in\partial i\setminus j}\int_0^t ds J_{ik} R_{k\setminus i}(t,s) J_{ki} C_{i\setminus j}(s,t') + 2D R_{i\setminus j}(t',t) \\ & + \sum_{k\in\partial i\setminus j}\int_0^{t'} ds R_{i\setminus j}(t',s) J_{ik}^2 C_{k\setminus i}(t,s).\end{aligned} \tag{3}$$

Numerical solution

A numerical solution of Eqs. (1), (2) and (3) can be found by discretizing time with a timestep Δ, i.e. $t = n\Delta$, $n = 0,\ldots,T$ with $T = \mathcal{T}/\Delta$. Within this discretization the cavity means become a time vector with T components $\mu_{i\setminus j}^n = \mu_{i\setminus j}(t = n\Delta)$, and the cavity correlation and response functions become time matrices with $(T+1)\times(T+1)$ components $C_{i\setminus j}^{n,n'} = C_{i\setminus j}(t = n\Delta, t' = n'\Delta)$ and $R_{i\setminus j}^{n,n'} = R_{i\setminus j}(t = n\Delta, t' = n'\Delta)$. Then a discretized version of GECaM equations is

$$\mu_{i\setminus j}^{n+1} = (1-\lambda_i\Delta)\,\mu_{i\setminus j}^{n} + \Delta\sum_{k\in\partial i\setminus j} J_{ik}\mu_{k\setminus i}^{n} + \Delta^2\sum_{m=0}^{n-1}\mu_{i\setminus j}^{m}\sum_{k\in\partial i\setminus j} J_{ik}J_{ki}R_{k\setminus i}^{n,m} \tag{4}$$

$$R_{i\setminus j}^{n+1,n'} = (1-\lambda_i\Delta)\,R_{i\setminus j}^{n,n'} + \delta_{n,n'} + \Delta^2\sum_{m=n'+1}^{n-1} R_{i\setminus j}^{m,n'}\sum_{k\in\partial i\setminus j} J_{ik}J_{ki}R_{k\setminus i}^{n,m} \tag{5}$$

$$\begin{aligned} C_{i\setminus j}^{n+1,n'} &= (1-\lambda_i\Delta)\,C_{i\setminus j}^{n,n'} + 2\Delta D R_{i\setminus j}^{n',n} + \Delta^2\sum_{m=0}^{n-1} C_{i\setminus j}^{m,n'}\sum_{k\in\partial i\setminus j} J_{ik}J_{ki}R_{k\setminus i}^{n,m} \\ &\quad + \Delta^2\sum_{m=0}^{n'-1} R_{i\setminus j}^{n',m}\sum_{k\in\partial i\setminus j} J_{ik}^2 C_{k\setminus i}^{n,m} \end{aligned} \tag{6}$$

where $R_{i\setminus j}^{n,n} = 0$ due to causality and $C_{i\setminus j}^{n,n'} = C_{i\setminus j}^{n',n}$.
The full marginals can be obtained as

$$\mu_{i}^{n+1} = (1-\lambda_i\Delta)\,\mu_{i}^{n} + \Delta\sum_{k\in\partial i} J_{ik}\mu_{k\setminus i}^{n} + \Delta^2\sum_{m=0}^{n-1}\mu_{i}^{m}\sum_{k\in\partial i\setminus j} J_{ik}J_{ki}R_{k\setminus i}^{n,m} \tag{7}$$

$$R_{i}^{n+1,n'} = (1-\lambda_i\Delta)\,R_{i}^{n,n'} + \delta_{n,n'} + \Delta^2\sum_{m=n'+1}^{n-1} R_{i}^{m,n'}\sum_{k\in\partial i\setminus j} J_{ik}J_{ki}R_{k\setminus i}^{n,m} \tag{8}$$

$$\begin{aligned} C_{i}^{n+1,n'} &= (1-\lambda_i\Delta)\,C_{i}^{n,n'} + 2\Delta D R_{i}^{n',n} + \Delta^2\sum_{m=0}^{n-1} C_{i}^{m,n'}\sum_{k\in\partial i\setminus j} J_{ik}J_{ki}R_{k\setminus i}^{n,m} \\ &\quad + \Delta^2\sum_{m=0}^{n'-1} R_{i}^{n',m}\sum_{k\in\partial i\setminus j} J_{ik}^2 C_{k\setminus i}^{n,m} \end{aligned} \tag{9}$$