Model Predictive Control

Lecture: MPC Stability Theory

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Outline

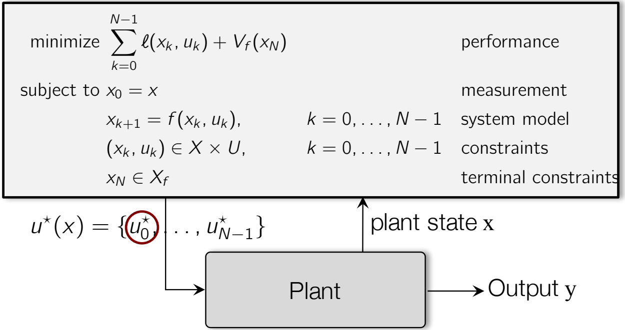
1. From Dynamic Programming to MPC

2. Motivating example

3. Infinite-Horizon Optimal Control

4. Recap of Infinite-Horizon Dynamic Programming

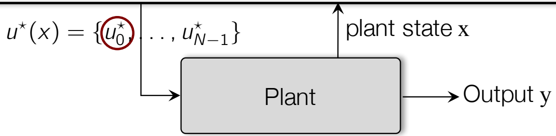
5. VI: Convergence from above



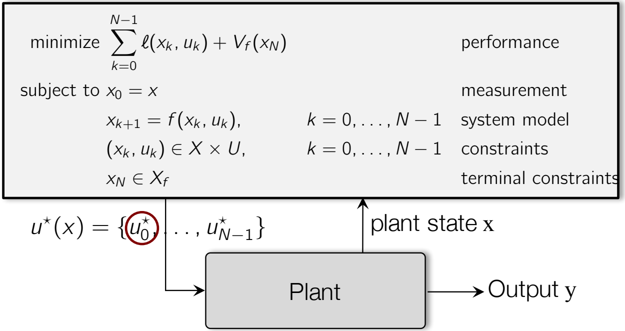
- In MPC we solve $\mathbf{OCP}_N(x)$ for the current state x, in real time.
- This is because we cannot afford to solve $\mathbf{OCP}_\infty(x)$.
- We already saw that the value function of $\mathbf{OCP}_N(x)$, $V_N^\star(x)$, is the N-th iterate of VI for solving $\mathbf{OCP}_\infty(x)$, starting from

$$V_0^\star(x) = \begin{cases} V_f(x), & x \in X_f \\ +\infty, & \text{otherwise} \end{cases}$$

$$
\begin{array}{llll}
\text{minimize} & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N) & & \text{performance} \\
\text{subject to} & x_0 = x & & \text{measurement} \\
& x_{k+1} = f(x_k, u_k), & k = 0, \dots, N-1 & \text{system model} \\
& (x_k, u_k) \in X \times U, & k = 0, \dots, N-1 & \text{constraints} \\
& x_N \in X_f & & \text{terminal constraints}
\end{array}
$$



- We have seen that VI converges **"from below"**: Whenever $V_0^\star \leq V_\infty^\star$.
- "In the limit", $\mathbf{OCP}_N(x)$ to inherit all nice properties of $\mathbf{OCP}_\infty(x)$
  - invariance (recursive feasibility): $x \in X_\infty^\star \implies f(x, \kappa_\infty^\star(x)) \in X_\infty^\star$
  - Origin globally attractive in $X_\infty^\star$: $\lim_{k \to \infty} \phi(k; x, \kappa_\infty^\star) = 0, \ \forall x \in X_\infty^\star$
  - Under a local constrained stabilizability, origin is GAS in $X_\infty^\star$.
- But for finite N, we have seen that this is not the case even for LQR!



Question

Can we guarantee that $\mathbf{OCP}_N(x)$ inherits all nice properties of $\mathbf{OCP}_\infty(x)$?

- **Yes**: By selecting appropriate terminal ingredients V_f, X_f.
- **Alternative view**: By selecting appropriate initial estimate $V_0^\star$ for VI.
- We should choose $V_0^\star \geq V_\infty^\star$: Does VI converge "from above"?

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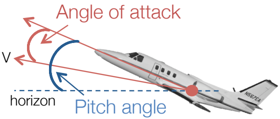
Example: Cessna Citation Aircraft

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.349rad/s ($\pm 39°$/s), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: $0, 0, -1.5594 \pm 2.29i$)

LQR and Linear MPC with Quadratic Cost

- Quadratic performance measure
- Linear system dynamics
- Linear constraints on inputs and states

LQR	**MPC**
$$\begin{aligned} V_\infty^\star(x) = \min_{x,u} & \sum_{k=0}^{\infty} x_k^\top Q x_k + u_k^\top R u_k \\ \text{s.t.} \quad & x_0 = x \\ & x_{k+1} = A x_k + B u_k \end{aligned}$$	$$\begin{aligned} V_N^\star(x) = \min_{x,u} & \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k \\ \text{s.t.} \quad & x_0 = x \\ & x_{k+1} = A x_k + B u_k \\ & b \geq C x_k + D u_k \end{aligned}$$

Assume: $Q = Q^\top \succeq 0$, $R = R^\top \succ 0$

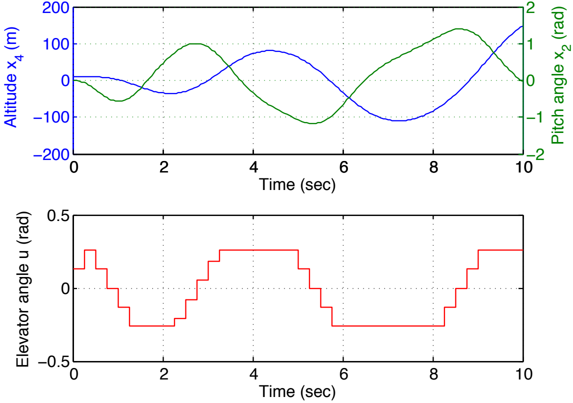
Example: LQR with saturation

Linear quadratic regulator with saturated inputs.

At time $t = 0$ the plane is flying with a deviation of $10m$ of the desired altitude, i.e. $x_0 = (0, 0, 0, 10)$

Problem parameters:

$$\boxed{\text{Sampling time 0.25sec, } Q = I,\ R = 10}$$



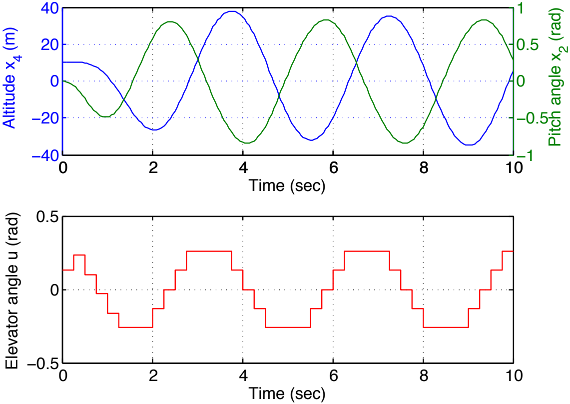
- Closed-loop system is unstable
- Applying LQR control and saturating the controller can lead to instability!

Example: MPC with Bound Constraints on Inputs

MPC controller with input constraints $|u_k| \leq 0.262$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time 0.25sec,}\\ Q = I,\ R = 10,\ N = 10\end{array}}$$



The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

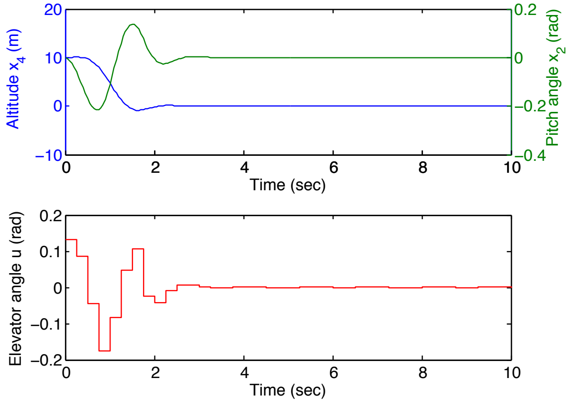
$\Rightarrow$ System does not converge to desired steady-state but to a limit cycle

Example: MPC with all Input Constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time 0.25sec,}\\ Q = I,\ R = 10,\ N = 10\end{array}}$$



The MPC controller considers all constraints on the actuator

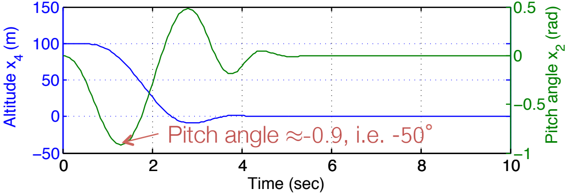
- Closed-loop system is stable
- Efficient use of the control authority

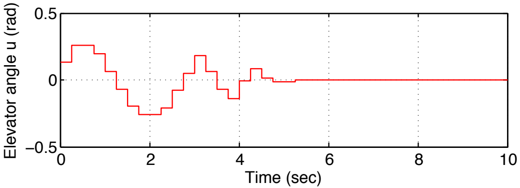
Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec}, \\ Q = I,\ R = 10,\ N = 10\end{array}}$





Increase step:
At time $t = 0$ the plane is flying with a deviation of 100m of the desired altitude, i.e. $x_0 = (0, 0, 0, 100)$

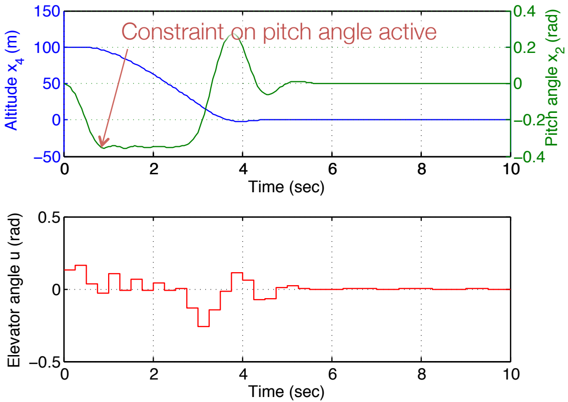
- Pitch angle too large during transient

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec},\\ Q = I,\ R = 10,\ N = 10\end{array}}$$



Add state constraints for passenger comfort:

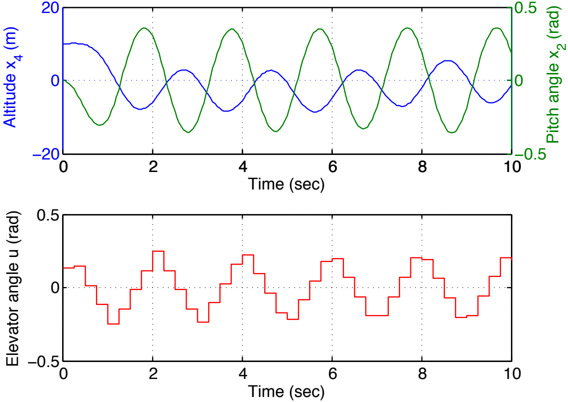
$$|x_2| \leq 0.349$$

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$, $N = 4$



Decrease in the prediction horizon causes loss of the stability properties

Next: How to ensure stability and constraint satisfaction for all choices of Q, R and N.

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Infinite-Horizon Optimal Control

$$\begin{array}{rll} \text{minimize} & \sum_{k=0}^{\infty} \ell(x_k, u_k) & \\ \text{subject to} & x_0 = x & \\ & x_{k+1} = f(x_k, u_k), & k \in \mathbb{N} \\ & x_k \in X, \quad u_k \in U, & k \in \mathbb{N} \end{array} \qquad \mathbf{OCP}_\infty(x)$$

- $\mathbf{OCP}_\infty(x)$: "golden standard" in terms of performance and region of attraction.
- However solving $\mathbf{OCP}_\infty(x)$ is almost always intractable.
- MPC can be seen as a **finite horizon approximation** of $\mathbf{OCP}_\infty(x)$.
- MPC corresponds to finite number of iterations of algorithm for solving $\mathbf{OCP}_\infty(x)$.
- Value function of $\mathbf{OCP}_\infty(x)$ as a **fixed point** of the **Bellman operator**.
- This is the so called **Bellman equation**, the cornerstone of DP.
- **Value iteration (VI)**: Fixed point iteration for solving Bellman's equation.
- N-th iterate of VI is value function of $\mathbf{OCP}_N(x)$, problem we solve within MPC.

Infinite-Horizon Optimal Control–Some Notation

$$
\begin{aligned}
\text{minimize} \quad & \sum_{k=0}^{\infty} \ell(x_k, u_k) \\
\text{subject to} \quad & x_0 = x \\
& x_{k+1} = f(x_k, u_k), \qquad k \in \mathbb{N} \\
& x_k \in X, \quad u_k \in U, \qquad k \in \mathbb{N}
\end{aligned}
\qquad \mathbf{OCP}_\infty(x)
$$

- $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ (takes nonnegative values).
- Minimization is carried over input sequences of infinite length

$$\mathbf{u} = \{u_k\}_{k \in \mathbb{N}}.$$

- Given an initial state x and an input sequence $\mathbf{u}$, the solution of $x^+ = f(x, u)$ at time k is denoted by $\phi(k; x, \mathbf{u})$.
- Given an initial state x and a control law $\kappa : X \to U$, the solution of $x^+ = f(x, u)$ at time k is denoted by $\phi(k; x, \kappa)$.

Infinite-Horizon Optimal Control–Some Notation

$$\begin{aligned} \text{minimize} \quad & \sum_{k=0}^{\infty} \ell(x_k, u_k) \\ \text{subject to} \quad & x_0 = x \\ & x_{k+1} = f(x_k, u_k), \qquad k \in \mathbb{N} \\ & x_k \in X, \quad u_k \in U, \qquad k \in \mathbb{N} \end{aligned} \qquad \mathbf{OCP}_\infty(x)$$

- Let

$$V_\infty(x, \mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_k, u_k), \qquad \mathcal{U}_\infty(x) = \{\mathbf{u} \mid x_k \in X, \quad u_k \in U, \ k \in \mathbb{N}\},$$

  where $x_k = \phi(k; x, \mathbf{u})$.
- $\mathcal{U}_\infty(x)$ is the **set of admissible inputs** corresponding to x.
- Infinite horizon **admissible state set**

$$X_\infty = \{x \in \mathbb{R}^n \mid \mathcal{U}_\infty(x) \neq \emptyset\} = \operatorname{dom} \mathcal{U}_\infty. \tag{1}$$

- Note that $X_\infty \subset X$ and the inclusion can be strict: there might not exist an admissible input sequence for each $x \in X$.

Infinite-Horizon Optimal Control–Some Notation

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=0}^{\infty} \ell(x_k, u_k) \\ \text{subject to} & x_0 = x \\ & x_{k+1} = f(x_k, u_k), \qquad k \in \mathbb{N} \\ & x_k \in X, \quad u_k \in U, \qquad k \in \mathbb{N} \end{array} \qquad \mathbf{OCP}_{\infty}(x)$$

- Value function and solution mapping of $\mathbf{OCP}_{\infty}(x)$

$$\begin{aligned} V_{\infty}^{\star}(x) &= \inf\{V_{\infty}(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_{\infty}(x)\} \\ \mathcal{U}_{\infty}^{\star}(x) &= \operatorname{argmin}\{V_{\infty}(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_{\infty}(x)\}. \end{aligned} \qquad \mathbf{OCP}_{\infty}(x)$$

- $\mathbf{OCP}_{\infty}(x)$ expressed equivalently as

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=0}^{\infty} \bar{\ell}(x_k, u_k) \\ \text{subject to} & x_0 = x \\ & x_{k+1} = f(x_k, u_k), \qquad k \in \mathbb{N} \end{array} \qquad \mathbf{OCP}_{\infty}(x)$$

where $\bar{\ell}(x, u) = \begin{cases} \ell(x, u) & (x, u) \in X \times U \\ +\infty & \text{otherwise} \end{cases}$

Outline

Bellman Equation

Theorem: Bellman

The value function of $\mathbf{OCP}_\infty(x)$, $V_\infty^\star : \mathbb{R}^n \to \overline{\mathbb{R}}_+$, satisfies

$$V_\infty^\star(x) = \inf_u \left\{\bar{\ell}(x,u) + V_\infty^\star(f(x,u))\right\}. \tag{\textbf{BE}}$$

Proposition: Necessary and sufficient optimality condition

A time-invariant feedback control law $\kappa_\infty^\star$ is optimal if and only if

$$\kappa_\infty^\star(x) \in \operatorname*{argmin}_u \{\bar{\ell}(x,u) + V_\infty^\star(f(x,u))\} \tag{1}$$

- **Bellman operator** (maps functions to functions)

$$(TV)(x) = \inf_u \left\{\bar{\ell}(x,u) + V(f(x,u))\right\}. \tag{2}$$

- Bellman equation can be expressed compactly as

$$V_\infty^\star = TV_\infty^\star.$$

- (**BE**) is a necessary condition for optimality.

Bellman and Value Iteration

$$V_\infty^\star(x) = \inf_u \left\{\bar{\ell}(x,u) + V_\infty^\star(f(x,u))\right\}. \qquad (\mathbf{BE})$$

- $V_\infty^\star$ is **fixed point** of **Bellman operator**: $V_\infty^\star = TV_\infty^\star$.
- Simplest algorithm for finding a fixed point of T is **fixed point iteration**.
- Starting from some initial estimate $V_0^\star : X \to \overline{\mathbb{R}}_+$ we iterate

$$V_{j+1}^\star = TV_j^\star, \qquad j = 0, 1, 2, \ldots. \qquad (\mathbf{VI})$$

- (**VI**) is called **value iteration**.
- The standard algorithm in DP for solving infinite horizon OCPs.
- Spelling out explicitly the Bellman operator, we can write (**VI**) as

$$V_{j+1}^\star(x) = \inf_{u \in U}\{\bar{\ell}(x,u) + V_j^\star(f(x,u))\}, \qquad j = 0, 1, 2, \ldots. \qquad (\mathbf{VI})$$

Bellman and Value Iteration

$$V_{j+1}^\star(x) = \inf_{u \in U}\{\bar{\ell}(x,u) + V_j^\star(f(x,u))\}, \qquad j = 0,1,2,\ldots. \qquad \textbf{(VI)}$$

- In (**VI**), the value functions are extended-real-valued: $V_j^\star : \mathbb{R}^n \to \overline{\mathbb{R}}_+$.
- Their effective domains are given by the recursion

$$X_{j+1} = \{x \in X \mid \exists u \in U \text{ s.t. } f(x,u) \in X_j\},\ j \in \mathbb{N}$$

  with $X_0 \triangleq \mathrm{dom}\, V_0^\star$.
- X_j is the set of states that can be steered to the terminal set X_0 by a feasible control sequence in j steps.
- (**VI**) in constrained form:

$$V_{j+1}^\star(x) = \inf_{u \in U}\{\ell(x,u) + V_j^\star(f(x,u)) \mid f(x,u) \in X_j\},\ x \in X_{j+1},$$

  for $j \in \mathbb{N}$.
- $V_j^\star$ are real-valued functions over their domain: $V_j : X_j \to \mathbb{R}_+$, $j \in \mathbb{N}_{[0,N]}$.

Bellman and Value Iteration

$$V^\star_{j+1}(x) = \inf_{u \in U} \{\bar{\ell}(x,u) + V^\star_j(f(x,u))\}, \qquad j = 0, 1, 2, \ldots. \qquad \textbf{(VI)}$$

- Initial "estimate"

$$V^\star_0(x) = \begin{cases} V_f(x) & x \in X_f \\ +\infty & \end{cases}$$

- Then $V^\star_j = T^j V^\star_0$ is value function of

$$\begin{array}{rll} \text{minimize} & \sum_{k=0}^{j-1} \ell(x_k, u_k) + V_f(x_j) & \\ \text{subject to} & x_0 = x & \\ & x_{k+1} = f(x_k, u_k) & \qquad\qquad \mathbf{OCP}_j(x) \\ & x_k \in X,\ u_k \in U, \qquad k = 0, \ldots, j-1 & \\ & x_j \in X_f & \end{array}$$

- This is the problem we solve in MPC at every sampling instant!

Convergence of Value Iteration

$$V_{j+1}^{\star}(x) = \inf_{u \in U} \{\bar{\ell}(x, u) + V_j^{\star}(f(x, u))\}, \qquad j = 0, 1, 2, \ldots. \qquad (\mathbf{VI})$$

- Fixed point iteration converges whenever T is a contraction mapping.
- But certainly **this is not the case** for the Bellman operator.
- However, T satisfies one key property, namely **monotonicity**.

Proposition: Monotonicity of Bellman operator

Let $V_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, 2$ satisfy $V_1 \le V_2$. Then

$$TV_1 \le TV_2.$$

In other words, if $V_1(x) \le V_2(x)$ for all x then

$$\min_u \{\bar{\ell}(x, u) + V_1(f(x, u))\} \le \min_u \{\bar{\ell}(x, u) + V_2(f(x, u))\}, \qquad \forall x$$

Convergence of Value Iteration

Assumption: Problem regularity

The following hold:

1. Function $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ is continuous,
2. sets $X \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$ are closed,
3. the system dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous
4. one of the following conditions is satisfied,
   4.1 $\ell(x, u) \geq \sigma(\|u\|)$ for all $(x, u) \in X \times U$, where $\sigma \in \mathcal{K}_\infty$, or
   4.2 U is compact.

- Monotonicity property and problem regularity are sufficient for VI convergence.

Convergence of Value Iteration

Theorem: Convergence of VI from below

Suppose that

$$V_0^\star(x) = \begin{cases} V_f(x) & x \in X_f \\ +\infty & \text{otherwise} \end{cases} \tag{3}$$

where $V_f : \mathbb{R}^n \to \mathbb{R}_+$ is continuous and $X_f \subset \mathbb{R}^n$ is closed and that the **regularity** assumption holds. If

$$V_0^\star \le V_\infty^\star$$

then

$$\lim_{j\to\infty} V_j^\star(x) = V_\infty^\star(x) \qquad \forall x \in X$$

Furthermore, the minimum in the right-hand side of the Bellman equation (**BE**) is attained:

$$\mathsf{dom}\, \mathcal{U}_\infty^\star = X_\infty^\star.$$

- Theorem shows convergence of VI whenever $V_0^\star$ lower bounds $V_\infty^\star$.
- A natural choice of such a $V_0^\star$ is that of (3) with $V_f(x) = 0$ and $X_f = X$.

Stability Properties of $\mathrm{OCP}_\infty(x)$

Assumption: Attractivity assumption

The following hold:

1. $f(0,0) = 0$,
2. X and U contain the origin,
3. $\ell(0,0) = 0$ and there exists an $\underline{\alpha}_\ell \in \mathcal{K}_\infty$ such that

$$\ell(x,u) \geq \underline{\alpha}_\ell(\|x\|), \qquad \forall (x,u) \in X \times U.$$

- Last item says that the equilibrium corresponds to the minimum cost ($\ell(0,0) = 0$) while the stage cost becomes larger as we move away from the origin.
- It is satisfied by the usual quadratic stage cost $\ell(x,u) = \frac{1}{2}(x^\top Q x + u^\top R u)$ provided that $Q \succ 0$. In that case $\underline{\alpha}_\ell(s) = \frac{1}{2}\lambda_{\min}(Q)s^2$.

Infinite horizon OCP and Attractivity

Theorem: Infinite horizon OCP and attractivity

Let the regularity and attractivity assumptions hold and consider the nonlinear system $x^+ = f(x, u)$ in closed loop with

$$\kappa_\infty^\star(x) \in \operatorname{argmin}\{V_\infty(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_\infty(x)\}.$$

Then $X_\infty^\star$ is positive invariant for

$$x^+ = f(x, \kappa_\infty^\star(x)) \tag{$\mathbf{CL}_\infty$}$$

and the origin is globally attractive in $X_\infty^\star$.

Infinite horizon OCP and GAS

Assumption: Constrained local stabilizability

There exist $\overline{\alpha} \in \mathcal{K}_\infty$ and $r > 0$ such that for every $x \in X_\infty^\star \cap \mathrm{B}_r(0)$ there exists an admissible input sequence $\mathbf{u} = \{u_k\}_{k \in \mathbb{N}} \in \mathcal{U}_\infty(x)$ with

$$V_\infty(x, \mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_k, u_k) \leq \overline{\alpha}(\|x\|)$$

where $x_k = \phi(k; x, \mathbf{u})$.

Theorem: Infinite horizon OCP and attractivity

Let the regularity, attractivity and constrained local stabilizability assumptions hold. Then the origin is globally asymptotically stable in $X_\infty^\star$ for

$$x^+ = f(x, \kappa_\infty^\star(x)). \qquad (\mathbf{CL}_\infty)$$

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Convergence of VI from above

Theorem: VI convergence from above

Suppose that the **regularity** and **attractivity** assumptions hold, $V_0^\star : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ is given by

$$V_0^\star(x) = \begin{cases} V_f(x) & x \in X_f \\ +\infty & \text{otherwise} \end{cases}$$

where $V_f : \mathbb{R}^n \to \mathbb{R}_+$ is continuous, $X_f \subset \mathbb{R}^n$ is closed and

$$V_\infty^\star \leq V_0^\star \tag{4}$$

If X_f **contains the origin in its interior** and there exists an $\overline{\alpha}_0 \in \mathcal{K}_\infty$ such that

$$V_f(x) \leq \overline{\alpha}_0(\|x\|) \qquad \forall x \in X_f,$$

then

$$\lim_{j \to \infty} V_j^\star(x) = V_\infty^\star(x) \qquad \forall x \in X.$$

Proof of VI Convergence from Above

From $V_\infty^\star \le V_0^\star$, **Bellman equation** and **monotonicity** of the Bellman operator, we easily deduce

$$V_\infty^\star(x) \le V_j^\star(x), \qquad \forall j \in \mathbb{N}. \tag{5}$$

If $x \notin \mathbf{dom}\, V_\infty^\star$ there is nothing to show, since, according to (5), $V_\infty^\star(x) = V_j^\star(x) = \infty$, $j \in \mathbb{N}$. Therefore we assume that $x \in \mathbf{dom}\, V_\infty^\star$. Due to **regularity** and **attractivity**, the origin is attractive in $X_\infty^\star$ for $x^+ = f(x, \kappa_\infty^\star(x))$:

$$\lim_{k\to\infty} \phi(k; x, \kappa_\infty^\star) = 0. \tag{6}$$

Let $\bar{N} \in \mathbb{N}$ (depending on x) be so large that $\phi(j; x, \kappa_\infty^\star)$ belongs to X_f (this will eventually happen due to (6) and since X_f **contains the origin in its interior**), for all $j \ge \bar{N}$. Now since $V_j^\star$ is the value function of $\mathbf{OCP}_j(x)$, for any $j \ge \bar{N}$

$$\begin{aligned} V_\infty^\star(x) \le V_j^\star(x) &\le \sum_{k=0}^{j-1} \ell(x_k, \kappa_\infty^\star(x_k)) + V_f(\phi(j; x, \kappa_\infty^\star)) \\ &\le V_\infty^\star(x) + \bar{\alpha}_0(\|\phi(j; x, \kappa_\infty^\star)\|) \end{aligned}$$

where $x_k = \phi(k; x, \kappa_\infty^\star)$, $k \in \mathbb{N}_{[0,j]}$. Taking the limit as $j \to \infty$ and using (6), we obtain that $V_\infty^\star(x) = \lim_{j\to\infty} V_j^\star(x)$.

VI Convergence from above–Remarks

- $V_\infty^\star \leq V_0^\star$ implies that $X_f = \mathbf{dom}\, V_0^\star \subseteq \mathbf{dom}\, V_\infty^\star = X_\infty^\star \subseteq X$.
- Instead of **"X_f contains the origin in its interior"**, we may assume
- $V_\infty^\star \leq V_0^\star$, $V_0^\star(0) = 0$ and the following (strong) assumption holds:

Assumption: Constrained local controllability

There exists a positive integer $\bar{N}$ such that for every $x \in X_\infty^\star \cap \mathrm{B}_r(0)$ there exists an admissible input sequence $\mathbf{u} = \{u_k\}_{k\in\mathbb{N}} \in \mathcal{U}_\infty(x)$ which satisfies $x_N = \phi(N; x, \mathbf{u}) = 0$, $N \geq \bar{N}$ and

$$V_\infty(x, \mathbf{u}) = \sum_{k=0}^{\bar{N}-1} \ell(x_k, u_k) \leq \overline{\alpha}(\|x\|)$$

where $x_k = \phi(k; x, \mathbf{u})$.

- Then **trivial upper bound** can be used: $V_0^\star(x) = \begin{cases} 0, & x = 0 \\ +\infty, & \text{otherwise} \end{cases}$

namely, $V_f \equiv 0$, $X_f = \{0\}$.

Generating meaningful upper bounds

- The tighter the upper bound on $V_\infty^\star$, the faster the convergence of VI.
- However, computing tighter upper bounds than the trivial one is not simple.

Generating meaningful upper bounds

- Recall the following **key result**

Proposition: Control Lyapunov functions and performance bounds

Suppose that $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ is such that

$$V(x) \geq \min_u \left\{ \bar{\ell}(x, u) + V(f(x, u)) \right\} \tag{7}$$

(the same as saying $V \geq TV$) and let

$$\kappa(x) \in \operatorname*{argmin}_u \left\{ \bar{\ell}(x, u) + V(f(x, u)) \right\}, \tag{8}$$

and $x_k = \phi(k; x, \kappa)$. Then

$$V_\infty^\star(x) \leq \sum_{k=0}^{\infty} \bar{\ell}(x_k, \kappa(x_k)) \leq V(x)$$

Generating meaningful upper bounds

- If we can find $V_f : \mathbb{R}^n \to \mathbb{R}_+$ and $0 \in X_f \subset \mathbb{R}^n$ with

$$V_f(x) \geq \min_{u \in U} \{\ell(x,u) + V_f(f(x,u)) \mid f(x,u) \in X_f\} \qquad \forall x \in X_f \tag{7}$$

then we have $V_0^\star \geq V_\infty^\star$ where $V_0^\star(x) = \begin{cases} V_f(x), & x \in X_f \\ +\infty, & \text{otherwise} \end{cases}$

- We call such $V_0^\star$ ($V_0^\star \geq TV_0^\star$) a **control Lyapunov function**.
- VI iterates converge to $V_\infty^\star$, they are control Lyapunov functions and **monotone decreasing**

$$V_j^\star \geq TV_j^\star = V_{j+1}^\star$$

- Moreover

$$V_\infty^\star(x) \leq \sum_{k=0}^{\infty} \ell(x_k, \kappa_{j+1}^\star(x)) \leq V_j^\star(x).$$

where $\kappa_{j+1}^\star(x) \in \operatorname{argmin}_u \left\{\bar{\ell}(x,u) + V_j(f(x,u))\right\}$.

- This is the **key to MPC stability theory!**

Time for MPC stability theory

- Recall that in MPC we measure state x at time t and we solve

$$\begin{array}{rll} \text{minimize} & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N) & \\ \text{subject to} & x_0 = x & \\ & x_{k+1} = f(x_k, u_k) & \\ & x_k \in X,\ u_k \in U, \qquad k = 0, \ldots, N-1 & \\ & x_N \in X_f & \end{array} \qquad \mathbf{OCP}_N(x)$$

  to obtain an optimal input sequence

$$\mathbf{u}^\star(x) = \{u_0^\star(x), \ldots, u_{N-1}^\star(x)\}.$$

- The MPC control law is

$$\kappa_N^\star(x) = u_0^\star(x).$$

- The value function of $\mathbf{OCP}_N(x)$ is the N-th iterate of VI
- ...and the MPC control law satisfies

$$\kappa_N^\star(x) \in \operatorname{argmin}\{\bar{\ell}(x, u) + V_{N-1}^\star(f(x, u))\}.$$

MPC stability theorem

Assumption: MPC stability assumption

Assume the following:

1. $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous with $f(0,0) = 0$
2. $X, X_f \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ are closed sets containing the origin
3. $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ is continuous with $\ell(0,0) = 0$ and either
   - 3.1 U is bounded or
   - 3.2 there exists $\sigma \in \mathcal{K}_\infty$ such that $\ell(x,u) \geq \sigma(\|u\|)$, $\forall (x,u) \in X \times U$
4. there exists $\underline{\alpha}_\ell \in \mathcal{K}_\infty$ such that $\ell(x,u) \geq \underline{\alpha}_\ell(\|x\|)$, $\forall (x,u) \in X \times U$
5. $V_f : \mathbb{R}^n \to \mathbb{R}_+$ is continuous, $V_f(0) = 0$ and satisfies

$$V_f(x) \geq \min_{u \in U} \{\ell(x,u) + V_f(f(x,u)) \mid f(x,u) \in X_f\} \qquad \forall x \in X_f$$

In terms of **DP** and **VI**, last condition means

$$V_0^\star \geq TV_0^\star \qquad \text{for} \qquad V_0^\star(x) = \begin{cases} V_f(x), & x \in X_f \\ +\infty, & \text{otherwise} \end{cases}$$

MPC stability theorem

$$\begin{aligned}
V_N^\star(x) &= \min\{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} && \text{MPC problem} \\
X_N &= \operatorname{dom}\mathcal{U}_N = \{x \in X \mid \exists u \in \mathcal{U}_N(x)\} && \text{feasible state set} \\
(u_0^\star(x), \dots, u_{N-1}^\star(x)) &\in \operatorname{argmin}\{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} && \text{optimal input sequence} \\
\kappa_N^\star(x) &= u_0^\star(x) && \text{MPC control law}
\end{aligned}$$

Theorem: MPC stability theorem I

Suppose that the **MPC stability assumption** holds. Then **for any** $N \in \mathbb{N}$

1. $X_N \subseteq X$ is **positively invariant** for $x^+ = f(x, \kappa_N^\star(x))$,
2. the origin is **globally attractive** in X_N for $x^+ = f(x, \kappa_N^\star(x))$,
3. $V_\infty^\star(x) \le \sum_{k=0}^\infty \ell(x_k, \kappa_N^\star(x)) \le V_N^\star(x)$, performance bound
4. $V_{N+1}^\star \le V_N^\star$ and $X_{N+1} \supseteq X_N$, performance improves with N

MPC stability theorem

$$\begin{aligned} V_N^\star(x) &= \min\{V_N(x,\mathbf{u}) \mid \mathbf{u}\in\mathcal{U}_N(x)\} && \text{MPC problem}\\ X_N &= \operatorname{dom}\mathcal{U}_N = \{x\in X \mid \exists u\in\mathcal{U}_N(x)\} && \text{feasible state set}\\ (u_0^\star(x),\ldots,u_{N-1}^\star(x)) &\in \operatorname{argmin}\{V_N(x,\mathbf{u}) \mid \mathbf{u}\in\mathcal{U}_N(x)\} && \text{optimal input sequence}\\ \kappa_N^\star(x) &= u_0^\star(x) && \text{MPC control law} \end{aligned}$$

Proof. Let $V_0^\star(x) = \begin{cases} V_f(x), & x\in X_f \\ +\infty, & \text{otherwise}\end{cases}$. Then $V_0^\star \geq TV_0^\star$ and this implies $V_0^\star \geq V_1^\star$ (recall $V_1^\star = TV_0^\star$). By **monotonicity** of Bellman operator: $V_0^\star \geq V_1^\star \implies V_1^\star = TV_0^\star \geq TV_1^\star$. Using induction $V_N^\star \geq TV_N^\star = V_{N+1}^\star$. This implies that for any N:

$$V_N^\star(x) \geq \ell(x,\kappa_N(x)) + V_N^\star(f(x,\kappa_N^\star(x))) \qquad \forall x\in X_N$$

Thus X_N is positive invariant for $x^+ = f(x,\kappa_N^\star(x))$. By summing up along closed-loop trajectories we obtain

$$\sum_{k=0}^{\infty} \ell(x_k,\kappa_N^\star(x_k)) \leq V_N^\star(x).$$

MPC stability theorem

$$\begin{aligned} V_N^\star(x) &= \min\{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} && \text{MPC problem} \\ X_N &= \operatorname{dom} \mathcal{U}_N = \{x \in X \mid \exists u \in \mathcal{U}_N(x)\} && \text{feasible state set} \\ (u_0^\star(x), \ldots, u_{N-1}^\star(x)) &\in \operatorname{argmin}\{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} && \text{optimal input sequence} \\ \kappa_N^\star(x) &= u_0^\star(x) && \text{MPC control law} \end{aligned}$$

Theorem: MPC stability theorem II

Suppose that the **MPC stability assumption** holds, X_f **contains the origin in its interior** and there exists an $\overline{\alpha}_0 \in \mathcal{K}_\infty$ such that

$$V_f(x) \leq \overline{\alpha}_0(\|x\|) \qquad \forall x \in X_f.$$

Then the claims of **MPC stability theorem I** are valid and moreover

1. the origin is **globally asymptotically stable** in X_N for $x^+ = f(x, \kappa_N^\star(x))$,
2. $V_N^\star$ converges to $V_\infty^\star$ as N tends to infinity.

MPC stability theorem–Remarks

- Condition on V_f in **MPC stability theorem II** can be replaced by the following **constrained stabilizability assumption**: There exists a $r > 0$ such that
$$V_N^\star(x) \leq \overline{\alpha}_0(\|x\|) \qquad \forall x \in X_N \cap \mathsf{B}_r(0).$$
- Moreover, if the **constrained controllability assumption** holds then $V_N^\star$ converges to $V_\infty^\star$ as N tends to infinity.

MPC stability–Recap

- The key to MPC stability is proper choice of terminal conditions

$$V_f(x) \geq \min_{u \in U} \{\ell(x,u) + V_f(f(x,u)) \mid f(x,u) \in X_f\} \qquad \forall x \in X_f \qquad (\mathbf{CLF})$$

- The most obvious choice is the trivial terminal conditions:

$$V_f(x) = 0 \qquad \text{and} \qquad X_f = \{0\}$$

- But X_f large $\implies$ X_N large $\implies$ larger region of attraction (set of states for which MPC works)
- We want to satisfy (**CLF**) with X_f as large and V_f as small as possible.

Terminal Sets and Control Invariance

- $X_f \subset X$ should be a control invariant set for $x^+ = f(x,u)$

$$x \in X_f \implies \exists u \in U \text{ s.t. } f(x,u) \in X_f.$$

- Largest such X_f is the maximal control invariant set for $x^+ = f(x,u)$.
- We say that $\mathcal{C}_\infty \subset X$ is **maximal control invariant set** if
  1. $\mathcal{C}_\infty$ is control invariant
  2. If $\mathcal{C}$ is control invariant then $\mathcal{C} \subseteq \mathcal{C}_\infty$
- $\mathcal{C}_\infty$ can be calculated conceptually via VI: $\mathcal{C}_0 = X$

$$\mathcal{C}_{k+1} = \{x \in X \mid \exists u \in U \text{ s.t. } f(x,u) \in \mathcal{C}_k\}, \quad k \in \mathbb{N}$$

- For nonlinear systems, as hard as solving $\mathbf{OCP}_\infty$ (it is $\mathbf{OCP}_\infty$ with $\ell \equiv 0$).
- **Not a practical choice**

Terminal Sets and Positive Invariance

Simpler choice

- Compute a "local stabilizing controller" κ and consider closed-loop system

$$x^+ = f_\kappa(x) \equiv f(x, \kappa(x)).$$

- Let $X_\kappa = \{x \in X \mid \kappa(x) \in U\}$.
- Choose X_f as a **positive invariant** subset of X_κ for $x^+ = f_\kappa(x)$

$$x \in X_f \implies \kappa(x) \in U \quad \text{and} \quad f(x, \kappa_f(x)) \in X_f$$

- Ideally, X_f is the maximal positive invariant set for $x^+ = f(x, \kappa(x))$ in X_κ.
- $\mathcal{O}_\infty$ is a **maximal positive invariant** set for $x^+ = f_\kappa(x)$ in X_κ if
  1. $\mathcal{O}_\infty \subseteq X_\kappa$ is positive invariant,
  2. If $\mathcal{O} \subseteq X_\kappa$ is positive invariant then $\mathcal{O} \subseteq \mathcal{O}_\infty$.

Maximal Positive Invariant Set

Algorithm 1: Computation of $\mathcal{O}_\infty$

$\mathcal{O}_0 = X_\kappa$
for $k = 0, 1 \ldots$ **do**
  $\mathcal{O}_{k+1} = \{x \in X_\kappa \mid f_\kappa(x) \in \mathcal{O}_k\}$
  if $\mathcal{O}_{k+1} = \mathcal{O}_k$ **then**
    $\mathcal{O}_\infty = \mathcal{O}_k$
  end
end

- $\mathcal{O}_k$: set of states that stay in X_κ for k steps

$$\begin{aligned}
\mathcal{O}_k &= \{x \in X_\kappa \mid f_\kappa(x) \in \mathcal{O}_{k-1}\} \\
&= \{x \in X_\kappa \mid f_\kappa(x) \in X_\kappa,\ f_\kappa(f_\kappa(x)) \in \mathcal{O}_{k-2}\} \\
&\ \vdots \\
&= \{x \in X_\kappa \mid f_\kappa(x) \in X_\kappa,\ f_\kappa(f_\kappa(x)) \in X_\kappa, \ldots,\ f_\kappa^k(x) \in X_\kappa\}
\end{aligned}$$

  where $f_\kappa^k(x) = \underbrace{f_\kappa(\cdots(f_\kappa}_{k \text{ times}}(x)\cdots)$ returns state of $x^+ = f_\kappa(x)$ at time k, starting from $x_0 = x$.

- Apparently $\mathcal{O}_{k+1} \subseteq \mathcal{O}_k$

Maximal Positive Invariant Set

Algorithm 2: Computation of $\mathcal{O}_\infty$

$\mathcal{O}_0 = X_\kappa$
for $k = 0, 1 \ldots$ **do**
  $\mathcal{O}_{k+1} = \{x \in X_\kappa \mid f_\kappa(x) \in \mathcal{O}_k\}$
  if $\mathcal{O}_{k+1} = \mathcal{O}_k$ **then**
    $\mathcal{O}_\infty = \mathcal{O}_k$
  end
end

- $\mathcal{O}_{k+1}$ can be expressed as $\mathcal{O}_{k+1} = f_\kappa^{-1}(\mathcal{O}_k) \cap X_\kappa$
- $f_\kappa^{-1}(\mathcal{O}_k)$ is called the **inverse image** of $\mathcal{O}_k$ under $f_\kappa : \mathbb{R}^n \to \mathbb{R}^n$

$$f_\kappa^{-1}(\mathcal{O}_k) = \{x \in \mathbb{R}^n \mid f_\kappa(x) \in \mathcal{O}_k\}$$

- $f_\kappa^{-1}(\mathcal{O}_k)$: states that will be in $\mathcal{O}_k$ in one step.
- A set $\mathcal{O}$ is positive invariant if and only if $\mathcal{O} \subset f_\kappa^{-1}(\mathcal{O})$
- $f_\kappa^{-1}(\mathcal{O}_k)$ is also called the **pre-set** in the MPC community.

Maximal Positive Invariant Set for Linear Systems

Algorithm 3: Computation of $\mathcal{O}_\infty$

$\mathcal{O}_0 = X_\kappa$
for $k = 0, 1 \ldots$ **do**
  $\mathcal{O}_{k+1} = \{x \in X_\kappa \mid f_\kappa(x) \in \mathcal{O}_k\}$
  if $\mathcal{O}_{k+1} = \mathcal{O}_k$ **then**
    $\mathcal{O}_\infty = \mathcal{O}_k$
  end
end

1. Linear system: $x^+ = Ax + Bu$
2. Linear controller: $\kappa(x) = Kx$
3. $x^+ = f_\kappa(x) \equiv (A + BK)x$ is stable
4. X_κ is compact
5. X_κ contains the origin in its interior

then Algorithm terminates after a finite number of steps.

Moreover, if X, U are polyhedral sets

$$x \in X = \{x \mid H_x x \leq g_x\}, \quad u \in U = \{u \mid H_u u \leq g_u\}$$

then $\mathcal{O}_\infty$ is a polyhedral set.

Computing $\mathcal{O}_\infty$ for linear systems

$$x^+ = A_K x \equiv (A + BK)x$$

$$x \in X = \{x \mid H_x x \leq g_x\}, \quad u \in U = \{u \mid H_u u \leq g_u\}$$

- $X_\kappa = \{x \in X \mid Kx \in U\}$ is a polyhedral set

$$X_\kappa = \{x \in \mathbb{R}^n \mid H_x x \leq g_x,\ H_u K x \leq g_u\} = \{x \in \mathbb{R}^n \mid Hx \leq g\}$$

where $H = \begin{bmatrix} H_x \\ H_u K \end{bmatrix}$, $g = \begin{bmatrix} g_x \\ g_u \end{bmatrix}$.

Computing $\mathcal{O}_\infty$ for linear systems

$$x^+ = A_K x \equiv (A + BK)x$$

$$x \in X = \{x \mid H_x x \le g_x\}, \quad u \in U = \{u \mid H_u u \le g_u\}$$

- $X_\kappa = \{x \in \mathbb{R}^n \mid Hx \le g\}$
- Inverse image $f_\kappa^{-1}(\mathcal{O}_k)$ is polyhedral if $\mathcal{O}_k = \{x \mid H_k x \le g_k\}$ is polyhedral

$$f_\kappa^{-1}(\mathcal{O}_k) = \{x \in \mathbb{R}^n \mid H_k A_K x \le g_k\}$$

Computing $\mathcal{O}_\infty$ for linear systems

$$x^+ = A_K x \equiv (A + BK)x$$
$$x \in X = \{x \mid H_x x \leq g_x\}, \quad u \in U = \{u \mid H_u u \leq g_u\}$$

- $X_\kappa = \{x \in \mathbb{R}^n \mid Hx \leq g\}$
- $\mathcal{O}_k = \{x \mid H_k x \leq g_k\} \implies f_\kappa^{-1}(\mathcal{O}_k) = \{x \in \mathbb{R}^n \mid H_k A_K x \leq g_k\}$
- $\mathcal{O}_{k+1} = f_\kappa^{-1}(\mathcal{O}_k) \cap X_\kappa$ is polyhedral

$$\mathcal{O}_{k+1} = \{x \in \mathbb{R}^n \mid Hx \leq g, H_k A_K x \leq g_k\} = \{x \in \mathbb{R}^n \mid H_{k+1} x \leq g_{k+1}\}$$

where $H_{k+1} = \begin{bmatrix} H \\ H_k A_K \end{bmatrix}$, $g_{k+1} = \begin{bmatrix} g \\ g_k \end{bmatrix}$.

- Number of linear inequalities in definition of $\mathcal{O}_k$ increases with k.
- Eliminate redundant inequalities to arrive at minimal representation.

Removing redundant inequalities in polyhedral sets

- Given: Polyhedral set $P = \{x \in \mathbb{R}^n \mid h_i^\top x \leq g_i,\ i = 1, \ldots, d\}$
- Question: Which of the d linear inequalities are essential for describing P?
- Express $P = \bigcap_{i=1}^d P_i$, $P_i = \{x \in \mathbb{R}^n \mid h_i^\top x \leq g_i\}$ (halfspaces)
- Let $P_{\setminus i} = \bigcap_{\substack{j=1 \\ j \neq i}}^d P_j$ (intersection of all halfspaces but the i-th).
- Does $x \in P_{\setminus i} \implies x \in P_i$? Equivalently, does $P_{\setminus i} \subseteq P_i$ hold?
- We can check by solving the linear program

$$\max\{h_i^\top x \mid h_j^\top x \leq g_j, j \in \mathbb{N}_{[1,d]} \setminus \{i\}\}$$

- If optimal value $\leq g_i$, then i-th constaint is redundant and can be dropped.
- In terms of **support function** $\sigma_D(v) = \max_{x \in D} v^\top x$:

$$\sigma_{P_{\setminus i}}(h_i) \leq g_i \implies i\text{-th inequality is redundant}$$

Algorithm 4: Elimination of redundant constraints in P

Input: $h_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$, $i \in \mathcal{I} = \{1, \ldots, d\}$
Output: $h_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$, $i \in \mathcal{I}^\star$
$\mathcal{I}^\star \leftarrow \mathcal{I}$
for $i = 1, 2, \ldots, d$ **do**
  $\mathcal{I}^\star \leftarrow \mathcal{I}^\star \setminus \{i\}$
  $P_{\mathcal{I}^\star} = \{x \in \mathbb{R}^n \mid h_j^\top x \leq g_j,\ j \in \mathcal{I}^\star\}$
  if $\sigma_{P_{\mathcal{I}^\star}}(h_i) > g_i$ **then**
    $\mathcal{I}^\star \leftarrow \mathcal{I}^\star \cup \{i\}$
  end
end

- Algorithm returns a **minimal representation** of P

$$P = \{x \in \mathbb{R}^n \mid h_i^\top x \leq g_i,\ \ i \in \mathcal{I}^\star\}.$$

- If we remove any of the indices in $\mathcal{I}^\star$ the set changes (grows).
- Important for keeping number of inequalities small when running invariant set computations
- Call the algorithm after set $\mathcal{O}_{k+1}$ is formed.

Stabilizing Terminal Conditions for Linear Systems

$x^+ = Ax + Bu \quad x \in X, u \in U$ (X, U polyhedral sets) $\quad \ell(x,u) = \frac{1}{2}(x^\top Qx + u^\top Ru)$

1. Choose a stabilizing feedback gain K (pair (A, B) is assumed stabilizable)
2. Find P_f that satisfies Lyapunov equation ($A_K = A + BK, Q_K = Q + K^\top RK$)

$$P_f = Q_K + A_K^\top P_f A_K$$

3. Compute maximal positive invariant set X_f for

$$x^+ = A_K x \qquad \text{in} \quad X_\kappa = \{x \in X \mid Kx \in U\}$$

$V_f(x) = \frac{1}{2}x^\top P_f x$, X_f satisfy the MPC stability conditions: For $x \in X_f$

$$\begin{aligned} V_f(x) &= \ell(x, Kx) + V_f(f(x, Kx)) && f(x, Kx) \in X_f \\ &\geq \min_{u \in U}\{\ell(x,u) + V_f(f(x,u)) \mid f(x,u) \in X_f\} \end{aligned}$$

Stabilizing Terminal Conditions for Linear Systems

$x^+ = Ax + Bu \quad x \in X, u \in U$ (X, U polyhedral sets) $\quad \ell(x,u) = \frac{1}{2}(x^\top Q x + u^\top R u)$

Typical choice

- K, P_f: Unconstrained infinite horizon LQR

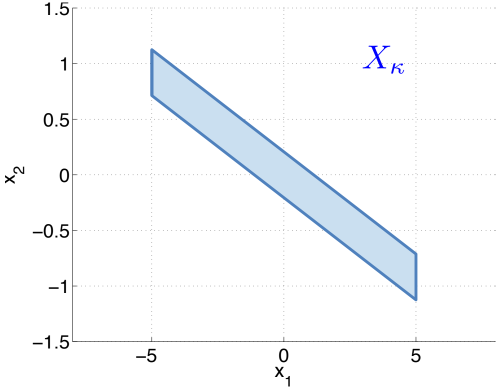
$$P_f = A^\top P_f A - (A^\top P_f B)(R + B^\top P_f B)^{-1}(B^\top P_f A) + Q$$
$$K = -(R + B^\top P_f B)^{-1}(B^\top P_f A) \quad \texttt{([Pf,\textasciitilde,K] = dare(A,B,Q,R); K=-K)}$$

- $X_f = \mathcal{O}_\infty$ for $x^+ = A_K x \equiv (A + BK)x$ in $X_\kappa = \{x \in X \mid Kx \in U\}$
- Gives optimal performance in X_f:

$$V_N^\star(x) = V_f(x) = V_\infty^\star(x) \qquad \text{for all } x \in X_f$$

- But X_f can be small (depending on the stage cost)

Computing Invariant Sets



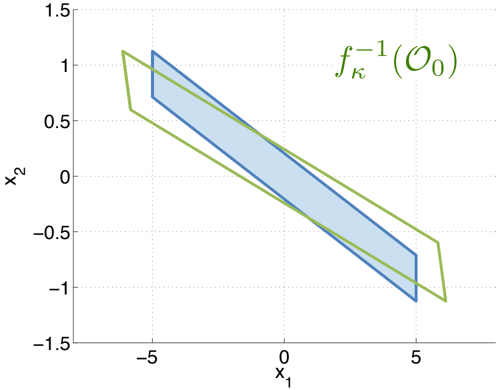
$\mathcal{O}_0 = X_\kappa$
for $k = 0, 1 \ldots$ **do**
  $\mathcal{O}_{k+1} = f_\kappa^{-1}(\mathcal{O}_k) \cap X_\kappa$
  if $\mathcal{O}_{k+1} = \mathcal{O}_k$ **then**
    $\mathcal{O}_\infty = \mathcal{O}_k$
  end
end

System:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u \qquad \begin{bmatrix} -5 \\ -10 \end{bmatrix} \leq x \leq \begin{bmatrix} 5 \\ 10 \end{bmatrix} \qquad -0.1 \leq u \leq 0.1$$

Where $u = Kx$, with K the optimal LQR controller for $Q = I$, $R = 90$.

Computing Invariant Sets



$\mathcal{O}_0 = X_\kappa$
for $k = 0, 1 \ldots$ **do**
  $\mathcal{O}_{k+1} = f_\kappa^{-1}(\mathcal{O}_k) \cap X_\kappa$
  if $\mathcal{O}_{k+1} = \mathcal{O}_k$ **then**
    $\mathcal{O}_\infty = \mathcal{O}_k$
  end
end

System:

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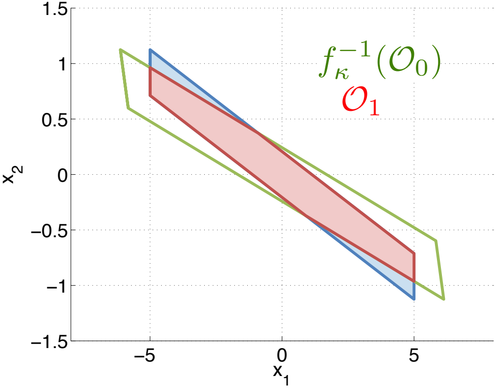
Where $u = Kx$, with K the optimal LQR controller for $Q = I$, $R = 90$.

Computing Invariant Sets



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for $k = 0, 1 \ldots$ **do**
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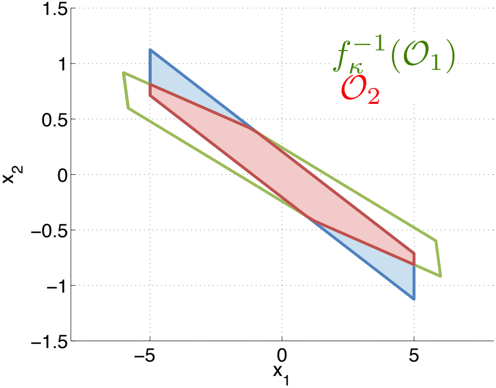

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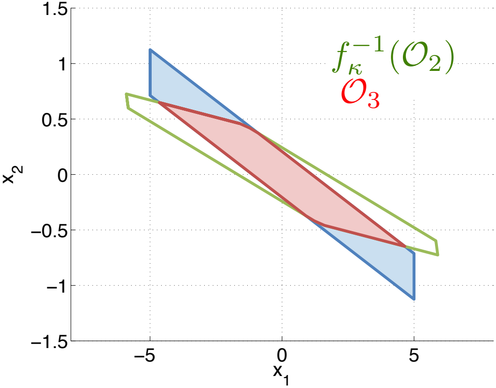

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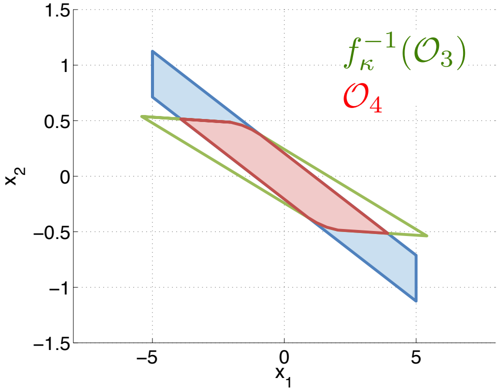

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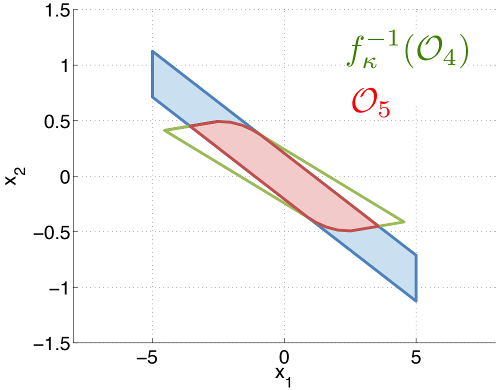

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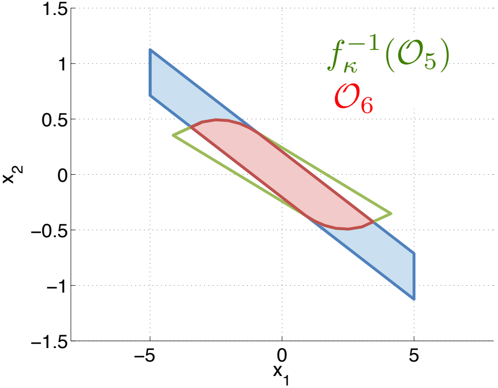

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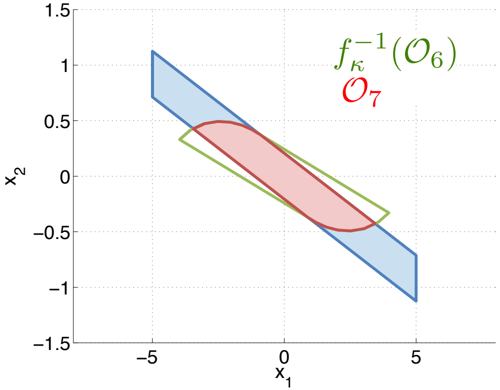

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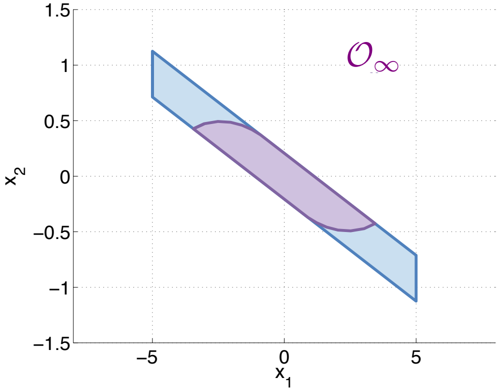

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Ellipsoid Invariant Sets

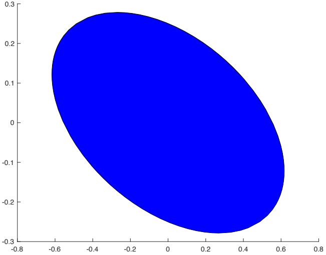
- Polyhedral invariant sets can be quite complex.
- $\mathcal{O}_\infty$ might contain an excessive number of linear inequalities.
- Number of constraints in $\mathbf{OCP}_N$ increases considerably.

Alternative choice: Ellipsoid invariant sets

- An ellipsoid is described by a **single (quadratic inequality**

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid x^\top P x \leq \alpha\}$$

where $P \in \mathbb{R}^{n \times n}$ symmetric positive definite and $\alpha > 0$.



Ellipsoid Invariant Sets

- Solve DLQR to determine P and K. Every level set of $x^\top P x$

$$\mathcal{E}(\alpha) = \{x \in \mathbb{R}^n \mid x^\top P x \leq \alpha\}$$

  is a positive invariant set for $x^+ = (A + BK)x$.
- Want to compute largest level set contained in $X_\kappa = \{x \in X \mid Kx \in U\}$.

$$\begin{aligned} &\text{maximize } \alpha \\ &\text{subject to } \mathcal{E}(\alpha) \subset X_\kappa \end{aligned}$$

Ellipsoid Invariant Sets

$$
\begin{aligned}
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\end{aligned}
$$

- X, U polyhedral: X_κ also polyhedral $X_\kappa = \{x \in \mathbb{R}^n \mid Hx \leq g\}$, $H \in \mathbb{R}^{d \times n}$.
- Problem becomes

$$
\begin{array}{ll}
\text{maximize} & \alpha \\
\text{subject to} & \sigma_{\mathcal{E}(\alpha)}(h_i) \leq g_i, \qquad i = 1, \ldots, d
\end{array}
$$

  where the support function of the ellipsoid is

$$\sigma_{\mathcal{E}(\alpha)}(v) = \alpha^{1/2} \| P^{-1/2} v \|_2$$

  or

$$
\begin{aligned}
&\text{maximize } \alpha \\
&\text{subject to } \alpha \leq \frac{g_i^2}{h_i^\top P^{-1} h_i}, \qquad i = 1, \ldots, d
\end{aligned}
$$

  Therefore $\alpha^\star = \min \left\{ \frac{g_i^2}{h_i^\top P^{-1} h_i} \mid i = 1, \ldots, d \right\}$.

Terminal Conditions for Nonlinear Systems

- Determining appropriate V_f, X_f is much harder for nonlinear systems.
- Trivial choice

$$V_f(x) = 0, \qquad\qquad X_f = \{0\}$$

  works under a local constrained controllability assumption.
- But this choice leads to small region of attraction.
- Another choice is based on linearization.

Terminal Conditions for Nonlinear Systems

$$x^+ = f(x, u) \qquad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \text{ smooth}$$

- Define the linearization

$$x^+ = Ax + Bu + \Delta f(x, u).$$

where $A = \frac{\partial f}{\partial x}(0)$, $B = \frac{\partial f}{\partial u}(0)$ and $\Delta f(x, u) = f(x, u) - Ax - Bu$ is such that

$$\|\Delta f(x, u)\| \leq L_f \|(x, u)\|^2$$

- Cost is given by $\ell(x, u) = \frac{1}{2}\left(x^\top Q x + u^\top R u\right)$ with Q, R positive definite.
- Let K be such that $A_K = A + BK$ is stable and let $Q_K = Q + K^\top R K$.
- We can next find the unique $P_f \succ 0$ such that

$$A_K^\top P_f A_K - P_f + 2 Q_K = 0.$$

Terminal Conditions for Nonlinear Systems

- Consider the linearization of f around x, in closed-loop with $u = Kx$:

$$x^+ = A_K x + \Delta f_K(x)$$

where $\Delta f_K(x) = \Delta f(x, Kx)$.

- We now have $\|\Delta f_K(x)\| \leq L_{f_K} \|x\|^2$, where $L_{f_K} = L_f(1 + \|K\|^2)$.
- We can show that for $\|x\| < \eta$ with

$$\eta = \frac{\sqrt{\|A_K\|^2 + \frac{\lambda_{\min}(Q_K)}{\lambda_{\max}(P)}} - \|A\|}{L_{f_K}}$$

we have

$$V_f(f(x, Kx)) - V_f(x) \leq -\tfrac{1}{4} x^\top (2Q_K) x = -\tfrac{1}{2} x^\top (Q + K^\top R K) x = -\ell(x, Kx)$$

where the terminal cost is $V_f(x) = \frac{1}{2} x^\top P_f x$.

Terminal Conditions for Nonlinear Systems

- The terminal set is chosen to be a level set of V_f, namely

$$X_f = \text{lev}_{\leq \alpha/2} V_f = \{x \in \mathbb{R}^n \mid V_f(x) \leq \tfrac{1}{2}\alpha\}$$

where $\alpha \in (0, \lambda_{\min}(P)\eta^2]$. Note that with this choice of α we have

$$x \in X_f \implies \|x\| \leq \eta,$$

and therefore X_f is control invariant.

- If state, input constraints are present, need to further reduce α in order to satisfy

$$X_f \subset X_\kappa = \{x \in X \mid Kx \in U\}$$

- In general it is computationally hard to find such an α, unless linear constraints.
- If $X = \{x \in \mathbb{R}^n \mid H_x x \leq g_x\}$, $U = \{u \in \mathbb{R}^m \mid H_u u \leq g_u\}$ we can set

$$\alpha = \min\left\{\lambda_{\min}(P_f)\eta^2, \min\left\{\frac{g_i^2}{h_i^\top P_f^{-1} h_i} \mid i = 1, \ldots, d\right\}\right\}$$

where $h_i^\top$, g_i are the i-th row and i-th element of $H = \begin{bmatrix} H_x \\ H_u K \end{bmatrix}, g = \begin{bmatrix} g_x \\ g_u \end{bmatrix}$.