GECaM equations for equilibrium

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We consider the GECaM equations for a linearly-coupled system of Langevin equations with additive noise, i.e. a system of linearly interacting Ornstein-Uhlenbeck processes.

$$\frac{d}{dt}\mu_{i\setminus j}(t) = -\lambda_i \mu_{i\setminus j}(t) + \sum_{k\in\partial i\setminus j} J_{ik}\mu_{k\setminus i}(t) + \sum_{k\in\partial i\setminus j}\int_0^t ds J_{ik} R_{k\setminus i}(t,s) J_{ki}\mu_{i\setminus j}(s), \tag{1}$$

$$\frac{\partial}{\partial t}R_{i\setminus j}(t,t') = -\lambda_i R_{i\setminus j}(t,t') + \sum_{k\in\partial i\setminus j}\int_{t'}^t ds J_{ik} R_{k\setminus i}(t,s) J_{ki} R_{i\setminus j}(s,t') + \delta(t-t'), \tag{2}$$

$$\begin{aligned}\frac{\partial}{\partial t}C_{i\setminus j}(t,t') = {} & -\lambda_i C_{i\setminus j}(t,t') + \sum_{k\in\partial i\setminus j}\int_0^t ds J_{ik} R_{k\setminus i}(t,s) J_{ki} C_{i\setminus j}(s,t') + 2D R_{i\setminus j}(t',t) \\ & + \sum_{k\in\partial i\setminus j}\int_0^{t'} ds R_{i\setminus j}(t',s) J_{ik}^2 C_{k\setminus i}(t,s).\end{aligned} \tag{3}$$

In the long-time limit $t, t' \to \infty$ the correlations and responses become time-translational invariant (TTI), i.e. $C_{i\setminus j}(t,t') = C_{i\setminus j}(\tau = t-t')$ and $R_{i\setminus j}(t,t') = R_{i\setminus j}(\tau = t-t')$ keeping time differences $\tau = t - t'$ finite. The equations for the responses and correlations read

$$\dot{R}_{i\setminus j}(\tau) = -\lambda_i R_{i\setminus j}(\tau) + \sum_{k\in\partial i\setminus j} J_{ik}J_{ki}\int_{t'}^t ds R_{k\setminus i}(t-s) R_{i\setminus j}(s-t') + \delta(\tau), \tag{4}$$

$$\begin{aligned}\dot{C}_{i\setminus j}(\tau) = {} & -\lambda_i C_{i\setminus j}(\tau) + \sum_{k\in\partial i\setminus j} J_{ik}J_{ki}\int_0^t ds R_{k\setminus i}(t-s) C_{i\setminus j}(s-t') + 2D R_{i\setminus j}(-\tau) \\ & + \sum_{k\in\partial i\setminus j} J_{ik}^2 \int_0^{t'} ds R_{i\setminus j}(t'-s) C_{k\setminus i}(t-s).\end{aligned} \tag{5}$$

The three integrals can be written as, respectively,

$$\int_{t'}^t ds R_{k\setminus i}(t-s) R_{i\setminus j}(s-t') = \int_0^\tau du R_{k\setminus i}(u) R_{i\setminus j}(\tau-u), \tag{6}$$

$$\begin{aligned}\int_0^t ds R_{k\setminus i}(t-s) C_{i\setminus j}(s-t') &= \int_0^{t'} ds R_{k\setminus i}(t-s) C_{i\setminus j}(s-t') + \int_{t'}^t ds R_{k\setminus i}(t-s) C_{i\setminus j}(s-t') \\ &= \int_\tau^t du R_{k\setminus i}(u) C_{i\setminus j}(\tau-u) + \int_0^\tau du R_{k\setminus i}(u) C_{i\setminus j}(\tau-u),\end{aligned} \tag{7}$$

$$\int_0^{t'} ds R_{i\setminus j}(t'-s) C_{k\setminus i}(t-s) = \int_\tau^t du C_{k\setminus i}(u) R_{i\setminus j}(u-\tau). \tag{8}$$

Eqs. (4) and (5) become

$$\dot{R}_{i\setminus j}(\tau) = -\lambda_i R_{i\setminus j}(\tau) + \sum_{k\in\partial i\setminus j} J_{ik}J_{ki}\int_0^\tau du R_{k\setminus i}(u)R_{i\setminus j}(\tau-u) + \delta(\tau), \tag{9}$$

$$\begin{aligned}\dot{C}_{i\setminus j}(\tau) = &-\lambda_i C_{i\setminus j}(\tau) + \sum_{k\in\partial i\setminus j} J_{ik}J_{ki}\left(\int_\tau^t du R_{k\setminus i}(u)C_{i\setminus j}(\tau-u) + \int_0^\tau du R_{k\setminus i}(u)C_{i\setminus j}(\tau-u)\right)\\ &+ 2DR_{i\setminus j}(-\tau) + \sum_{k\in\partial i\setminus j} J_{ik}^2\int_\tau^t du C_{k\setminus i}(u)R_{i\setminus j}(u-\tau).\end{aligned} \tag{10}$$

If the interaction matrix $\boldsymbol{J}$ is symmetric, i.e. $J_{ij} = J_{ji}$ for every $i,j = 1,\ldots,N$, the system satisfies detailed balance and it eventually reaches equilibrium after a sufficient long time. Within this regime the Fluctuation Dissipation Theorem (FDT) holds,

$$DR^{\rm eq}_{i\setminus j}(\tau) = -\dot{C}^{eq}_{i\setminus j}(\tau)\Theta(\tau). \tag{11}$$

Thus, by substituting the FDT into Eq. (10) we obtain an equation for the equilibrium correlations only,

$$\begin{aligned}\dot{C}^{eq}_{i\setminus j}(\tau)\left(1-2\Theta(-\tau)\right) = &-\lambda_i C^{eq}_{i\setminus j}(\tau) - \sum_{k\in\partial i\setminus j}\frac{J_{ik}^2}{D}\left(\int_\tau^t du \dot{C}^{eq}_{k\setminus i}(u)C^{eq}_{i\setminus j}(\tau-u) + \int_0^\tau du\dot{C}^{eq}_{k\setminus i}(u)C^{eq}_{i\setminus j}(\tau-u)\right)\\ &- \sum_{k\in\partial i\setminus j}\frac{J_{ik}^2}{D}\int_\tau^t du C^{\rm eq}_{k\setminus i}(u)\dot{C}^{eq}_{i\setminus j}(u-\tau).\end{aligned} \tag{12}$$

Integrating by parts the last integral

$$\int_\tau^t du C^{\rm eq}_{k\setminus i}(u)\dot{C}^{eq}_{i\setminus j}(u-\tau) = C^{\rm eq}_{k\setminus i}(t)C^{eq}_{i\setminus j}(t') - C^{\rm eq}_{k\setminus i}(\tau)C^{eq}_{i\setminus j}(0) - \int_\tau^t du\dot{C}^{\rm eq}_{k\setminus i}(u)C^{eq}_{i\setminus j}(u-\tau).$$

Taking the limit $t,t'\to\infty$ the first term $C^{\rm eq}_{k\setminus i}(t)C^{eq}_{i\setminus j}(t')\to C^{\rm eq}_{k\setminus i}(\infty)C^{eq}_{i\setminus j}(\infty)$ vanishes, since the equilibrium correlation decays to zero for long time differences.

The cavity equilibrium correlations are therefore obtained by solving the set of equations

$$\mathrm{sgn}(\tau)\dot{C}^{eq}_{i\setminus j}(\tau) = -\lambda_i C^{eq}_{i\setminus j}(\tau) + \sum_{k\in\partial i\setminus j}\frac{J_{ik}^2}{D}\left(C^{\rm eq}_{k\setminus i}(\tau)C^{eq}_{i\setminus j}(0) - \int_0^\tau du\dot{C}^{eq}_{k\setminus i}(u)C^{eq}_{i\setminus j}(\tau-u)\right). \tag{13}$$

The full equilibrium correlations are obtained from the cavity ones as

$$\mathrm{sgn}(\tau)\dot{C}^{eq}_{i}(\tau) = -\lambda_i C^{eq}_{i}(\tau) + \sum_{k\in\partial i}\frac{J_{ik}^2}{D}\left(C^{\rm eq}_{k\setminus i}(\tau)C^{eq}_{i}(0) - \int_0^\tau du\dot{C}^{eq}_{k\setminus i}(u)C^{eq}_{i}(\tau-u)\right). \tag{14}$$

Numerical solution

A numerical solution of Eq. (13) can be found by discretizing time with a timestep Δ, i.e. $t = n\Delta$, $n = 0,\ldots,T$ with $T = \mathcal{T}/\Delta$. Within this discretization the equilibrium correlation function becomes a time vector with $T+1$ components $C^{eq,n}_{i\setminus j} = C^{eq}_{i\setminus j}(t = n\Delta)$. Then a discretized version of Eq. (13) is

$$\begin{aligned}C^{eq,n+1}_{i\setminus j} &= \left(1-\lambda_i\Delta\right)C^{eq,n}_{i\setminus j} + \Delta\sum_{k\in\partial i\setminus j}\frac{J_{ik}^2}{D}C^{\rm eq,n}_{k\setminus i}C^{eq,0}_{i\setminus j}\\ &\quad-\Delta\sum_{k\in\partial i\setminus j}\frac{J_{ik}^2}{D}\sum_{m=0}^{n-1}\left(C^{eq,m+1}_{k\setminus i} - C^{eq,m}_{k\setminus i}\right)C^{eq,n-m}_{i\setminus j}. \end{aligned}\tag{15}$$

$$\begin{aligned}&= \left(1-\lambda_i\Delta\right)C^{eq,n}_{i\setminus j} + \Delta\frac{C^{eq,0}_{i\setminus j}}{D}\sum_{k\in\partial i\setminus j}J_{ik}^2 C^{\rm eq,n}_{k\setminus i}\\ &\quad-\Delta\sum_{m=0}^{n-1}\frac{C^{eq,n-m}_{i\setminus j}}{D}\sum_{k\in\partial i\setminus j}J_{ik}^2\left(C^{eq,m+1}_{k\setminus i} - C^{eq,m}_{k\setminus i}\right).\end{aligned} \tag{16}$$

while for the full correlation we obtain

$$\begin{aligned} C_i^{eq,n+1} &= (1-\lambda_i\Delta)\, C_i^{eq,n} + \Delta \frac{C_i^{eq,0}}{D} \sum_{k\in\partial i\setminus j} J_{ik}^2 C_{k\setminus i}^{\mathrm{eq,n}} \\ &\quad - \Delta \sum_{m=0}^{n-1} \frac{C_i^{eq,n-m}}{D} \sum_{k\in\partial i\setminus j} J_{ik}^2 \left(C_{k\setminus i}^{eq,m+1} - C_{k\setminus i}^{eq,m} \right). \end{aligned} \tag{17}$$