MATH 2940 Study Guide

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Overview

1 Textbook Information

Lay, Lay, McDonald, "Linear Algebra with Applications", 6th edition

2 Aids

Matrix Calculator

3 Sources

1. Lay, Lay, McDonald, "Linear Algebra with Applications", 6th edition
2. Cornell Spring 2023 semester Canvas notes (Professor Frans Schalekamp)

4 Syntax

- **linear equations** are basically a diagonal line on a graph
- a **system of linear equations (SoE)** is a group of linear equations with the same variables

4.1 Solving Linear Equations

- SoEs can have one of three solutions:
  1. exactly one solution
  2. no solutions
  3. infinitely many solutions
- linear equations can be quite long and cumbersome to write out, so we want a way to make the method for solving them more precise, and a shorter notation for writing them. Thus, **augmented** and **coefficient matrices**

  Example 4.1.1.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

  - augmented matrix (includes the right-hand sides of the equations):

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -8 & 0 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

  - coefficient matrix (does not include right-hand sides of equations)

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

4.2 Matrices

Elementary Row operations (EROs)

1. **Replacement**

   Replacing row by sum of itself and multiple of another row
2. **Scaling**

   Multiply all entries in a row with a nonzero constant
3. **Interchange**

   Swap two rows

Solving SoEs with Matrices

- use Gauss-Jordan Elimination to get matrices in reduced echelon form and solve them
- once the **forward phase of G-J elimination** is done, the resulting matrix can be used to determine the solutions of the system:
  - if there is a row such that $0 = \text{b}$ (where b is nonzero), i.e. a row that looks like $\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & b \end{bmatrix}$, then the system is inconsistent. Otherwise, the system is consistent
- after **backward phase** is done, **parametric description** of solution of system can be determined
  - translate the matrix back into SoE. variables corresponding to pivot columns are **basic variables**, and the other variables are **free variables**.
  - Write the basic variables in terms of the free variables. This is known as the parametric description of the solution set.

Example 4.2.1. Reduce $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$ to reduced row echelon form.

In the below example, pivot positions are circled.

$$
\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}
$$

$$
\xrightarrow{①\leftrightarrow④} \begin{bmatrix} ① & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}
$$

$$
\xrightarrow{②\leftarrow②+1①} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}
$$

$$
\xrightarrow{③\leftarrow③+2①} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & ② & 4 & 6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}
$$

$$
\xrightarrow{③\leftarrow③+\text{-}(5/2)②} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}
$$

$$
\xrightarrow{④\leftarrow④+\text{-}(3/2)②} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}
$$

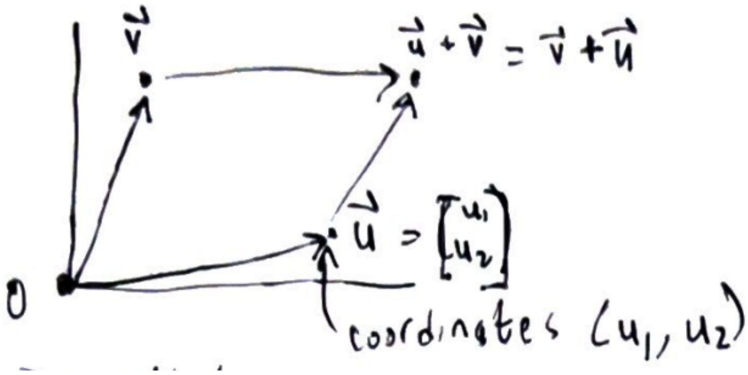
$$
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}
$$

4.3 Vectors

- solution to matrix can be written as a list of vectors $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
- vectors are equal when all entries are the same

Operations

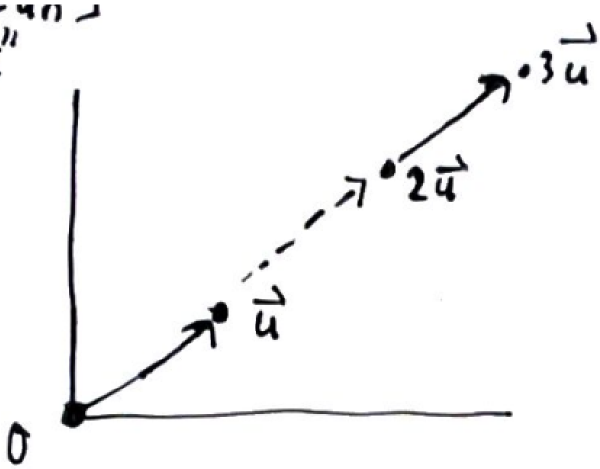
- Addition
  - $\vec{u} + \vec{v}$ defined if $\vec{u}$ and $\vec{v}$ have the same number of entries
  - $\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$



- Multiplication
  - scalar (real number) times vector only
  - $c\vec{u} = c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix} = \begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$

c is called "scalar"

• geom. intuition



Properties

- zero vector $= \vec{0} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$
- $-\vec{u} = (-1)\vec{u}$
- $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
- $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
- $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$
- $-\vec{u}+\vec{u} = \vec{0}$
- $(c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

- $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
- $c(d\vec{u}) = (cd)\vec{u}$
- $1\vec{u} = \vec{u}$

Linear Combinations

- a **linear combination** is a vector written as the sum of other vectors
- the equation $\vec{y} = c_1\vec{v_1} + \ldots + c_p\vec{v_p}$
- $v_1, v_2, ..., v_p$ are vectors $c_1, c_2, ..., c_p$ are **weights**
- $\vec{y}$ is written as a linear combination of $v_1, v_2, ..., v_p$ with weights $c_1, c_2, ..., c_p$

  Example 4.3.1. $\begin{bmatrix}4\\5\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ because $\begin{bmatrix}4\\5\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}2\\3\end{bmatrix}$. The weights are 2 for $\begin{bmatrix}1\\1\end{bmatrix}$ and 1 for $\begin{bmatrix}2\\3\end{bmatrix}$.

4.4 Span

- because of our definition of linear combinations, we have the following relation:

  a vector equation
  $$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = \vec{b}$$
  has the same solution set as the linear system with augmented matrix
  $$\begin{bmatrix}\vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} & \vec{b}\end{bmatrix}$$

- solving this matrix is equivalent to finding weights in a linear combination
- the **span** of a set of vectors is set of all possible linear combinations of those vectors
- if an augmented matrix $\begin{bmatrix}\vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} & \vec{b}\end{bmatrix}$ is consistent, then $\vec{b}$ is in the span of $\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}\}$, or, in other words, $\vec{b} \in span\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}\}$
- the span of two vectors is a plane
- geometric intuition:

4.5 The Matrix Equation $A\vec{x} = \vec{b}$

- fundamental idea of linear algebra is to view linear combo of vectors as product of matrix and vector (matrix vector multiplication)
- $A\vec{x}$ only defined when # columns of A = # entries of $\vec{x}$
- $A\vec{x}$ is a vector

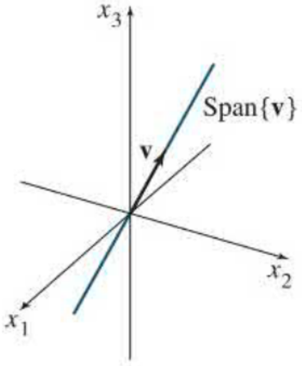


FIGURE 10 Span{**v**} as a line through the origin.

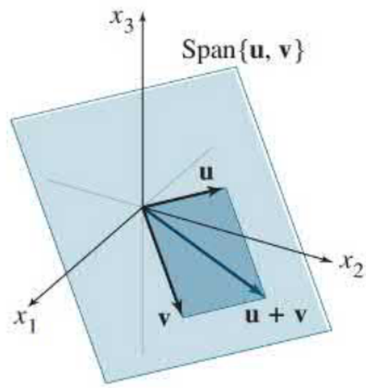


FIGURE 11 Span{**u**, **v**} as a plane through the origin.

- Ex:

$$\begin{aligned}\begin{bmatrix}1 & 2 & -1\\ 5 & 6 & 3\end{bmatrix}\begin{bmatrix}2\\7\\3\end{bmatrix} &= 2\begin{bmatrix}1\\5\end{bmatrix}+7\begin{bmatrix}2\\6\end{bmatrix}+3\begin{bmatrix}-1\\3\end{bmatrix}\\ &= \begin{bmatrix}2\\10\end{bmatrix}+\begin{bmatrix}14\\42\end{bmatrix}+\begin{bmatrix}-3\\9\end{bmatrix}\\ &= \begin{bmatrix}13\\61\end{bmatrix}\end{aligned}$$

- See Theorem 2 and Theorem 3
- use Gauss-Jordan Elimination see if system is consistent or not
- Properties of $A\vec{x}$ multiplication: Theorem 4

4.6 Homogenous and Nonhomogenous SoEs, and Parametric Vector Form

- **homogenous** SoEs are SoEs hat cna be written as $A\vec{x} = \vec{0}$
- homogenous SoEs are always consistent ($\vec{a} = \vec{0}$ is always solution, which is called the **trivial solution**)
- care about nontrivial solutions to the equation since we know that there is always a solution
- can write solution set of matrix equation as a vector using parametric description of solution set, known as **parametric vector form**
- to write in parametric vector form, first write out $\vec{x}$ as a vector, then "extract" free variables (look at math notes for continutation)

  Example 4.6.1. The vector form of the parametric description

$$\begin{aligned}x_1 &= (4/3)x_3\\ x_2 &= 0\\ x_3 &\, is\ free\end{aligned}$$

is

$$\vec{x} = \begin{bmatrix} (4/3)x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

5 Important Theorems

Theorem 1. Each matrix is row equivalent to one and only one reduced echelon matrix.

Theorem 2. When A is $m \times n$ matrix with columns $a_1, \ldots, a_n$, and $\vec{b}$ in $\mathbb{R}^m$,

$$A\vec{x} = \vec{b}$$

has same solution set as

$$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = \vec{b}$$

which has same solution set as

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} & \vec{b} \end{bmatrix}$$

Theorem 3. Let A be an $m \times n$ matrix (coefficient matrix). Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.

b. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Theorem 4. Let A be an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

b. $A(c\vec{u}) = c(A\vec{u})$

6 Toolbox

Make sure to check your work after using any of these methods!

6.1 Linear Equations

To solve systems of linear equations:

1. Variable Elimination
   - System of 2 variables
     (a) Eliminate one of the variables by adding or subtracting the equations together. This will give you can equation to solve for the other variable. Solve for this value.
     (b) Plug the value for the variable that was just solved for into either of the equations to solve for the value of the other variable.
   - System of 3 or more variables
     (a) Choose a variable in one equation to eliminate from the other equations. Eliminate this variable by adding or subtracting the equations together. Repeat the first step until you are able to solve for one variable.
     (b) Plug the value for the variable that was just solved for into either of the equations to solve for the value of the another variable. Repeat this step until all variables are solved for.
2. Substitution
   (a) Define one variable in terms of another, and plug it into all other equations. Use this equation to solve for this variable. If the system of equations has 3 or more equations, use all the other equations to solve for one variable in terms of another.

6.2 Gauss-Jordan Elimination

1. Create augmented matrix for the SoE.
2. Use first nonzero entry in the leftmost column as first pivot. If necessary, interchange rows so that there is a nonzero entry in the top entry of the first column.
3. Use EROs to create all 0s below the pivot position in the first column.
4. Ignore the row with the pivot position, and apply steps 2-3 to the submatrix that remains until there are no more nonzero rows to reduce.
   - after this step, one can determine the pivot positions and pivot columns of the matrix
5. Create zeroes above each pivot, starting with the rightmost pivot. If a pivot is not a 1, make it a 1 with a scaling operation.

7 Terminology

linear equation
: a linear equation in vars $x_1, x_2, \ldots, x_n$ is an equation that can be written as

$$a_1x_1 + a_2x_2 + \ldots a_nb_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real (or complex) numbers.

system of linear equations (SoE) or linear system
: one or more linear equations with the same variables.

consistent system
: an SoE that has one, or infinitely many solutions

inconsistent system
: an SoE that has no solution

augmented matrix
: a representation of a system of equations that includes the right-hand sides of the equations.

coefficient matrix
: a representation of a system of equations that does not include the right-hand sides of the equations.

leading entry
: The first nonzero entry of a matrix.

basic vairable
: variables in an SoE that correspond to pivot columns in the REF of a matrix.

free vairable
: variables in an SoE that are not basic variables. These variables can take on any value and have the equations of the SoE hold true.

echelon form (EF)
: A matrix is in echelon form if it has the following properties:

1. All nonzero rows are above any rows of all zeroes
2. Each leading entry of a row is in a column to the right of the leading entry above it
3. all entries in a column below a leading entry are zeroes

$$\begin{bmatrix} \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form (REF)
: A matrix is in reduced echelon form if it is in reduced echelon form and has the following additional properties:

1. The leading entry of each nonzero row is 1
2. Each leading 1 is the only nonzero entry in its column

$$\begin{bmatrix} 1 & * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot position
given a matrix in echelon or reduced form, a pivot position is a position with a leading number in a row.

pivot columns
given a matrix in echelon or reduced form, a pivot column is a column with a pivot columns.

forward phase of G-J Elimination
getting the matrix into echelon form (steps 1-4 of G-J elimination)

backward phase of G-J Elimination
getting the matrix into echelon form (step 5 of G-J elimination)

parametric description
a description of all the variables involved in a system with their associated values

linear combination
a description of a vector as a sum of vectors multiplied by scalars. The scalars the vectors are multiplied by are called **weights**.

span
the set of all possible linear combinations of a set of vectors (span$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_p}\}$ is the set of all possible linear combinations of $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_p}\}$)

homogenous system of linear equations
an SoE that can be written as $A\vec{x} = \vec{0}$.

trivial solution
a solution to the matrix equation $A\vec{x} = \vec{0}$ where $\vec{x} = \vec{0}$.

nontrivial solution
a solution to the matrix equation $A\vec{x} = \vec{0}$ where $\vec{x} \neq \vec{0}$.

parametric vector form
a way of representing the solution set to a matrix equation where $\vec{x}$ is a sum of vectors with its free variables as weights.