MATH 2940 Study Guide

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Overview

1 Textbook Information

Lay, Lay, McDonald, “Linear Algebra with Applications”, 6th edition

2 Aids

Matrix Calculator

3 Sources

1. Lay, Lay, McDonald, “Linear Algebra with Applications”, 6th edition
2. Cornell Spring 2023 semester Canvas notes (Professor Frans Schalekamp)

4 Syntax

- **linear equations** are basically a diagonal line on a graph
- a **system of linear equations (SoE)** is a group of linear equations with the same variables

4.1 Solving Linear Equations

- SoEs can have one of three solutions:
  1. exactly one solution
  2. no solutions
  3. infinitely many solutions
- linear equations can be quite long and cumbersome to write out, so we want a way to make the method for solving them more precise, and a shorter notation for writing them. Thus, **augmented** and **coefficient matrices**

  Example 4.1.1.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

  – augmented matrix (includes the right-hand sides of the equations):

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -8 & 0 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

  – coefficient matrix (does not include right-hand sides of equations)

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

4.2 Matrices

Elementary Row operations (EROs)

1. **Replacement**

   Replacing row by sum of itself and multiple of another row
2. **Scaling**

   Multiply all entries in a row with a nonzero constant
3. **Interchange**

   Swap two rows

Solving SoEs with Matrices

- use Gauss-Jordan Elimination to get matrices in reduced echelon form and solve them
- once the **forward phase of G-J elimination** is done, the resulting matrix can be used to determine the solutions of the system:
  - if there is a row such that 0 = b (where b is nonzero), i.e. a row that looks like $\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & b \end{bmatrix}$, then the system is inconsistent. Otherwise, the system is consistent
- after **backward phase** is done, **parametric description** of solution of system can be determined
  - translate the matrix back into SoE. variables corresponding to pivot columns are **basic variables**, and the other variables are **free variables**.
  - Write the basic variables in terms of the free variables. This is known as the parametric description of the solution set.

Example 4.2.1. Reduce $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$ to reduced row echelon form.

In the below example, pivot positions are circled.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow{①\leftrightarrow④} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{②\leftarrow②+1①} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow{③\leftarrow③+2①} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{③\leftarrow③+\text{-}(5/2)②} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow{④\leftarrow④+\text{-}(3/2)②} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\xrightarrow{③\leftrightarrow④} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in echelon form. To reduce to reduced row echelon form, we need to use row operations to create 0s above the pivots. The following row operations achieve that.

$$
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow{3 \leftarrow (-1/5) 3}
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\xrightarrow{2 \leftarrow 2 + 6 \, 3}
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow{1 \leftarrow 1 + 9 \, 3}
\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

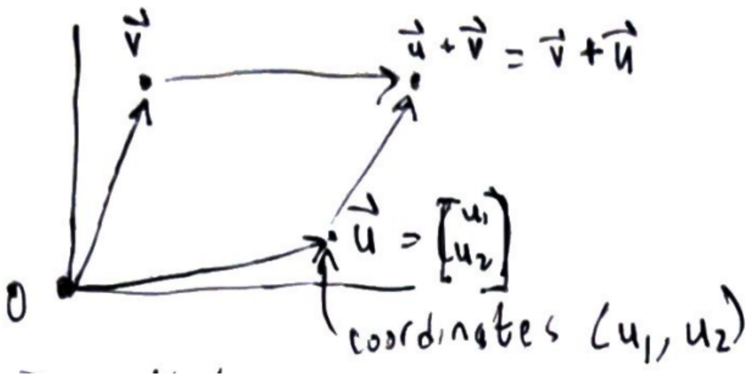
$$
\xrightarrow{2 \leftarrow (1/2) 2}
\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow{1 \leftarrow 1 + -4 \, 2}
\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

4.3 Vectors

- solution to matrix can be written as a list of vectors $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
- vectors are equal when all entries are the same

Operations

- Addition
  - $\vec{u} + \vec{v}$ defined if $\vec{u}$ and $\vec{v}$ have the same number of entries
  - $\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$

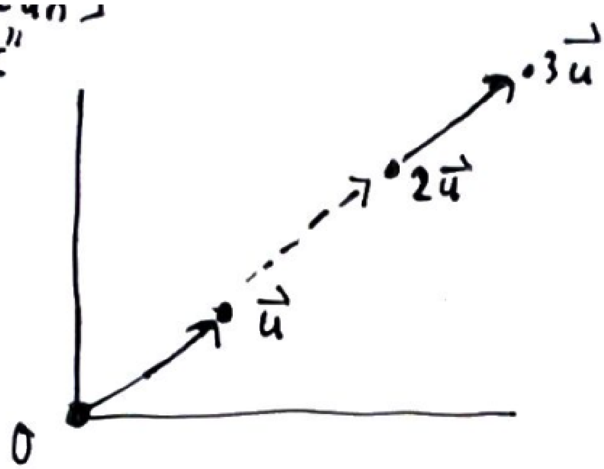


- Multiplication
  - scalar (real number) times vector only

- $c\vec{u} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$

c is called "scalar"

• geom. intuition



Properties

- zero vector $= \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$
- $-\vec{u} = (-1)\vec{u}$
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
- $-\vec{u} + \vec{u} = \vec{0}$
- $(c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
- $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
- $c(d\vec{u}) = (cd)\vec{u}$
- $1\vec{u} = \vec{u}$

Linear Combinations

- a **linear combination** is a vector written as the sum of other vectors
- the equation $\vec{y} = c_1\vec{v_1} + \ldots + c_p\vec{v_p}$
- $v_1, v_2, ..., v_p$ are vectors $c_1, c_2, ..., c_p$ are **weights**
- $\vec{y}$ is written as a linear combination of $v_1, v_2, ..., v_p$ with weights $c_1, c_2, ..., c_p$

Example 4.3.1. $\begin{bmatrix}4\\5\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ because $\begin{bmatrix}4\\5\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}2\\3\end{bmatrix}$. The weights are 2 for $\begin{bmatrix}1\\1\end{bmatrix}$ and 1 for $\begin{bmatrix}2\\3\end{bmatrix}$.

4.4 Span

- because of our definition of linear combinations, we have the following relation:

  a vector equation
  $$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = \vec{b}$$
  has the same solution set as the linear system with augmented matrix
  $$\begin{bmatrix}\vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} & \vec{b}\end{bmatrix}$$
- solving this matrix is equivalent to finding weights in a linear combination
- the **span** of a set of vectors is set of all possible linear combinations of those vectors
- if an augmented matrix $\begin{bmatrix}\vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} & \vec{b}\end{bmatrix}$ is consistent, then $\vec{b}$ is in the span of $\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}\}$, or, in other words, $\vec{b} \in span\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}\}$
- the span of two vectors is a plane
- geometric intuition:

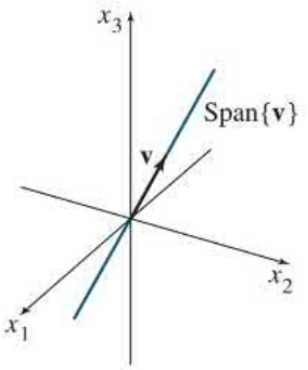


FIGURE 10 Span{**v**} as a line through the origin.

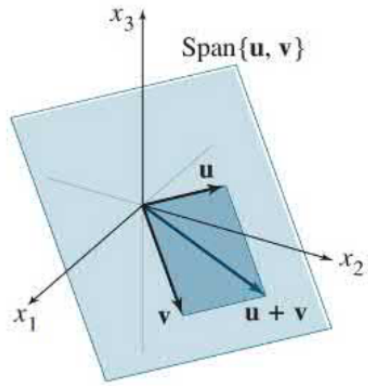


FIGURE 11 Span{**u**, **v**} as a plane through the origin.

4.5 The Matrix Equation $A\vec{x} = \vec{b}$

- fundamental idea of linear algebra is to view linear combo of vectors as product of matrix and vector (matrix vector multiplication)
- $A\vec{x}$ only defined when # columns of A = # entries of $\vec{x}$
- $A\vec{x}$ is a vector

Example 4.5.1.

$$\begin{bmatrix} 1 & 2 & -1 \\ 5 & 6 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 7 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} 14 \\ 42 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$
$$= \begin{bmatrix} 13 \\ 61 \end{bmatrix}$$

- See Theorem 2 and Theorem 3
- use Gauss-Jordan Elimination see if system is consistent or not
- Properties of $A\vec{x}$ multiplication: Theorem 4

4.6 Homogenous and Nonhomogenous SoEs, and Parametric Vector Form

- **homogenous** SoEs are SoEs hat cna be written as $A\vec{x} = \vec{0}$
- homogenous SoEs are always consistent ($\vec{a} = \vec{0}$ is always solution, which is called the **trivial solution**)
- care about nontrivial solutions to the equation since we know that there is always a solution
- can write solution set of matrix equation as a vector using parametric description of solution set, known as **parametric vector form**
- to write in parametric vector form, see Parametric Vector Form

  Example 4.6.1. The vector form of the parametric description

$$x_1 = (4/3)x_3$$
$$x_2 = 0$$
$$x_3\, is\, free$$

  is

$$\vec{x} = \begin{bmatrix} (4/3)x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

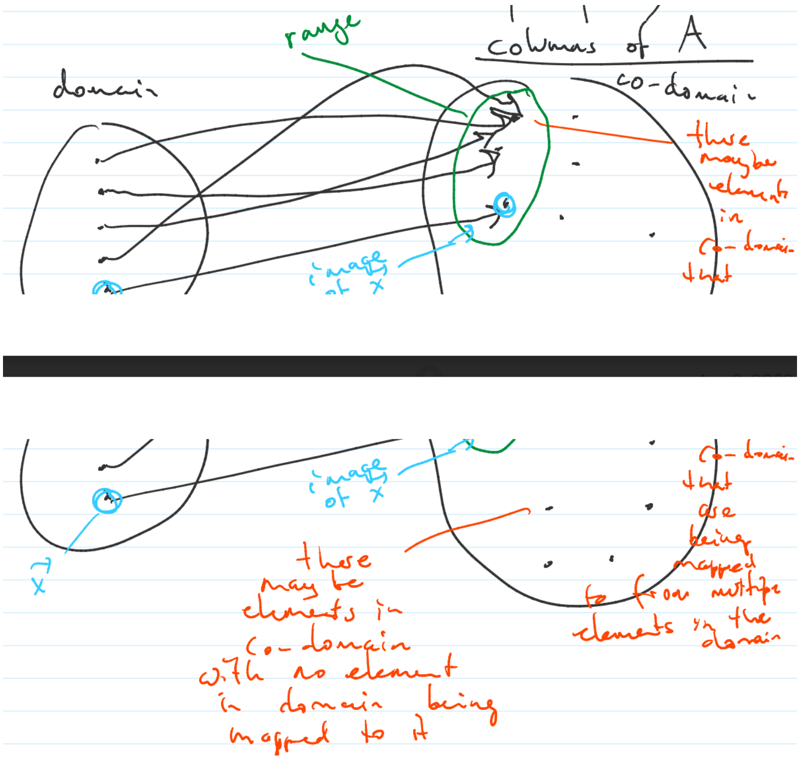
- Non-homogenous SoEs have same parametric vector form solution as $A\vec{x} = \vec{0}$, but with extra vector added that comes from $\vec{b}$ (see Theorem 5)

4.7 Linear Dependence

- see definitions for **linear independence and linear dependence**
- columns of a matrix A are linearly indep. $\Leftrightarrow A\vec{x} = \vec{0}$ has only trivial solution
- see Checking Linear Independence, Theorem 6, Theorem 7, and Theorem 8

4.8 Transformations

- a transformation T is a function that maps a vector $\vec{x}$ to a vector $A\vec{x}$
- $T(\vec{x}) = A\vec{x}$ for a matrix A. Notation: $T : \mathbb{R}^n \to \mathbb{R}^m$
- T has a **domain** and **co-domain**. For a $\vec{x}$, the **image** of $\vec{x}$ is $T(\vec{x}) = A\vec{x}$, and the **range** is all possible images under T.



- $\vec{x} \mapsto A\vec{x}$ means $\vec{x}$ maps to $A\vec{x}$
- Properties of linear transformations:
  1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
  2. $T(c\vec{u}) = cT(\vec{u})$
- these properties lead to the following facts:
  1. $T(\vec{0}) = \vec{0}$ (though the $\vec{0}$ vectors can be in different spaces)
  2. $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$.
  3. $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots + c_pT(\vec{v}_p)$
- linear transformations can also be **onto** (all points in co-domain have at least one arrow pointing in) and/or **one-to-one** (no points in co-domain with more than 1 arrow pointing in)
- See Theorem 9, Theorem 10, Theorem 11

4.9 Key Matrices and Matrix Algebra

- think of matrices as structure with vectors as columns

$$A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \dots & \vec{a_n} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{11} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- **diagonal entries** are $a_{11}, a_{22}, a_{33}, \dots$, which form **main diagonal** of A.
- **diagonal matrix** (only diagonal entries are nonzero) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, **identity matrix** (diagonal matrix where diagonal entries are 1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and **zero matrix** (all entries 0) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are also important to know about
- Addition:
  - $A + B$ only defined if A and B are the same size
  - result is sum of corresponding entries in A and B
- Scalar Multiplication:
  - $rA = \begin{bmatrix} r\vec{a_1} & r\vec{a_2} & \dots r\vec{a_n} \end{bmatrix}$
- See Theorem 12 for properties of the operations
- Matrix Multiplication:
  - only defined if # rows of B = # columns of A
  - $AB = \begin{bmatrix} A\vec{b_1} & A\vec{b_2} & \dots & A\vec{b_p} \end{bmatrix}$
  - if A is $m \times n$ matrix and B is $n \times p$ matrix, size of $AB = m \times p$
  - ORDER MATTERS!!!!! ($AB \neq BA$ generally, even if both multiplications are defined)
  - if $AB = 0$ one cannot conclude that $A = 0$ or $B = 0$
  - See Theorem 13 for properties of matrix multiplication
- Powers of a square matrix:
  - $A^k = A \times A \times A \times \dots \times A$ (k times)
  - $A^0 = I$
- Transpose:
  - transpose of matrix is matrix where rows of original matrix become columns of the new matrix and vice versa

Example 4.9.1. $\begin{bmatrix} 1 & 7 & 9 \\ 6 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 6 & 3 \\ 7 & 4 & 4 \\ 9 & 2 & 3 \end{bmatrix}$

– See Theorem 14 for properties of matrix transposes

4.10 Invertible Matrices

- invertibility only applies to square matrices, so matrices in this section are all square ($n \times n$)
- a matrix is invertible if there exists a matrix C such that $AC = I$ and $CA = I$. C is the inverse of A (denoted as A^{-1}) in this case.
- non-invertible matrices are sometimes called **singular matrices**, and invertible matrices are sometimes called **nonsingular matrices**.
- the matrix $\begin{bmatrix} 0 \end{bmatrix}$ is not invertible
- see Theorem 15 and Theorem 16 and Theorem 17
- To find inverse of matrix: see Finding the Inverse of an Invertible Matrix
- IMPORTANT THEOREM: Theorem 18
- a transformation T is invertible if there exists if there exists a transformation S such that
  i. $S(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ in the domain of T
  ii. $T(S(\vec{b})) = \vec{b}$ for all $\vec{b}$ in the domain of S
- see Theorem 19

4.11 LU Factorization

- factorization that makes it easier to solve for $A\vec{x} = \vec{b}$ faster
- $LandU$ have a special structure:
  a. L is "lower triangular" with 1s on diagonal
  b. U is the echelon form of A (so "upper triangular", 0s below diagonal)

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U} \tag{1}$$

- Method: see LU Factorization Algorithm and Using LU to solve $A\vec{x} = \vec{b}$

5 Important Theorems

Theorem 1. Each matrix is row equivalent to one and only one reduced echelon matrix.

Theorem 2. When A is $m \times n$ matrix with columns $a_1, \ldots, a_n$, and $\vec{b}$ in $\mathbb{R}^m$,

$$A\vec{x} = \vec{b}$$

has same solution set as

$$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = \vec{b}$$

which has same solution set as

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} & \vec{b} \end{bmatrix}$$

Theorem 3. Let A be an $m \times n$ matrix (coefficient matrix). Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.

b. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Theorem 4. Let A be an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

b. $A(c\vec{u}) = c(A\vec{u})$

Theorem 5. Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$, and let $\vec{p}$ be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{w} = \vec{p} + \vec{v_h}$, where $\vec{v_h}$ is any solution of the homogeneous equation $A\vec{x} = \vec{0}$.

Theorem 6. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent if one of the vectors is a linear combination of the others. $(v_j \in span v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_p)$ for some j.

Theorem 7. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent if there are more vectors than entries in each vector. Any set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem 8. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent if it contains the zero vector.

Theorem 9. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there exists a unique matrix A such that

$$T(\vec{x}) = A\vec{x}$$

namely

$$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \ldots & T(\vec{e_n}) \end{bmatrix}$$

where

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \vec{e_j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \leftarrow j^{th} entry \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{e}_1, \ldots, \vec{e}_j$ are known as **standard basis vectors**. The matrix A is the **standard matrix for the linear transformation T**.

Theorem 10. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, T is one-to-one exactly when $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Theorem 11. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A:

a. T is onto $\Leftrightarrow$ span of columns of $A = \mathbb{R}^m$.

b. T is one-to-one $\Leftrightarrow$ $A\vec{x} = \vec{0}$ only has trivial solution, $\Leftrightarrow$ columns of A are linearly independent.

Theorem 12. Given matrices A, B, C of the same size, and r, s are scalars:

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Theorem 13. Given A, B, C matrices such that sums and products below are defined:

a. $A(BC) = (AB)C$

b. $A(B + C) = AB + AC$

c. $(B + C)A = BA + CA$

d. $r(AB) = (rA)B = A(rB)$

e. $IA = A = AI$

Theorem 14. Given a matrix A:

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Theorem 15. Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $ad - bc \neq 0$ then A is invertible and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Theorem 16. If A is invertible $n \times n$ matrix, then for every $\vec{b}$ in $\mathbb{R}^n$, $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

Theorem 17. Given A, B invertible matrices, then

a. A^{-1} is also invertible and $(A^{-1})^{-1} = A$

b. AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

Theorem 18 (Invertible Matrix Theorem). If A is an $n \times n$ matrix, then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix

b. $A\vec{x} = \vec{b}$ has unique solution for every $\vec{b} \in \mathbb{R}^n$.

c. A has n pivot positions, and a pivot position in every row and column. (because A is square)

d. A has reduced echelon form I

e. $A\vec{x} = \vec{0}$ only has trivial solution

f. the columns of A are linearly independent

g. columns of A span $\mathbb{R}^n$

h. A^T is invertible

i. the transformation $\vec{x} \mapsto A\vec{x}$ is onto

j. the transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one

k. there exists an $n \times n$ matrix C such that $CA = I$

l. there exists an $n \times n$ matrix D such that $AD = I$

Theorem 19. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with standard matrix A is invertible $\Leftrightarrow$ A is an invertible matrix. In that case, $\vec{b} \mapsto A^{-1}\vec{b}$ is the inverse function of T.

6 Toolbox

Make sure to check your work after using any of these methods!

6.1 Linear Equations

To solve systems of linear equations:

1. Variable Elimination
   - System of 2 variables
     (a) Eliminate one of the variables by adding or subtracting the equations together. This will give you can equation to solve for the other variable. Solve for this value.
     (b) Plug the value for the variable that was just solved for into either of the equations to solve for the value of the other variable.
   - System of 3 or more variables
     (a) Choose a variable in one equation to eliminate from the other equations. Eliminate this variable by adding or subtracting the equations together. Repeat the first step until you are able to solve for one variable.
     (b) Plug the value for the variable that was just solved for into either of the equations to solve for the value of the another variable. Repeat this step until all variables are solved for.
2. Substitution
   (a) Define one variable in terms of another, and plug it into all other equations. Use this equation to solve for this variable. If the system of equations has 3 or more equations, use all the other equations to solve for one variable in terms of another.

6.2 Gauss-Jordan Elimination

1. Create augmented matrix for the SoE.
2. Use first nonzero entry in the leftmost column as first pivot. If necessary, interchange rows so that there is a nonzero entry in the top entry of the first column.
3. Use EROs to create all 0s below the pivot position in the first column.
4. Ignore the row with the pivot position, and apply steps 2-3 to the submatrix that remains until there are no more nonzero rows to reduce.
   - after this step, one can determine the pivot positions and pivot columns of the matrix
5. Create zeroes above each pivot, starting with the rightmost pivot. If a pivot is not a 1, make it a 1 with a scaling operation.

6.3 Parametric Vector Form

1. Find the parametric description of the solution set for the SoE. Write all the basic variables in terms of the free variables.

2. Write the parametric description as a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, substituting each variable for its actual value.

3. Write each free variable as the product of itself and a vector $\vec{v_n} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, where each entry of $\vec{v_n}$ is the corresponding coefficient of the variable in the vector $\vec{x}$.

4. Write all the free variables and the vectors they are multiplied by as a sum expression.

6.4 Checking Linear Independence

Given a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$:

1. Put the vectors into a matrix A

2. Solve for $A\vec{x} = \vec{0}$. If there is only the trivial solution, the S is linearly independent. Otherwise, it is linearly dependent.

6.5 Finding the Inverse of an Invertible Matrix

To find the inverse of a matrix A:

1. create the matrix $M = \begin{bmatrix} A \mid I \end{bmatrix}$.

2. perform Gauss-Jordan Elimination on M. If the matrix is invertible, the reduced echelon form of M will be $\begin{bmatrix} I \mid A^{-1} \end{bmatrix}$.

In other words,

$$\begin{bmatrix} A \mid I \end{bmatrix} \xrightarrow{G.J} \begin{bmatrix} I \mid A^{-1} \end{bmatrix}$$

6.6 LU Factorization Algorithm

1. Reduce A to echelon form. As you are row reducing, when you get 0s underneath a column, place entries in the next column of L such that the same sequence of row operations will reduce L to the corresponding column of I. L will have as many columns as pivots of A. Note: Generally, the entries of L are the corresponding entries of A divided by the top entry of that column, with 1s along the main diagonal (see 6.6.1 below).

2. U is the echelon form of A.

Example 6.6.1.

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 8 & 11 \\ 4 & 10 & 16 \end{bmatrix} \xrightarrow[\text{②} \leftarrow \text{②} + \text{-2①}]{\text{③} \leftarrow \text{③} + \text{-2①}} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 4 & 8 \end{bmatrix}$$

We have created 0s under the first column. Now, we can create the first column of L. We ask ourselves, what column could be turned into $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with the row operations $\underset{\text{②} \leftarrow \text{②} + \text{-2①}}{\text{③} \leftarrow \text{③} + \text{-2①}}$? The answer is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, so this is our first column of L. We repeat this process for the other pivots of A:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 4 & 8 \end{bmatrix} \xrightarrow{\text{③} \leftarrow \text{③} + \text{-2②}} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

We have created 0s under the second column. Now, we can create the second column of L. We ask ourselves, what column could be turned into $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ with the row operations ③ $\leftarrow$ ③ + -2②? The answer is $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and this is our second column of L. For the last column, it is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, because L is lower triangular. Therefore, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$.

You will notice from this example that **the entries of L are the corresponding entries of A divided by the top entry of that column, with 1s along the main diagonal**. This is generally the case, but you should always go through this whole algorithm to ensure you don't make mistakes.

6.7 Using LU to solve $A\vec{x} = \vec{b}$

$A\vec{x} = \vec{b}$ can be written as $L(U\vec{x}) = \vec{b}$ with LU factorization. Therefore, to solve $A\vec{x} = \vec{b}$:

1. Solve $L\vec{y} = \vec{b}$ for $\vec{y}$.
2. Solve $U\vec{x} = \vec{y}$ for $\vec{x}$.

7 Terminology

7.1 Symbols

$\sim$ a symbol used to represent row equivalence between two matrices.

■ a symbol used to represent a nonzero number

* a symbol used to represent any number

7.2 Definitions

linear equation
a linear equation in vars $x_1, x_2, \ldots, x_n$ is an equation that can be written as

$$a_1x_1 + a_2x_2 + \ldots a_nb_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real (or complex) numbers.

system of linear equations (SoE) or linear system
one or more linear equations with the same variables.

consistent system
an SoE that has one, or infinitely many solutions

inconsistent system
an SoE that has no solution

augmented matrix
a representation of a system of equations that includes the right-hand sides of the equations.

coefficient matrix
a representation of a system of equations that does not include the right-hand sides of the equations.

leading entry
The first nonzero entry of a matrix.

basic vairable
variables in an SoE that correspond to pivot columns in the REF of a matrix.

free vairable
variables in an SoE that are not basic variables. These variables can take on any value and have the equations of the SoE hold true.

echelon form (EF)
A matrix is in echelon form if it has the following properties:

1. All nonzero rows are above any rows of all zeroes
2. Each leading entry of a row is in a column to the right of the leading entry above it
3. all entries in a column below a leading entry are zeroes

$$\begin{bmatrix} \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form (REF)
A matrix is in reduced echelon form if it is in reduced echelon form and has the following additional properties:

1. The leading entry of each nonzero row is 1
2. Each leading 1 is the only nonzero entry in its column

$$\begin{bmatrix} 1 & * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot position
given a matrix in echelon or reduced form, a pivot position is a position with a leading number in a row.

pivot columns
given a matrix in echelon or reduced form, a pivot column is a column with a pivot columns.

forward phase of G-J Elimination
getting the matrix into echelon form (steps 1-4 of G-J elimination)

backward phase of G-J Elimination
getting the matrix into echelon form (step 5 of G-J elimination)

parametric description
a description of all the variables involved in a system with their associated values

linear combination
a description of a vector as a sum of vectors multiplied by scalars. The scalars the vectors are multiplied by are called **weights**.

span
the set of all possible linear combinations of a set of vectors (span$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_p}\}$ is the set of all possible linear combinations of $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_p}\}$)

homogenous system of linear equations
an SoE that can be written as $A\vec{x} = \vec{0}$.

trivial solution
a solution to the matrix equation $A\vec{x} = \vec{0}$ where $\vec{x} = \vec{0}$.

nontrivial solution
a solution to the matrix equation $A\vec{x} = \vec{0}$ where $\vec{x} \neq \vec{0}$.

parametric vector form
: a way of representing the solution set to a matrix equation where $\vec{x}$ is a sum of vectors with its free variables as weights.

linear independence and dependence
: Given a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$, they are **linearly independent** if

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution. If there is a nontrivial solution to the equation, for example, a

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \vec{0}$$

where $\{c_1, \ldots, c_p\}$ are not all zero, the set is **linearly dependent**.

linear dependence relation
: Given a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$, a **dependence relation** among $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \vec{0}$$

where $\{c_1, \ldots, c_p\}$ are not all zero.

matrix transformation
: a function that maps a vector $\vec{x}$ to a vector $A\vec{x}$.

domain of matrix trnasformation
: the set of objects being mapped. If a matrix A has m rows and n columns, the domain is $\mathbb{R}^n$.

co-domain of matrix trnasformation
: the set of objects being mapped to. If a matrix A has m rows and n columns, the domain is $\mathbb{R}^m$.

image of matrix transformation
: for a particular $\vec{x} \in \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ is the image of $\vec{x}$ under T.

range of a matrix transformation
: the set of all images under T.

onto
: a transformation is onto when its range is its co-domain. In other words, for every $\vec{b}$ in the co-domain, there exists at least one $\vec{x}$ in the domain such that $T(\vec{x}) = \vec{b}$.

one-to-one
: a transformation is one-to-one when each $\vec{b}$ in the co-domain is mapped to at most 1 $\vec{x}$ in the domain.

standard basis vectors
: the set of vectors $\{\vec{e}_1, \ldots, \vec{e}_j\}$, where the vectors have 0s in all other positions besides their j^{th} position, which is a 1.

standard matrix for a linear transformation
: the matrix

$$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \ldots & T(\vec{e_n}) \end{bmatrix}$$

where $\vec{e}_1, \ldots, \vec{e}_n$ are standard basis vectors.

diagonal entries
: the entries on the diagonal that goes through the middle of a matrix.

main diagonal
: the diagonal on which the diagonal entries lie on a matrix.

diagonal matrix
: a square matrix where only its diagonal entries are nonzero.

identity matrix
: a special case of a diagonal matrix where its diagonal entries are all equal to 1 (denoted as I or I_n).

zero matrix
: a matrix whose entries are all 0 (denoted as 0 or $0_{m\times n}$).

invertible/nonsingular matrix
: a matrix is invertible when exists a matrix C such that $AC = I$ and $CA = I$. A non-invertible matrix is called a singular matrix. The inverse of tha matrix is denoted as A^{-1}, if the original matrix is A.

lower triangular matrix
: a matrix that only has nonzero numbers below its diagonal.

$$\begin{bmatrix} \blacksquare & 0 & 0 & 0 \\ * & \blacksquare & 0 & 0 \\ * & * & \blacksquare & 0 \\ * & * & * & \blacksquare \end{bmatrix}$$

unit lower triangular matrix
: a special case of the lower triangular matrix that has 1s on its diagonal.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}$$

upper triangular matrix
: a matrix that has only has nonzero numbers above its diagonal.

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

unit upper triangular matrix
: special case of the upper triangular matrix that has 1s on its diagonal.

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

triangular matrix
: a matrix that is either lower or upper triangular.