THE NUMBER OF HYPERPLANES SPANNED BY LINEAR INDEPENDENT ZERO-ONE VECTORS

Let $c \in \mathbb{R}^n$ and $T \in \mathbb{R}$. A **Boolean threshold function** $f\colon \{0,1\}^n \to \{0,1\}$ is a function defined by

$$f(x) = \begin{cases} 1, & \text{if } c^T x \geq T \\ 0, & \text{if } c^T x < T. \end{cases}$$

The hyperplane $\{x \in \mathbb{R}^n : c^T x = T\}$ divides the cube $[0,1]^n$ into two (possibly empty) polyhedra $S_1 := \{x \in [0,1]^n : c^T x \geq T\}$, and $S_2 := \{x \in [0,1]^n : c^T x < T\}$. Thus, f applied to any vertex of S_1 is 1 and applied to any vertex of S_2 is 0. We say that $\{x \in \mathbb{R}^n : c^T x = T\}$ is the **hyperplane induced by** f.

Consider the LP

$$\text{for all } x \in S_1 : c^T x \geq 0 \text{ and}$$
$$\text{for all } x \in S_2 : c^T x \leq -1$$

is a basis solution $v_1, \dots, v_i, v_{i+1}, \dots, v_n$ where $v_1, \dots, v_i \in S_1$ and $v_{i+1}, \dots, v_n \in S_2$ are linearly independent and, as basis solution they satisfy $c^T v_j = 0$ for all $j \in \{1, \dots, i\}$ and $c^T v_j = -1$ for all $j \in \{i+1, \dots, n\}$.

Consider the "lifted" hyperplane $\{(x^T, x_{n+1})^T \in \mathbb{R}^{n+1} : c^T x - T x_{n+1} \geq 0\}$ in $\mathbb{R}^{n+1}$ defining the polyhedra $S_1^{new} \subseteq \mathbb{R}^{n+1}$ and $S_2^{new} \subseteq \mathbb{R}^{n+1}$ as we defined S_1 and S_2 before. It follows that $S_1 = \{x \in \mathbb{R}^n : (x^T, 1)^T \in S_1^{new}\}$ and $S_2 = \{x \in \mathbb{R}^n : (x^T, 0)^T \in S_2^{new}\}$.

Consider the lifted basis solution

$$\begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} v_i \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} v_{i+1} \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} v_n \\ 1 \end{pmatrix}$$

This solution defines as well a hyperplane through 0 in $\mathbb{R}^{n+1}$ as a threshold function. Thus, the number of hyperplanes spanned by $(n-1)$ linear independent $0/1$ vectors is bigger or equal the number of Boolean threshold functions and thus also the number of facets of the White Whale $\mathscr{W}(m)$. A lower bound on the number of Boolean threshold functions is given by the following Proposition.

Proposition 1. *The number of Boolean threshold functions is bigger or equal* $2^{\frac{1}{2}n^2}$.

The proof follows by the following lemmas.

Lemma 2. *Let* $f, g\colon \{0,1\}^n \to \{0,1\}$ *and* $f', g'\colon \{0,1\}^n \to \{0,1\}$ *be Boolean threshold functions. Define two Boolean threshold functions* $h, h'\colon \{0,1\}^{n+1} \to \{0,1\}$ *as*

$$h(x, x_{n+1}) = f(x) x_{n+1} + g(x)(1 - x_{n+1})$$
$$h'(x, x_{n+1}) = f'(x) x_{n+1} + g'(x)(1 - x_{n+1}).$$

Then if $f \neq f'$ *or* $g \neq g'$ *it follows that* $h \neq h'$.

Lemma 3. *Let* $c \in \mathbb{R}^n$ *and* $T_f, T_g \in \mathbb{R}$ *be given. Let* $f, g\colon \{0,1\}^n \to \{0,1\}$ *be Boolean threshold functions defined as*

$$f(x) = \begin{cases} 1, & \text{if } c^T x \geq T_f \\ 0, & \text{if } c^T x < T_f \end{cases}$$

and

$$g(x) = \begin{cases} 1, & \textit{if } c^T x \geq T_g \\ 0, & \textit{if } c^T x < T_g. \end{cases}$$

Then h, as defined, in Lemma 2 is a threshold function.

Proof. If $x_{n+1} = 1$, then $h(x,1) = 1$ if and only if $c^T x \geq T_f x_{n+1}$. If $x_{n+1} = 0$, then $h(x,0) = 1$ if and only if $c^T x \geq T_g x_{n+1}$. Thus, $h(x, x_{n+1}) = 1$ if and only if $c^T x + (T_g - T_f) x_{n+1} \geq T_g$. □

Now let $f\colon \{0,1\}^n \to \{0,1\}$ be a given Boolean threshold function with $c \in \mathbb{R}^n$ and we assume w.l.o.g. that $c^T x \neq c^T y$ for all $x, y \in \{0,1\}^n$ with $x \neq y$. Define a series of Boolean threshold functions $g_i\colon \{0,1\}^n \to \{0,1\}$ such that the hyperplane induced by g_i cuts off $i \in \{0, \ldots, 2^n\}$ many integer points of $[0,1]^n$. So, there are $2^n + 1$ different g_i. By Lemma 3, the functions $h^{f,i}$ defined by $h^{f,i}(x, x_{n+1}) = f(x) x_{n+1} + g_i(x)(1 - x_{n+1})$ are Boolean threshold functions. By Lemma 2, $h^{f,i} \neq h^{f',i'}$ if and only if $f \neq f'$ or $i \neq i'$. Thus, for the number $T(n+1)$ of Boolean threshold functions we have $T(n+1) \leq (2^n + 1) T(n) \leq \ldots \leq 2^n \cdot 2^{n-1} \cdots 2^0 = 2^{\frac{1}{2} n^2}$, by the geometric series.

Thus, we know that both the number of facets and the number of vertices of the White Whale $\mathscr{W}(m)$ are in $2^{\Omega(n^2)}$.

Question 1. *So, the question arises how the rest of the face lattice behaves, i.e., if all numbers of faces are in* $2^{\Omega(n^2)}$ *or if there is a different behaviour for the other faces of* $\mathscr{W}(m)$ *(if the face lattice looks more "convex" or "concave")*