STEP Hints and Answers

June 2005

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Section A: Pure Mathematics

1   The simplest approach to this question is to use the result that if you have a collection of A identical objects of type a, B identical objects of type b, C identical objects of type c etc, then they can be rearranged in

$$\frac{(A+B+C+...)!}{A!B!C!...} \text{ ways}$$

(i) $43 = 9+9+9+9+7 = 5$ distinct rearrangements, since five objects of which four are 9 can be rearranged in $\frac{5!}{4!1!}$ ways, or

$43 = 9+9+9+8+8 = \frac{5!}{3!2!} = 10$ distinct rearrangements.

(ii) $39 = 9+9+9+9+3 = 5$ distinct rearrangements, or

$39 = 9+9+9+7+5 = 9+9+9+8+4 = 2 \times 20$ distinct rearrangements, or

$39 = 9+9+9+6+6 = 10$ distinct rearrangements, or

$39 = 9+9+8+8+5 = 30$ distinct rearrangements, or

$39 = 9+9+8+7+6 = 60$ distinct rearrangements, or

$39 = 9+9+7+7+7 = 10$ distinct rearrangements, or

$39 = 9+8+8+8+6 = 20$ distinct rearrangements, or

$39 = 9+8+8+7+7 = 30$ distinct rearrangements, or

$39 = 8+8+8+8+7 = 5$ distinct rearrangements

$= 210$ distinct rearrangements.

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2 $y^2 = 4x \Rightarrow 2yy' = 4 \Rightarrow y' = \dfrac{2}{y}$

Equation of tangent at P: $y - 2p = \dfrac{1}{p}\left(x - p^2\right) \Rightarrow py = x + p^2$.

Equation of tangent at Q: $qy = x + q^2$

Intersect where $qy - q^2 = py - p^2$

$\Rightarrow (q - p)\, y = q^2 - p^2$

$\Rightarrow y = p + q \Rightarrow x = pq$ by substitution. Hence $R\left(pq,\ p+q\right)$.

Equation of normal at P: $y - 2p = -p\left(x - p^2\right) \Rightarrow y + px = 2p + p^3$.

Equation of normal at Q: $y + qx = 2q + q^3$

Intersect where $2p + p^3 - px = 2q + q^3 - qx$

$\Rightarrow x\,(p - q) = 2\,(p - q) + (p - q)\left(p^2 + pq + q^2\right)$ using the identity $p^3 - q^3 \equiv (p - q)\left(p^2 + pq + q^2\right)$

$\Rightarrow x = p^2 + pq + q^2 + 2 \Rightarrow y = -pq\,(p + q)$ by substitution.

But $(1,\ 0)$ lies on PQ so the gradient of the line segement from P to $(1,\ 0)$ equals the gradient of the line segment from Q to $(1,\ 0)$.

$\Rightarrow \dfrac{2p}{p^2 - 1} = \dfrac{2q}{q^2 - 1}$

$\Rightarrow 2pq^2 - 2p = 2qp^2 - 2q \Rightarrow pq\,(q - p) = p - q \Rightarrow pq = -1$

$\Rightarrow S$, where the normals intersect, has coordinates $\left(p^2 + q^2 + 1,\ p + q\right)$.

Obviously, SP is perpendicular to PR and QS is perpendicular to QR, because each of these is a tangent-normal pair.

Furthermore, the gradient of PR × the gradient of $QR = \frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = -1$, so PR is perpendicular to QR.

Also, the gradient of PS × the gradient of $QS = -p \times -q = pq = -1$, so PS is perpendicular to QS.

Therefore all four angles are right angles, proving that PSQR is a rectangle. It is not sufficient to consider only the lengths of the sides, since the quadrilateral could be a parallelogram.

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3   (i) $\dfrac{x}{x-a} + \dfrac{x}{x-b} = 1 \Rightarrow x^2 - bx + x^2 - ax = x^2 - ax - bx + ab$

$\Rightarrow x^2 = ab$ which has two distinct real solutions since $ab > 0$.

(ii) $\dfrac{x}{x-a} + \dfrac{x}{x-b} = 1 + c \Rightarrow 2x^2 - (a+b)x = (c+1)\left(x^2 - (a+b)x + ab\right)$

$\Rightarrow (c-1)x^2 - c(a+b)x + (c+1)ab = 0.$

This has one real solution if its discriminant equals 0

$\Rightarrow c^2(a+b)^2 - 4(c-1)(c+1)ab = 0$

$\Rightarrow c^2\left[(a+b)^2 - 4ab\right] = -4ab \Rightarrow c^2 = \dfrac{-4ab}{(a-b)^2} = 1 - \dfrac{a^2 + 2ab + b^2}{a^2 - 2ab - b^2}.$

Since a, b and c are real, $c^2 \geqslant 0$ and $\left(\dfrac{a+b}{a-b}\right)^2 \geqslant 0$. Therefore clearly $0 \leqslant c^2 \leqslant 1$.

However, $c^2 = 0 \Rightarrow (a+b)^2 = (a-b)^2$

$\Rightarrow a + b = \pm(a-b)$

$\Rightarrow b = -b$ or $a = -a$

$\Rightarrow b = 0$ or $a = 0$ neither of which is permitted.

Therefore $0 < c^2 \leqslant 1$.

4   (a) $\cos\theta = \dfrac{3}{5} \Rightarrow \sin\theta = \pm\dfrac{4}{5}$, since $\sin^2\theta = 1 - \cos^2\theta$.

Since $\sin\theta$ is negative in the given domain, $\sin 2\theta \equiv 2\cos\theta\sin\theta = 2 \times \dfrac{3}{5} \times -\dfrac{4}{5} = -\dfrac{24}{25}$

Also, $\cos 3\theta \equiv \cos(\theta + 2\theta) \equiv \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$

$= \dfrac{3}{5} \times \left[2 \times \left(\dfrac{3}{5}\right)^2 - 1\right] - \left(-\dfrac{4}{5}\right) \times -\dfrac{24}{25} = -\dfrac{117}{125}$

(b) $\tan(\theta + 2\theta) \equiv \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} \equiv \dfrac{\tan\theta + \frac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta \times \frac{2\tan\theta}{1-\tan^2\theta}} = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$

So $\dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \dfrac{11}{2}$

$\Rightarrow 6t - 2t^3 = 11 - 33t^2$ where $t \equiv \tan\theta$

$\Rightarrow 2t^3 - 33t^2 - 6t + 11 = 0$

$\Rightarrow (2t - 1)\left(t^2 - 16t - 11\right) = 0$

$\Rightarrow t = 0$ or $t = \dfrac{1}{2}\left(16 \pm \sqrt{16^2 + 44}\right) = 8 \pm \sqrt{75}$

so $\tan\theta = 8 + \sqrt{75}$ since $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{2} \Rightarrow \tan\theta \geqslant 1$.

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(i)   If $k \neq 0$ the given integral $= \left[\dfrac{(x+1)^k}{k} \right]_0^1 = \dfrac{2^k - 1}{k}$

If $k = 0$ then the integral $= [\ln(x+1)]_0^1 = \ln 2$

As k tends to 0, the two integrals become closer in value, so $\dfrac{2^k - 1}{k} \approx \ln 2$ when $k \approx 0$.

You might like to consider whether this is always true: if a sequence of functions converges to a certain limit function, must some definite integral of each function in the sequence converge to the same integral of the limit function? Simple questions such as these have motivated many of the great discoveries in Mathematics.

(ii)   The simplest way to integrate $x(x+1)^m$ is to notice that

$$x(x+1)^m \equiv (x+1-1)(x+1)^m \equiv (x+1)^{m+1} - (x+1)^m.$$

This is true for all values of m. Of course, it can be integrated by parts or with a substitution if preferred.

Assume $m \neq -1, -2$. Therefore the given integral equals $\left[\dfrac{(x+1)^{m+2}}{m+2} - \dfrac{(x+1)^{m+1}}{m+1} \right]_0^1$

$= \left(\dfrac{2^{m+2}}{m+2} - \dfrac{2^{m+1}}{m+1} \right) - \left(\dfrac{1}{m+2} - \dfrac{1}{m+1} \right) = \dfrac{m2^{m+1} + 1}{(m+2)(m+1)}.$

If $m = -1$, $x(x+1)^{-1} \equiv 1 - (x+1)^{-1}$.
Hence the given integral equals $[x - \ln(x+1)]_0^1 = 1 - \ln 2$.

If $m = -2$, $x(x+1)^{-2} \equiv (x+1)^{-1} - (x+1)^{-2}$.
Hence the given integral equals $\left[\ln(x+1) + (x+1)^{-1} \right]_0^1 = \ln 2 - \tfrac{1}{2}$

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6   (i) $PA = 2PB \Rightarrow (x-5)^2 + (y-16)^2 = 4\left((x+4)^2 + (y-4)^2\right)$

$\Rightarrow x^2 + y^2 - 10x - 32y + 281 = 4x^2 + 4y^2 + 32x - 32y + 128$

$\Rightarrow 3x^2 + 3y^2 + 42x - 153 = 0$

$\Rightarrow x^2 + y^2 + 14x - 51 = 0$

$\Rightarrow (x+7)^2 - 49 + y^2 - 51 = 0$

$\Rightarrow (x+7)^2 + y^2 = 100$

which is a circle centre $(-7,\, 0)$ with radius 10.

   (ii) $QC = k \times QD \Rightarrow (x-a)^2 + y^2 = k^2(x-b)^2 + k^2 y^2$

$\Rightarrow x^2\left(k^2 - 1\right) + y^2\left(k^2 - 1\right) + x\left(2a - 2k^2 b\right) + \left(k^2 b^2 - a^2\right) = 0$

If this locus is the same as the locus of P, then the ratios of the coefficients must be the same.

$\Rightarrow \dfrac{2a - 2k^2 b}{k^2 - 1} = 14$ and $\dfrac{k^2 b^2 - a^2}{k^2 - 1} = -51.$

Notice that you **cannot** conclude that $k^2 - 1 = 1$.

$\Rightarrow k^2 = \dfrac{a+7}{b+7}$ and $k^2 = \dfrac{a^2 + 51}{b^2 + 51}$

$\Rightarrow \dfrac{a+7}{b+7} = \dfrac{a^2 + 51}{b^2 + 51}$

$\Rightarrow (a+7)\left(b^2 + 51\right) = (b+7)\left(a^2 + 51\right)$

$\Rightarrow ab^2 - a^2 b = 7\left(a^2 - b^2\right) + 51(b - a)$

$\Rightarrow ab(b - a) = 7(a - b)(a + b) + 51(b - a)$

$\Rightarrow ab = 51 - 7(a + b)$ since $a \neq b \Rightarrow a - b \neq 0$

$\Rightarrow ab + 7(a + b) = 51$

$\Rightarrow ab + 7(a + b) + 49 = 51 + 49$

$\Rightarrow (a + 7)(b + 7) = 100$

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7   (i) $\displaystyle\prod_{r=1}^{n}\left(\frac{r+1}{r}\right) = \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times ... \times \frac{n+1}{n} = n+1$

(ii) $\displaystyle\prod_{r=2}^{n}\left(\frac{r^2-1}{r^2}\right) = \prod_{r=2}^{n}\left(\frac{r-1}{r}\right)\left(\frac{r+1}{r}\right) = \prod_{r=2}^{n}\left(\frac{r-1}{r}\right)\prod_{r=2}^{n}\left(\frac{r+1}{r}\right)$

$\displaystyle = \left[\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times ... \times \frac{n-1}{n}\right] \times \left[\frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times ... \times \frac{n+1}{n}\right] = \frac{n+1}{2n}$

(iii) $\displaystyle\prod_{r=1}^{n}\left(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n}\cot\frac{(2r-1)\pi}{n}\right)$ where n is even

$\displaystyle = \prod_{r=1}^{n}\left(\cos\frac{2\pi}{n} + \frac{\sin\frac{2\pi}{n}\cos\frac{(2r-1)\pi}{n}}{\sin\frac{(2r-1)\pi}{n}}\right)$

$\displaystyle = \prod_{r=1}^{n}\left(\frac{\cos\frac{2\pi}{n}\sin\frac{(2r-1)\pi}{n} + \sin\frac{2\pi}{n}\cos\frac{(2r-1)\pi}{n}}{\sin\frac{(2r-1)\pi}{n}}\right)$

$\displaystyle = \prod_{r=1}^{n}\left(\frac{\sin\left[\frac{(2r-1)\pi}{n} + \frac{2\pi}{n}\right]}{\sin\frac{(2r-1)\pi}{n}}\right)$

$\displaystyle = \prod_{r=1}^{n}\left(\frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{(2r-1)\pi}{n}}\right) = \frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}} \times \frac{\sin\frac{5\pi}{n}}{\sin\frac{3\pi}{n}} \times \frac{\sin\frac{7\pi}{n}}{\sin\frac{5\pi}{n}} \times ... \times \frac{\sin\frac{(2n+1)\pi}{n}}{\sin\frac{(2n-1)\pi}{n}}$

$\displaystyle = \frac{\sin\frac{(2n+1)\pi}{n}}{\sin\frac{\pi}{n}} = \frac{\sin\left(2\pi + \frac{\pi}{n}\right)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}} = 1$ using the periodicity of $\sin x$.

You should consider why it was necessary to require that n was even.

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8 $y^2 = x^k \mathrm{f}(x) \Rightarrow 2yy' = x^k \mathrm{f}'(x) + kx^{k-1}\mathrm{f}(x)$

$\Rightarrow 2xyy' = x^{k+1}\mathrm{f}'(x) + kx^k\mathrm{f}(x) = x^{k+1}\mathrm{f}'(x) + ky^2$

(i) $k = 1 \Rightarrow 2xyy' = x^2\mathrm{f}'(x) + y^2$

$\Rightarrow x^2\mathrm{f}'(x) + y^2 = y^2 + x^2 - 1$

$\Rightarrow \mathrm{f}'(x) = 1 - \dfrac{1}{x^2}$

$\Rightarrow \mathrm{f}(x) = x + \dfrac{1}{x} + c$

$\Rightarrow y^2 = x^2 + 1 + cx$ since $y^2 = x\mathrm{f}(x)$.

But $x = 1, y = 2 \Rightarrow y^2 = 4 = 1 + 1 + c \Rightarrow c = 2$

$\Rightarrow y^2 = x^2 + 2x + 1$, which is the pair of straight lines $y = \pm(x+1)$.

(ii)   Since $2xyy' = 2\dfrac{\ln x}{x} - y^2$ let $k = -1$.

$\Rightarrow 2xyy' = -y^2 + x^0\mathrm{f}'(x) = 2\dfrac{\ln x}{x} - y^2$

$\Rightarrow \mathrm{f}'(x) = 2\dfrac{\ln x}{x}$

$\Rightarrow \mathrm{f}(x) = (\ln x)^2 + c$

$\Rightarrow y^2 = \dfrac{(\ln x)^2 + 1}{x}$ since $y = 1$ when $x = 1$, and $y^2 = \dfrac{\mathrm{f}(x)}{x}$.

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Section B: Mechanics

9   Let the centre of mass of the rod be a distance x from A and let the tension in the string attached at B be S. Let moments be taken about A.

$\Rightarrow S + T = W$ and $Wx = 3Sl$.

Therefore, $Wx = 3l\,(W - T)$

Let the upward force supplied by the pivot be Q, and let moments be taken about A.

$\Rightarrow T + Q = W$ and $Ql + 3Tl = Wx$.

$\Rightarrow l\,(W - T) + 3Tl = 3l\,(W - T) \Rightarrow 5Tl = 2Wl \Rightarrow 5T = 2W$.

To determine x:

$$\frac{5T}{2}x = 3l\left(\frac{5T}{2} - T\right) \Rightarrow \frac{5x}{2} = 3l \times \frac{3}{2} \Rightarrow x = \frac{9l}{5}$$

Let the upwards reaction force acting on the rod from the ground be R, and let the frictional force acting on the rod be F. It is essential **not** to assert that the frictional force is μR: this would only be true in limiting equilibrium. A common error is to assume that $F = \mu R$ rather than $F \leqslant \mu R$.

$\Rightarrow R + \dfrac{T}{2}\cos\theta = W$ and $\dfrac{T}{2}\sin\theta = F$.

Also, taking moments about B, $\dfrac{T}{2} \times 3l = W \times \left(3l - \dfrac{9l}{5}\right)\cos\theta = \dfrac{5T}{2} \times \dfrac{6l}{5}\cos\theta$

$\Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = 60°$

$\Rightarrow R = W - \dfrac{T}{2} \times \dfrac{1}{2} = \dfrac{5T}{2} - \dfrac{T}{4} = \dfrac{9T}{4}$

and $F = \dfrac{T}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{T\sqrt{3}}{4} \leqslant \mu R$

$\Rightarrow \mu \geqslant \dfrac{\sqrt{3}}{9}$

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10   Let the velocity of mass A be x after it collides with B.

Let the velocity of mass B be v after it collides with A.

Let the velocity of mass B be y after it collides with C.

Let the velocity of mass C be w after it collides with B.

Using the fact that linear momentum is conserved in collisions, and Newton's Law of Restitution (in the form $e \times$ approach speed = separation speed)

$\Rightarrow au + 0 = ax + bv$ and $eu = v - x$

$\Rightarrow au + aeu = bv + av$ since $x = v - eu$

$\Rightarrow v = \dfrac{au(1 + e)}{a + b}$

Also, $au - beu = ax + bx \Rightarrow x = \dfrac{u(a - be)}{a + b}$

Similarly, to find y:

$bv + 0 = by + cw$ and $ev = w - y$

$\Rightarrow bv + bev = cw + bw \Rightarrow w = \dfrac{bv(1 + e)}{b + c} = \dfrac{abu(1 + e)^2}{(a + b)(b + c)}$

Also, $bv - cev = by + cy \Rightarrow y = \dfrac{v(b - ce)}{b + c}$

(i)   If the masses a, b and c are such that $\dfrac{a}{b} = \dfrac{b}{c} = e \Rightarrow a = be$ and $b = ce$

$\Rightarrow x = y = 0$ and $w = \dfrac{abu(1 + e)^2}{(be + b)(ce + c)} = \dfrac{abu}{bc} = e^2 u$

i.e. A and B are at rest and C has velocity $e^2 u$.

(ii)   If the masses a, b and c are such that $\dfrac{b}{a} = \dfrac{c}{b} = e \Rightarrow b = ae$ and $c = be$

$\Rightarrow x = \dfrac{u(a - ae^2)}{a + ae} = u(1 - e)$

and $v = \dfrac{au(1 + e)}{a + ae} = u \Rightarrow y = \dfrac{u(b - be^2)}{b + be} = u(1 - e)$

and $w = \dfrac{abu(1 + e)^2}{(a + ae)(b + be)} = u$

i.e. C has velocity u and A and B each has velocity $u(1 - e)$.

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11 (i) $\mathbf{r} = 0 \Rightarrow \sin 2t = 0$ and $\cos t = 0 \Rightarrow t = \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$

(ii) $\mathbf{r} = \begin{pmatrix} \sin 2t \\ 2\cos t \end{pmatrix} \Rightarrow \mathbf{v} = \begin{pmatrix} 2\cos 2t \\ -2\sin t \end{pmatrix}$

If $\mathbf{r}$ is perpendicular to $\mathbf{v}$ then $\begin{pmatrix} \sin 2t \\ 2\cos t \end{pmatrix} \cdot \begin{pmatrix} 2\cos 2t \\ -2\sin t \end{pmatrix} = 0$

$\Rightarrow 2\sin 2t \cos 2t - 4\sin t \cos t = 0$

$\Rightarrow 2\sin 2t \left(\cos 2t - 1 \right) = 0$

$\Rightarrow t = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$ (recall that $t < 2\pi$).

(iii) $\begin{pmatrix} \sin 2t \\ 2\cos t \end{pmatrix}$ is parallel to $\begin{pmatrix} 2\cos 2t \\ -2\sin t \end{pmatrix} \Rightarrow \dfrac{\sin 2t}{2\cos t} = \dfrac{2\cos 2t}{-2\sin t}$

$\Rightarrow -2\sin t \sin 2t = 4\cos t \cos 2t$

$\Rightarrow -\sin^2 t \cos t = \cos t \cos 2t$

$\Rightarrow \cos t \left(\cos 2t + \sin^2 t \right) = 0$

$\Rightarrow \cos t = 0 \Rightarrow \mathbf{r} = \mathbf{0}$ or $1 - 2\sin^2 t + \sin^2 t = 0$

$\Rightarrow 1 = \sin^2 t \Rightarrow \cos^2 t = 0 \Rightarrow \mathbf{r} = \mathbf{0}$

(iv) The distance of the particle from the origin $= \sqrt{\sin^2 2t + 4\cos^2 t}$

$= \sqrt{4\sin^2 t \cos^2 t + 4\cos^2 t}$

$= \sqrt{8\cos^2 t - 4\cos^4 t}$

Using differentiation to find the maximum or minimum distance

$\Rightarrow \dfrac{16\cos t \sin t - 16\cos^3 t \sin t}{2\sqrt{8\cos^2 t - 4\cos^4 t}} = 0$

$\Rightarrow \cos t \sin t \left(1 - \cos^2 t \right) = 0$

$\Rightarrow \sin t = 0 \Rightarrow \text{distance} = 2$

or $\cos t = 0 \Rightarrow \text{distance} = 0$

or $\cos^2 t = 1 \Rightarrow \text{distance} = 2$.

The path of the particle is a "figure of 8".

Alternatively, $\sqrt{\sin^2 2t + 4\cos^2 t}$

$= \sqrt{4\sin^2 t \cos^2 t + 4\cos^2 t}$

$= \sqrt{4\cos^2 t \left(\sin^2 t + 1 \right)}$

$= 2\sqrt{\left(1 - \sin^2 t \right) \left(1 + \sin^2 t \right)}$

$= 2\sqrt{1 - \sin^4 t} \leqslant 2$

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Section C: Probability and Statistics

12   Venn diagrams are very helpful when answering this question. This question **cannot** be answered using a tree diagram: you are not told the probability that a hobbit wears a hat given that he wears a pipe, so you cannot draw a tree with 0.4 on a branch subsequent to 0.7 (or vice versa: you are not told the probability that a hobbit smokes a pipe given that he wears a hat).

(a)   Since $P\,(\text{pipe but no hat}) = p$, we can state that $P\,(\text{pipe and hat}) = 0.7 - p$ and $P\,(\text{hat but no pipe}) = 0.4 - (0.7 - p) = p - 0.3$. Clearly therefore, $p \geqslant 0.3$.

Furthermore, $P\,(\text{pipe}) + P\,(\text{hat}) = 1.1 \Rightarrow P\,(\text{pipe and hat}) \geqslant 0.1$.

Therefore $0.7 - p \geqslant 0.1 \Rightarrow p \leqslant 0.6$

(b) $P\,(\text{wizard wears hat, cloak and ring}) = 0.1$ and $P\,(\text{wizard wears none}) = 0.05$

Let $x = P\,(\text{wizard wears hat and cloak but not ring})$.

Let $y = P\,(\text{wizard wears hat and ring but not cloak})$.

Let $z = P\,(\text{wizard wears ring and cloak but not hat})$.

Therefore none of the following can be negative:

$P\,(\text{wizard wears only hat}) = 0.6 - x - y$

$P\,(\text{wizard wears only cloak}) = 0.7 - x - z$

$P\,(\text{wizard wears only ring}) = 0.3 - y - z$

Since $0.1 + 0.05 + (0.6 - x - y) + (0.7 - x - z) + (0.3 - y - z) + x + y + z = 1$

$\Rightarrow x + y + z = 0.75 = P\,(\text{wizard wears exactly two items})$

$P\,(\text{wizard wears hat but not ring given that he is wearing a cloak}) = q = \dfrac{x}{0.8}$

To determine the range of x, let $x = 0.6 - k$

Then $P\,(\text{hat only}) = k - y$ and $P\,(\text{cloak only}) = k - z + 0.1$; remember that neither of these can be negative.

Therefore $0.95 = (k - y) + (0.6 - k) + (k - z + 0.1) + 0.1 + y + z + (0.3 - y - z)$

$\Rightarrow 0.95 = 1.1 + k - (y + z) \Rightarrow k = y + z - 0.15$

Since $k - y \geqslant 0 \Rightarrow z \geqslant 0.15$, and since $k - z + 0.1 \geqslant 0 \Rightarrow y - 0.05 \geqslant 0 \Rightarrow y \geqslant 0.05$

Therefore $0.2 \leqslant y + z \leqslant 0.3$

$\Rightarrow 0.05 \leqslant k \leqslant 0.15$

$\Rightarrow 0.45 \leqslant x \leqslant 0.55$

$\Rightarrow \dfrac{9}{16} \leqslant q \leqslant \dfrac{11}{16}$

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13   Although X is not Normally distributed, you can picture it as something similar: a sketch of a (symmetrical) graph will help you.

(a)   Analogously to a Normal distribution:

$$P\left(\mu - \tfrac{1}{2}\sigma \leqslant X \leqslant \mu + \sigma\right) = a - (1 - b) = a + b - 1$$

$$P\left(X \leqslant \mu + \tfrac{1}{2}\sigma \mid X \geqslant \mu - \tfrac{1}{2}\sigma\right) = \frac{b - (1 - b)}{b} = \frac{2b - 1}{b}.$$

(b)   A tree diagram is useful: the first pair of branches indicates the type of milk (skimmed with probability 0.6 or full-fat with probability 0.4), and then branches to represent the volume of the carton being more or less than the stated amounts.

(i)   It is important to recognise that you are being asked for a **conditional** probability.

$P(\text{volume} > 500 \text{ ml given that volume} < 505 \text{ ml})$

$$= \frac{0.6\,(b - 0.5) + 0.4\,(a - b)}{0.6b + 0.4a}$$

$$= \frac{0.4a + 0.2b - 0.3}{0.6b + 0.4a}$$

(ii)   The stated information is saying that $0.6b + 0.4a = 0.7$ and $\dfrac{0.4 \times \tfrac{1}{2}}{0.6b + 0.4 \times \tfrac{1}{2}} = \dfrac{1}{3}$

(again, notice the language of a **conditional** probability in the second statement).

Therefore $b = \dfrac{2}{3}$ and $a = \dfrac{3}{4}$.

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STEP I, 2005, Hints and Answers

14 (i) $m + P\left(0 \leqslant X < \infty\right) = 1 \Rightarrow k = 1 - m$.

(ii)   Since the cumulative distribution function of X is $k\left(1 - e^{-x}\right)$, the probability density function of X is ke^{-x}, the derivative of the cdf.

$$\Rightarrow E\left(X\right) = -1 \times m + \left(1 - m\right)\int_0^\infty xe^{-x}\,\mathrm{d}x = -m + \left(1 - m\right) \times 1 = 1 - 2m$$

(iii) $$\mathrm{var}\left(X\right) = \left[\left(-1\right)^2 \times m + \left(1 - m\right)\int_0^\infty x^2 e^{-x}\,\mathrm{d}x\right] - \left(1 - 2m\right)^2$$

$$= m + \left(1 - m\right) \times 2 - \left(1 - 4m + 4m^2\right) = 1 + 3m - 4m^2$$

Let the median value of X be T.

$$\Rightarrow k\left(1 - e^{-T}\right) = \frac{1}{2} - m$$

$$\Rightarrow 1 - e^{-T} = \frac{1 - 2m}{2 - 2m}$$

$$\Rightarrow e^{-T} = \frac{1}{2 - 2m}$$

$$\Rightarrow -T = -\ln\left(2 - 2m\right)$$

$$\Rightarrow T = \ln\left(2 - 2m\right)$$

(iv) $$E\left(\sqrt{|X|}\right) = m\sqrt{|-1|} + \left(1 - m\right)\int_0^\infty \sqrt{x}e^{-x}\,\mathrm{d}x$$

$$= m + \left(1 - m\right)\int_0^\infty ue^{-u^2}\,2u\,\mathrm{d}u \text{ using } u^2 = x$$

$$= m + 2\left(1 - m\right)\frac{\sqrt{\pi}}{4} \text{ using the given result}$$

$$= m + \left(1 - m\right)\frac{\sqrt{\pi}}{2}$$

Step II, Hints and Answers
June 2005

STEP MATHEMATICS PAPER 2: 9470: JULY 2005: HINTS AND ANSWERS

Q1 Differentiation leads to $f'(x) = 2xe^{-x^2} - 2x^3 e^{-x^2}$. Since $e^{-x^2} \neq 0$ for any finite x then $f'(x) = 0 \Rightarrow x - x^3 = 0 \Rightarrow x = 0, 1, -1$

For the rest of the question, observe first that $P'(x) - 2xP(x) \equiv x(x^2 - a^2)(x^2 - b^2) (*)$ within a multiplicative non - zero constant. Thus $P(x)$ can take the form $-x^4/2 + px^2 + q$ and hence substitution into (*) plus equating the coefficient of x^2 and constant terms leads to a possible result for $P(x)$.

A similar argument based on setting $P(x) = \sum_{i=0}^{4} c_i x^i$ is feasible, but it involves more working and so is correspondingly more error prone.

Alternatively, one can multiply (*) by e^{-x^2} and then integrate with respect to x to obtain $P(x)e^{-x^2} = \int x(x^2 - a^2)(x^2 - b^2)e^{-x^2} dx$. From $\int xe^{-x^2} dx = -(1/2)e^{-x^2}$ the integrals $\int x^3 e^{-x^2} dx$ and $\int x^5 e^{-x^2} dx$ can be evaluated by use of the integration by parts rule. It then only remains to cancel out the factor e^{-x^2} to obtain $P(x) = -x^4/2 + (a^2/2 + b^2/2 - 1)x^2 - 1 + a^2/2 + b^2/2 - a^2 b^2/2$.

Q2 (a) (i) Following the definition of $f(N)$, it is immediate that $f(12) = 12(1 - 1/2)(1 - 1/3) = 4$, and $f(180) = 180(1 - 1/2)(1 - 1/3)(1 - 1/5) = 48$.

(ii) The result may seem obvious but care must be taken in order to construct a complete proof. For example, $N = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \Rightarrow f(N) = p_1^{\alpha_1 - 1} \ldots p_k^{\alpha_k - 1}(p_1 - 1)\ldots(p_k - 1)$. Thus as p_i is a positive integer and $\alpha_i - 1$ is a non-negative integer for $1 \leq i \leq k$, then $f(N)$ is an integer.

(b) In each of (i),(ii),(iii), the conclusion must be made clear.

(i) As $f(3)f(9) = 2 \times 6 = 12 \neq f(27) = 18$, then the statement is false.

(ii)For any two primes p and q, $f(p)f(q) = p(1 - 1/p)q(1 - 1/q) = pq(1 - 1/p)(1 - 1/q) = f(pq)$. Hence the statement is true.

(iii) Consider $f(5) = 4$, $f(6) = 2$, $f(30) = 8 = 2 \times 4$. Then as 6 is not a prime it is clear that the statement is false.

(c)Start with $p^{m-1}(p - 1) = 146410$, then without difficulty it will be found that $p = 11$ and $m = 5$ (not 4).

Q3 Here $dy/dx = x \sin x$ which is zero at $x = 0$ and is positive for $0 < x \leq \pi/2$. A further differentiation will show that $d^2 y/dx^2 = 0$ at $x = 0$ and positive for $0 < x \leq \pi/2$. Also, as $y(0) = 0$ and $y(\pi/2) = 1$, then $0 \leq y \leq 1$ for $0 \leq x \leq \pi/2$, and the sketch can now be completed consistently with the above conclusions.

(i) $\int_0^{\pi/2} \sin x \, dx = \ldots = 1$ and use of the integration by parts rule will show that $\int_0^{\pi/2} x \cos x \, dx = \pi/2 - 1$. The displayed result for $\int_0^{\pi/2} y \, dx$ then follows immediately.

(ii) Start with $\int_0^{\pi/2} y^2\,dx = \int_0^{\pi/2} \sin^2 x\,dx - \int_0^{\pi/2} x\sin 2x\,dx + \int_0^{\pi/2} x^2\cos^2 x\,dx$.

Next, express $\sin^2 x$ and $\cos^2 x$ in terms of $\cos 2x$ so that now there are only two essentially different integrals involving trigonometric terms, namely, $\int_0^{\pi/2} x\sin 2x\,dx$ and $\int_0^{\pi/2} x^2\cos 2x\,dx$. The second of these can be obtained from the first, again by application of the integration by parts rule. A correct application of this rule to the first integral and the careful collection of terms will lead to the displayed result.

For the final result, begin with the observation that $y^2 < y$ for $0 < x < \pi/2$. From this, it is immediate that $\int_0^{\pi/2} y^2\,dx < \int_0^{\pi/2} y\,dx$ and hence that $\pi^3/48 - \pi/8 < 2 - \pi/2$. The final displayed result can then be obtained without difficulty.

Q4 The first two parts of this question depend on the identity $\tan^{-1} A + \tan^{-1} B = \tan^{-1}[(A + B)/(1 - AB)]$ which is simply another way of writing $\tan(\alpha + \beta) = [\tan\alpha + \tan\beta]/[1 - \tan\alpha\tan\beta]$.

In the second part, it follows from the data that

$$\tan^{-1}[1/(p+q+s)] + \tan^{-1}[1/(p+q+t)] = \tan^{-1}[1/(p+q)]$$

and

$$\tan^{-1}[1/(p+r+u)] + \tan^{-1}[1/(p+r+v)] = \tan^{-1}[1/(p+r)].$$

As also from the data,

$$\tan^{-1}[1/(p+q)] + \tan^{-1}[1/(p+r)] = \tan^{-1}(1/p),$$

then the proof is complete.

For the final part, it is clear that $p = 7$ and this leads to $st = (7+q)^2 + 1$, $uv = (7+r)^2$, $qr = 50$. From the second displayed result it is obvious that $q + s = 6$, $q + t = 14$, so that $(7+q)^2 + 1 = (6-q)(14-q) \Rightarrow q = 1$, and hence $s = 5$, $t = 13$. The values $r = 50$, $u = 25$, $v = 130$ can be obtained by a similar strategy.

The solution given above is not unique. Moreover, other plausible strategies may lead to incorrect solutions. It is important, therefore, to check that the solution obtained not only satisfies the displayed identity, but also the given conditions.

Q5 At the outset it should be emphasised that a large, well annotated diagram will enable insight into this question.

The first result may be obtained expeditiously by observing that if S_1 touches the sides BC, CA, AB at P, Q, R, respectively, then $AQ = AR = r \Rightarrow BR = c - r$, $CQ = b - r$. Thus $b - r + c - r = a \Rightarrow 2r = b + c - a$.

This result leads to $r = a(q - 1)/2$ which is the key to the remainder of the question. In fact $R = [2bc - \pi a^2(q - 1)^2]/\pi a^2$ (*).

From the data $a^2 = b^2 + c^2 \Rightarrow 2bc = (a + b + c)(b + c - a) \Rightarrow bc/a^2 = (q^2 - 1)/2$ which together with (*) leads to the second displayed result.

The obtaining of the turning value of a quadratic function is routine. In this context the method of completion of the square is to be preferred to the use of the calculus. Where the critical value of q is obtained from $dR/dq = 0$, it is important to give a reason as to why this defines an upper bound for R.

$Q6$ (i) The power series representations of $(1 + x)^{-k}$ for $k = 1, 2, 3$ are standard and should be well known by any candidate for this examination. In fact the general terms of these three series are x^n, $(n + 1)x^n$, $(1/2)(n + 1)(n + 2)x^n$, respectively.

The displayed series may be summed by using the general terms obtained. Thus,

$\sum_{n=1}^{\infty} n2^{-n} = (1/2)(1 - 1/2)^{-2} = 2$,

and as $\sum_{n=1}^{\infty} n(n + 1)2^{-n} = 8$, then $\sum_{n=1}^{\infty} n^2 2^{-n} = 8 - 2 = 6$.

(ii) The obtaining of the general term of the power series (*) for $(1 - x)^{-1/2}$ is a straightforward application of the binomial series for a general exponent.

To sum the penultimate series, put $x = 1/3$ in (*) and to sum the final series, first differentiate (*) with respect to x and then put $x = 1/3$. The sums will be found to be $\sqrt{3/2}$ and $(1/4)\sqrt{3/2}$, respectively.

$Q7$ (i) The absence of a $\mathbf{k}$ component in the specification of the locus of P, shows immediately that its motion takes place in the plane $z = 0$, i.e. in the $x - y$ plane. Also, it is obvious that $x^2 + y^2 = 1$. Hence P describes a circle centre O and radius 1 in the $x - y$ plane.

For the locus of Q, it is helpful to write $x = (3/2)\cos(t+\pi/4)$, $y = 3\sin(t+\pi/4)$, $z = (3\sqrt{3}/2)\cos(t+\pi/4)$. It is then evident that $\sqrt{3}x - z = 0$ and so this defines the plane in which the motion of Q takes place. Furthermore, it is clear that $x^2 + y^2 + z^2 = 9$ which shows that the distance of Q from O is constant and equal to 3. Hence Q describes the circle centre O and radius 3 in the plane $\sqrt{3}x - z = 0$.

(ii) Use of the scalar product leads to $\cos\theta = |(1/2)\cos t\cos(t + \pi/4) + \sin t\sin(t + \pi/4)| = \ldots = |3/4\sqrt{2} - (1/4)\cos(2t + \pi/4)|$.

(iii) From the result just obtained, it is immediate that $\theta \geq \pi/4 \Rightarrow -1/\sqrt{2} \leq 3/4\sqrt{2} - c/4 \leq 1/\sqrt{2}$ $(c \equiv \cos(2t + \pi/4) \Rightarrow -1/\sqrt{2} \leq c \leq 7/\sqrt{2}$,

and as $c \leq 1$, then c is restricted to the interval $-1/\sqrt{2} \leq c \leq 1$ so that $t \notin [\pi/4, \pi/2]$ and $t \notin [5\pi/4, 3\pi/2]$ are required. Hence $T = 3\pi/2$

Q8 Separation of variables will lead to

$$A - 1/y = \int x^3 (1 + x^2)^{-5/2}\, dx,$$

where A is a constant. There are several possible strategies for the evaluation of the integral on the right; by parts, or by any of such substitutions as $w = 1 + x^2$, $x = \tan t$, $x = \sinh v$. One way or another, the result of this integration correctly carried out will lead to the equivalent of

$$A - 1/y = -(1/3)x^2(1 + x^2)^{-3/2} - (2/3)(1 + x^2)^{-1/2}.$$

Use of the initial condition $y(0) = 1$ will then show that $A = 1/3$ and the required result follows at once.

$1/y = 1/3 + (2 + 3x^2)/[3(1 + x^2)^{3/2}]$.

To obtain the required approximation for y for large postive x, first write

$$1/y \approx 1/3 + (2 + 3x^2)(1 - 3/2x^2)/3x^3,$$

from which it follows that $1/y = 1/3 + 1/x + O(1/x^3)$ and this can easily be worked to the displayed approximation for y.

For the sketch, the main features are that it has a zero gradient at $x = 0$ and that for $x > 0$, it is monotonically increasing and has exactly one point of inflexion and that it is asymptotic to the line $y = 3$.

The two given differential equations are related by $y = z^2$. Thus it is unnecessary to solve the second differential equation independently of the first. In any case, the question does not require a formal solution for z. Nevertheless, it is helpful to obtain the approximation $z \approx \pm\sqrt{3} \mp 3\sqrt{3}/2x$ from $y \approx 3 - 9/x$, for large positive x.

All the main features of the $x - z$ sketch may be derived from those listed above for the $x - y$ sketch. The two curves which make up the sketch of z are reflections of each other in the $x-$ axis. They start at $(0, \pm 1)$ and are asymptotic to the lines $z = \pm\sqrt{3}$.

Q9(i) To begin with it is essential to draw a complete force diagram without omission or duplication of forces. In this respect, no particular direction, such as the horizontal or up the plane, should be assumed for the action of P. A convenient specification for the direction of P is $\theta + \pi/6$ with the horizontal, where at this stage, $0 < \theta < \pi/2$, but otherwise is general.

For motion up the slope it is necessary that

$$P\cos\theta \geq mg + mg/2 + (1/2\sqrt{3})(mg\sqrt{3}/2) + (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta).$$

From this inequality it follows that $P\cos(\theta - \pi/6) \geq (11\sqrt{3}/8)mg$ so that P_{min} is defined by $\theta = \pi/6$ and hence is equal to $(11\sqrt{3}/8)mg$.

Thus P_{min} acts at a direction making an angle of $\pi/3$ above the horizontal.

(ii) In order to clarify ideas in the second part of this question, it is advisable to draw a separate diagram. The friction, which again is limiting, acts up the slope so that now

$$P\cos\theta \geq mg + mg/2 - (1/2\sqrt{3})(mg\sqrt{3}/2) - (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta)$$

which implies $P\cos(\theta + \pi/6) \geq (\sqrt{3}/8)mg$. Thus in this case, $P_{min} = (\sqrt{3}/8)mg$ and is achieved when $\theta = -\pi/6$, so that P_{min} acts horizontally.

$Q10$ Take horizontal and vertical axes with origin at A and denote the position of the missile projected from A at time t by (x_1, y_1). Then prior to the collision, at time t_c,

$$x_1 = 80t, \ y_1 = 60t - 5t^2.$$

If (x_2, y_2) is the position of the anti-missile missile at time t, where $t_c \leq t \leq T$, then

$$x_2 = 180 - 120(t - T), \ y_2 = 160(t - T) - 5(t - T)^2.$$

At the collision,

$x_1 = x_2 \Rightarrow 200t_c = 120T + 180$ (1), $y_1 = y_2 \Rightarrow 60t_c - 5t_c^2 = 160(t_c - T) - 5(t_c - T)^2$ (2).

From (2) it follows that $T^2 + (32 - 2t)T - 20t = 0$ (3) and elimination of t from (1) and (3) yields $T^2 + [(151 - 6T)/5]T - 12T - 18 = 0$. Thus $T^2 - 91T + 90 = 0 \Rightarrow T = 1, 90$. However, in the absence of the collision, the flight time of the missile, would be 12 seconds, so that without ambiguity it may be concluded that $T = 1$.

$Q11$ Again, good supporting diagrams will enhance success with this question. The first result is standard. Beyond that, the motion of A up and down the slope need to be considered separately.

Let u be the velocity when the string breaks and T_1 be the time from this instant to when the particle A reaches its highest point. Thus $u = \lambda gT$, where $\lambda = (m_2 - m_1)/(m_2 + m_1)$, and as the deceleration of A during the time T_1 is g, then $T_1 = \lambda T$. Hence the total time taken by P to reach the highest point is $(1 + \lambda)T$.

For the downward motion of A, the acceleration is $g/10$, so that the data given in the penultimate sentence of the question implies

$$(g/10)(1 + \lambda)^2 T^2 = (\lambda g/2)T^2 + (\lambda^2 g/2)T^2.$$

From this it follows that $\lambda = 1/4$ and hence that $m_1/m_2 = 3/5$.

Q12 It is important to adopt an effective notation. Thus, for example, let $\alpha \sim$ heads , $\beta \sim$ tails , $T \sim$ true , $F \sim$ false .

(i) Use of the multiplication and addition laws of probability leads immediately to $P(\alpha) = ap + bq$. The coin is given to be fair so that $P(\alpha) = 1/2$. Hence $2(ap + bq) = 1$.

(ii) Write $G = ap + bq$, then,

$P(\alpha) = P(\alpha TF)/[P(\alpha TF) + P(\beta FT)] = [G(1 - G)/2]/[G(1 - G)/2 + (1 - G)G/2] = 1/2$,

independently of the value of G.

(iii) Here, it is given that $G = 1/2$. Although the argument below has some similarities with the previous working, there are important differences in the fine detail. Thus now,

$P(\alpha) = P(\alpha TT)/[P(\alpha TT) + P(\beta FF)] = [G^2/2]/[G^2/2 + (1 - G)^2/2] = 1/2$.

Q13 The introductory result at the end of the first paragraph is standard. For the approximations exhibited in (i),(ii) and (iii) it is important to ensure that enough terms are taken in the relevant expansions.

(i) $q = 1 - (1 + \lambda)e^{-\lambda} = 1 - (1 + \lambda)[1 - \lambda + \lambda^2/2 + O(\lambda^3)]$, as $\lambda \to 0$.

Thus $q = \lambda^2/2 + O(\lambda^3) \approx \lambda^2/2$, as $\lambda \to 0$.

(ii) $P(Y = n) = p^n > 1 - \lambda \Rightarrow e^{-n\lambda}(1 + \lambda)^n > 1 - \lambda$

$\Rightarrow [1 - n\lambda + n^2\lambda^2/2][1 + n\lambda + n(n - 1)\lambda^2/2] + O(\lambda^3) > 1 - \lambda$

which leads to $n < 2/\lambda$ within $O(\lambda^3)$

(iii) Write $P(Y > 1|Y > 0) = P(Y > 1)/P(Y > 0)$

$= (1 - q^n - npq^{n-1})/(1 - q^n) \approx 1 - np(\lambda^2/2)^{n-1}/[1 - (\lambda^2/2)^n]$,

from which follows the required result.

Q14 Standard methods lead to $k = 1/(1 + \lambda)$ and $\mu = \lambda^2/[2(1 + \lambda]$. respectively.

Also $E(X^2) = k + k\lambda^3/3 = (\lambda^3 + 3)/[3(1 + \lambda)]$, so that $\sigma^2 = (\lambda^3 + 3)/[3(1 + \lambda)] - \lambda^4/[4(1 + \lambda)^2)] \Rightarrow$

$\dots \Rightarrow$ required result.

For the remainder of the question $k = 1/3$.

(i) The graph is made up of three segments corresponding to $x < 0, 0 \leq x \leq 2$ and $x > 2$. In particular, the middle segment is a translation of the curve $y = \phi(x)/3$ upwards through $1/3$ parallel to the $f(x)$ axis.

(ii) If $F(x)$ is the CDF of X, then

$F(x) = \Phi(x)/3$ for $x \leq 0$,

$F(x) = \Phi(x)/3 + x/3$ for $0 < x \leq 2$,

$F(x) = \Phi(x)/3 + 2/3$ for $x > 2$.

(iii) Begin with $\mu = 2/3, \sigma^2 = 7/9$, then from the information given it is clear that

$P(0 < X < \mu + 2\sigma) = .9921/3 + 2/3 - 1/6 = 0.8307$.

Step III, Hints and Answers
June 2005

STEP Mathematics III 2005: Hints and Answers

Section A: Pure Mathematics

1   To prove the first part, use the results: $\cos B = \sin\left(\dfrac{\pi}{2} - B\right)$, whatever the value of B; and
$$\sin A = \sin Y \Leftrightarrow A = Y + 2n\pi \text{ or } A = \pi - Y + 2n\pi;$$
thus here, replacing Y by $\dfrac{\pi}{2} - B$, $A = 2n\pi + \dfrac{\pi}{2} \pm B$.

For the next part, it is probably easiest to use the fact that $a\sin x \pm b\cos x$ can be written in the form $R\sin(x \pm \alpha)$; here,
$$\sin x \pm \cos x = \sqrt{2}\left(\sin x \cos\frac{\pi}{4} \pm \cos x \sin\frac{\pi}{4}\right) = \sqrt{2}\sin\left(x \pm \frac{\pi}{4}\right)$$
so $|\sin x \pm \cos x| \leqslant \sqrt{2}$.

Now, from the first part,
$$\sin(\sin x) = \cos(\cos x) \Leftrightarrow \sin x = 2n\pi + \frac{\pi}{2} \pm \cos x$$
so
$$|\sin x \pm \cos x| \geqslant \left|2n\pi + \frac{\pi}{2}\right| \geqslant \frac{\pi}{2} > \sqrt{2},$$
which is a contradiction.

All the curves asked for have period 2π, so they will be sketched for x in this range only.

For $y = \sin(\sin x)$, $y = 0$ when $\sin x = 0$ only (since $|\sin x| < \pi$), so at 0, π and 2π; the turning points are at $\cos x \cos(\sin x) = 0$, so when $\cos x = 0$, that is at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$, or when $\cos(\sin x) = 0$, which is impossible since $|\sin x| < \dfrac{\pi}{2}$; the turning points are a maximum at $\left(\dfrac{\pi}{2}, \sin(1)\right)$ and a minimum at $\left(\dfrac{3\pi}{2}, -\sin(1)\right)$, where $\sin(1) \approx 0\!\cdot\!84$.

For $y = \cos(\cos x)$, $y > 0$ for all x, since $|\cos x| \leqslant 1 < \dfrac{\pi}{2}$; the turning points are at $\sin x \sin(\cos x) = 0$, so when either $\sin x = 0$ or $\cos x = 0$, that is at $x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$; the turning points are maxima at $\left(\dfrac{\pi}{2}, 1\right)$ and $\left(\dfrac{3\pi}{2}, 1\right)$, and minima at $(0, \cos(1))$, $(\pi, \cos(1))$, $(2\pi, \cos(1))$, where $\cos(1) \approx 0\!\cdot\!54$.
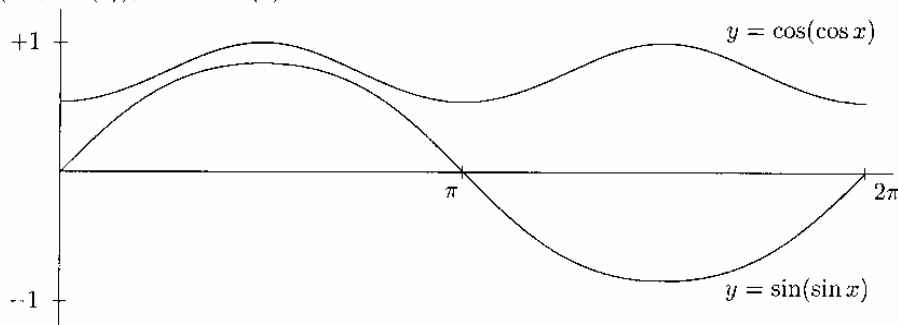
STEP Mathematics III 2005: Hints and Answers

For the curve $y = \sin(2\sin x)$, $y = 0$ if $2\sin x$ is a multiple of π, which is only possible if $\sin x = 0$ (since $|2\sin x| < \pi$), so when x is 0, π and 2π; the turning points are at $2\cos x \cos(2\sin x) = 0$; so when $\cos x = 0$, that is at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$, or when $2\sin x$ is an odd multiple of $\dfrac{\pi}{2}$, which is only possible if $2\sin x = \pm\dfrac{\pi}{2}$, so when $\sin x = \dfrac{\pi}{4} \approx \pm 0{\cdot}8$; the turning points are a minimum at $\left(\dfrac{\pi}{2},\, \sin 2\right)$, where $\sin 2 \approx 0{\cdot}91$ and maxima either side of this, with y-coordinates 1 and a maximum at $\left(\dfrac{3\pi}{2},\, -\sin 2\right)$ with minima either side with y-coordinates -1.
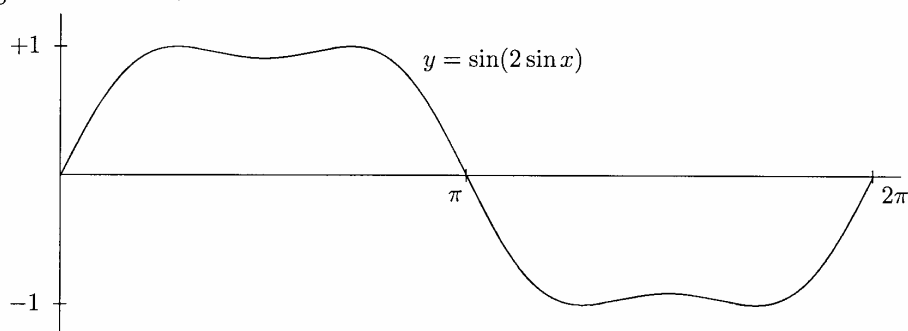
2   This equation can be solved by separating the variables:

$$\int \frac{2\,\mathrm{d}y}{y} = -\int \frac{2x\,\mathrm{d}x}{x^2 + a^2} \quad\text{so}\quad \ln(y^2) = -\ln(x^2 + a^2) + k \quad\text{or}\quad y^2(x^2 + a^2) = c^2.$$

The curve has two branches: one has $y > 0$, reflection symmetry about the y-axis, a maximum at $\left(0,\, \dfrac{c}{a}\right)$ and $y \longrightarrow 0$ as $|x| \longrightarrow \infty$; the other has $y < 0$ and is a reflection of the first branch in the x-axis.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = 2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{2xy^2}{x^2 + a^2} = 2x - \frac{2xc^2}{(x^2 + a^2)^2}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2 + y^2) = 2 - \frac{2c^2}{(x^2 + a^2)^2} + \frac{4xc^2 \cdot 2x}{(x^2 + a^2)^3} = 2\left(1 - \frac{c^2}{(x^2 + a^2)^2}\right) + \frac{8c^2x^2}{(x^2 + a^2)^3}$$

STEP Mathematics III 2005: Hints and Answers

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2)=0$ when $x=0$ and when $c^2=(x^2+a^2)^2$, but the latter is not possible if $0<c<a^2$. If $x=0$, $y=\pm\dfrac{c}{a}$ and $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2)=1-\dfrac{c^2}{a^4}$ which is positive if $0<c<a^2$, indicating a local minimum. Hence the points on the curve whose distance from the origin is least are $\left(0,\pm\dfrac{c}{a}\right)$.

(ii)   If $c>a^2$ then $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2)$ is negative at $x=0$, indicating a local maximum there; but in this case there are further stationary points at $x^2=c-a^2$, $y=\pm\sqrt{c}$ and at these points $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2)=\dfrac{8x^2}{c}>0$. Hence the points on the curve whose distance from the origin is least are $(\pm\sqrt{c-a^2},\pm\sqrt{c})$.

3   Direct substitution gives

$$\mathrm{f}\left(\mathrm{g}\left(x\right)\right)=(x^2+rx+s)^2+p(x^2+rx+s)+q$$

$$=x^4+2rx^3+(r^2+2s+p)x^2+(2rs+rp)x+s^2+ps+q.$$

If $x^4+ax^3+bx^2+cx+d$ is to have this form then it is necessary to choose $r=\dfrac{a}{2}$ and to choose s and p to satisfy $2s+p=b-r^2=b-\dfrac{a^2}{4}$ and $r(2s+p)=c$ or $a(2s+p)=2c$. Thus $a\left(b-\dfrac{a^2}{4}\right)=2c$ is a necessary condition for this to be possible.

It is also sufficient: in fact, pick $p=0$; then $s=\dfrac{4b-a^2}{8}$ and $q=d-s^2$ will do.

Expanding the second form gives

$$(x^2+vx+w)^2-k=x^4+2vx^3+(v^2+2w)x^2+2vwx+w^2-k,$$

but this is identical to $x^4+2rx^3+(r^2+2s+p)x^2+(2rs+rp)x+s^2+ps+q$ with $p=0$, $v=r$, $w=s$ and $k=-q$, and so, since the sufficiency demonstrated above allowed the choice $p=0$, the condition is the same.

To solve the final equation, write the quartic in the second form:

$$x^4-4x^3+10x^2-12x+4=(x^2-2x+3)^2-5=0$$

so

$$x^2-2x+3-\sqrt{5}=0 \text{ or } x^2-2x+3+\sqrt{5}=0$$

so

$$x=1\pm\sqrt{\sqrt{5}-2} \text{ or } 1\pm j\sqrt{\sqrt{5}+2}.$$

STEP Mathematics III 2005: Hints and Answers

4   For the base case you need to verify that $u_{2n} = \dfrac{b}{a} u_{2n-1}$ and $u_{2n+1} = cu_{2n}$ when $n = 1$:

$u_1 = a$, $u_2 = b$ so $u_{2n} = \dfrac{b}{a} u_{2n-1}$ when $n = 1$;

$u_3 = \dfrac{u_2}{u_1}(ku_1 - u_2) = u_2 \dfrac{ka - b}{a}$ so $u_{2n+1} = cu_{2n}$ when $n = 1$, provided $c = k - \dfrac{b}{a}$.

For the induction step, assume that $u_{2n} = \dfrac{b}{a} u_{2n-1}$ and $u_{2n+1} = cu_{2n}$ when $n = N$ then

$$u_{2N+2} = \frac{u_{2N+1}}{u_{2N}}(ku_{2N} - u_{2N+1})$$

$$= u_{2N+1}(k - c) \text{ (by the induction hypothesis)}$$

$$= \frac{b}{a} u_{2N+1} \text{ (by the definition of } c)$$

and

$$u_{2N+3} = \frac{u_{2N+2}}{u_{2N+1}}(ku_{2N+1} - u_{2N+2})$$

$$= u_{2N+2}\left(k - \frac{b}{a}\right) \text{ (by what has just been shown)}$$

$$= cu_{2N+2}.$$

which completes the induction.

Hence $u_{2n} = \dfrac{bc}{a} u_{2n-2} = \ldots = \left(\dfrac{bc}{a}\right)^{n-1} u_2 = b\left(\dfrac{bc}{a}\right)^{n-1}$

and $u_{2n-1} = \dfrac{bc}{a} u_{2n-3} = \ldots = \left(\dfrac{bc}{a}\right)^{n-1} u_1 = a\left(\dfrac{bc}{a}\right)^{n-1}$

(i)   For u_n to be geometric requires $\dfrac{u_{2n}}{u_{2n-1}} = \dfrac{u_{2n+1}}{u_{2n}}$; that is, $\dfrac{b}{a} = c = k - \dfrac{b}{a}$ or $ak = 2b$;

(ii)   For u_n to have period 2 requires $u_{2n+1} = u_{2n-1}$, but $u_{2n+1} = cu_{2n} = \dfrac{cb}{a} u_{2n-1}$, so it is

necessary that $\dfrac{cb}{a} = 1$ or $a^2 + b^2 = kab$;

(iii)   For u_n to have period 4 requires $u_{2n+3} = u_{2n-1}$ so, by the previous part, it is necessary

that $\left(\dfrac{cb}{a}\right)^2 = 1$ but $\dfrac{cb}{a} \neq 1$ (to avoid period 2) so $\dfrac{cb}{a} = -1$ or $b^2 - a^2 = kab$.

STEP Mathematics III 2005: Hints and Answers

5   The point on the curve with the required gradient is given by

$$\frac{dy}{dx} = 2ax + b = m \text{ or } x = \frac{m-b}{2a},$$

with

$$y = a\left(\frac{m-b}{2a}\right)^2 + b\left(\frac{m-b}{2a}\right) + c = \frac{m^2 - b^2}{4a} + c.$$

The equation of the tangent is therefore:

$$y - mx = a\left(\frac{m-b}{2a}\right)^2 + b\left(\frac{m-b}{2a}\right) + c - m\left(\frac{m-b}{2a}\right)$$

$$= c - \frac{(m-b)}{2a}\left(m - b - \frac{(m-b)}{2}\right) = c - \frac{(m-b)^2}{4a}.$$

The curves have a common tangent with gradient m if and only if the equations of the tangents to the two curves with gradient m are identical; that is, have the same intercept, so if and only if

$$c_1 - \frac{(m-b_1)^2}{4a_1} = c_2 - \frac{(m-b_2)^2}{4a_2}$$

that is,

$$4a_1a_2c_1 - a_2m^2 + 2a_2b_1m - a_2b_1^2 = 4a_1a_2c_2 - a_1m^2 + 2a_1b_2m - a_1b_2^2$$

which gives the quoted result.

There is exactly one common tangent when $a_1 \neq a_2$ when the quadratic equation for m has exactly one root, which occurs if and only if the discriminant of the equation is zero; that is

$$4(a_1b_2 - a_2b_1)^2 = 4(a_2 - a_1)(4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2)$$

$$\Leftrightarrow \quad 4a_1^2b_2^2 - 8a_1a_2b_1b_2 + 4a_2^2b_1^2 = 16a_1a_2(a_2 - a_1)(c_2 - c_1) + 4a_2^2b_1^2 + 4a_1^2b_2^2 - 4a_1a_2(b_1^2 + b_2^2)$$

$$\Leftrightarrow \quad 4a_1a_2(b_1^2 + b_2^2) - 8a_1a_2b_1b_2 = 16a_1a_2(a_2 - a_1)(c_2 - c_1)$$

$$\Leftrightarrow \quad b_1^2 + b_2^2 - 2b_1b_2 = 4(a_2 - a_1)(c_2 - c_1) \quad \text{(dividing by } 4a_1a_2 \text{ which is non-zero).}$$

The curves touch if there is exactly one solution to the simultaneous equations

$$y = a_1x^2 + b_1x + c_1 \text{ and } y = a_2x^2 + b_2x + c_2;$$

that is, if the equation $(a_2 - a_1)x^2 + (b_2 - b_1)x + (c_2 - c_1) = 0$ has exactly one root so, again using the discriminant condition, if and only if $(b_2 - b_1)^2 = 4(a_2 - a_1)(c_2 - c_1)$, which is the same condition.

If $a_1 = a_2$ the curves have exactly one common tangent if there is exactly one solution to

$$2m(b_2 - b_1) + 4a(c_2 - c_1) + (b_1^2 - b_2^2) = 0;$$

since this is just a linear equation, the only condition is that $b_2 - b_1 \neq 0$.

STEP Mathematics III 2005: Hints and Answers

6   Direct substitution of $x = 2a \cosh\left(\dfrac{T}{3}\right)$ into the left hand side of the equation gives

$$\left(2a\cosh\left(\frac{T}{3}\right)\right)^3 - 6a^3\cosh\left(\frac{T}{3}\right) = 2a^3\left(4\left(\cosh\left(\frac{T}{3}\right)\right)^3 - 3\cosh\left(\frac{T}{3}\right)\right) = 2a^3\cosh T$$

(by the first result given at the start of the question).

Let $a^2 = b$, which is possible since $b > 0$, and $\cosh T = \dfrac{c}{a^3}$, which requires $\dfrac{c}{a^3} \geqslant 1$; but this holds if you choose a to have the same sign as c, since then $\dfrac{c}{a^3} > 0$ and $c^2 > b^3 = a^6$.

Then, by the second result given at the start of the question,

$$T = \ln\left(\frac{c}{a^3} + \sqrt{\frac{c^2}{a^6} - 1}\right) = \ln\left(\frac{c + \sqrt{c^2 - b^3}}{a^3}\right) = 3\ln\left(\frac{u}{a}\right),$$

so one of the roots of the equation $x^3 - 3bx = 2c$ is

$$2a\cosh\left(\ln\left(\frac{u}{a}\right)\right) = 2a\,\frac{\dfrac{u}{a} + \dfrac{a}{u}}{2} = u + \frac{b}{u}.$$

Note that, since $u + \dfrac{b}{u}$ is a root of the equation $x^3 - 3bx = 2c$,

$$2c = \left(u + \frac{b}{u}\right)^3 - 3b\left(u + \frac{b}{u}\right) = \left(u + \frac{b}{u}\right)\left(u^2 + \frac{b^2}{u^2} - b\right)$$

and that

$$u^2 + \frac{b^2}{u^2} - b - \left(u + \frac{b}{u}\right)^2 = -3b,$$

so

$$x^3 - 3bx - 2c = \left(x - \left(u + \frac{b}{u}\right)\right)\left(x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b\right),$$

so the other two roots of $x^3 - 3bx = 2c$ are the roots of $x^2 + \left(u + \dfrac{b}{u}\right)x + u^2 + \dfrac{b^2}{u^2} - b = 0$, which are

$$\frac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{\left(u + \frac{b}{u}\right)^2 - 4\left(u^2 + \frac{b^2}{u^2} - b\right)}\right)$$

$$= \frac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{-3\left(u^2 + \frac{b^2}{u^2} - 2b\right)}\right) = \frac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{3}j\left(u - \frac{b}{u}\right)\right),$$

that is $\omega u + \omega^2\dfrac{b}{u}$ and $\omega^2 u + \omega\dfrac{b}{u}$.

In $x^3 - 6x = 6$, $b = 2$, $c = 3$, so $a = \sqrt{2}$ and so $u = \sqrt[3]{3+1} = 2^{\frac{2}{3}}$ and $\dfrac{b}{u} = 2^{\frac{1}{3}}$, so the solutions are $2^{\frac{1}{3}} + 2^{\frac{2}{3}}$, $\omega 2^{\frac{1}{3}} + \omega^2 2^{\frac{2}{3}}$ and $\omega^2 2^{\frac{1}{3}} + \omega 2^{\frac{2}{3}}$.

STEP Mathematics III 2005: Hints and Answers

7   Substituting $u = x^m$ gives

$$\int \frac{m\,\mathrm{d}x}{x\,\mathrm{f}\,(x^m)} = \int \frac{mx^{m-1}\mathrm{d}x}{x^m\,\mathrm{f}\,(x^m)} = \int \frac{\mathrm{d}u}{u\,\mathrm{f}\,(u)} = \mathrm{F}(u) + c = \mathrm{F}(x^m) + c$$

(i) $\displaystyle \int \frac{\mathrm{d}x}{x^n - x} = \int \frac{\mathrm{d}x}{x\,(x^{n-1} - 1)},$

so letting $u = x^{n-1}$ and $\mathrm{f}\,(u) = u - 1$,

$$\int \frac{(n-1)\mathrm{d}x}{x^n - x} = \int \frac{\mathrm{d}u}{u\,(u-1)} = \int \frac{1}{u-1} - \frac{1}{u}\,\mathrm{d}u = \ln\left|\frac{u-1}{u}\right|$$

so $\displaystyle \int \frac{\mathrm{d}x}{x^n - x} = \frac{1}{n-1}\ln\left|\frac{x^{n-1} - 1}{x^{n-1}}\right| + c.$

(ii) $\displaystyle \int \frac{\mathrm{d}x}{\sqrt{x^n + x^2}} = \int \frac{\mathrm{d}x}{x\sqrt{x^{n-2} + 1}}$ (for $x > 0$)

so letting $u = x^{n-2}$ and $\mathrm{f}\,(u) = \sqrt{u+1}$ (and assuming $n \neq 2$)

$$\int \frac{(n-2)\mathrm{d}x}{\sqrt{x^n + x^2}} = \int \frac{\mathrm{d}u}{u\,\sqrt{u+1}}.$$

Substituting $u = v^2 - 1$ with $v > 0$,

$$\int \frac{\mathrm{d}u}{u\,\sqrt{u+1}} = \int \frac{2v\,\mathrm{d}v}{(v^2 - 1)\,v} = \int \frac{1}{v-1} - \frac{1}{v+1}\,\mathrm{d}v = \ln\left|\frac{v-1}{v+1}\right|$$

so $\displaystyle \int \frac{\mathrm{d}x}{\sqrt{x^n + x^2}} = \frac{1}{n-2}\ln\left|\frac{\sqrt{x^{n-2} + 1} - 1}{\sqrt{x^{n-2} + 1} + 1}\right| + c.$

STEP Mathematics III 2005: Hints and Answers

8   Direct use of the important result $|z|^2 = zz^*$ gives

$$|a - c|^2 = (a - c)(a^* - c^*) = aa^* + cc^* - ac^* - ca^*.$$

OAC is a right angle if and only if $|AC|^2 + |OA|^2 = |OC|^2$; that is, $|a - c|^2 + |a|^2 = |c|^2$ or, using the result above, $2aa^* - ac^* - ca^* = 0$.

The circle has centre C and radius AC, so complex numbers representing points on the circle satisfy $|z - c|^2 = |a - c|^2$ or $zz^* - zc^* - cz^* = aa^* - ac^* - ca^*$.
Because OA is a tangent to the circle, angle OAC is a right angle and so $2aa^* - ac^* - ca^* = 0$ as above; thus the condition for points to lie on the circle becomes $zz^* - zc^* - cz^* + aa^* = 0$.

P lies on this circle if and only if

$$aba^*b^* - abc^* - ca^*b^* + aa^* = 0$$

and P$'$ lies on the circle if and only if

$$\frac{aa^*}{bb^*} - \frac{a}{b^*}c^* - c\frac{a^*}{b} + aa^* = 0$$

but multiplying this by bb^* (which is not equal to zero) gives the same condition.

Conversely, if the points lie on the circle represented by $|z - c|^2 = |a - c|^2$,

$$aba^*b^* - abc^* - ca^*b^* + aa^* = 2aa^* - ac^* - ca^*$$

and

$$\frac{aa^*}{bb^*} - \frac{a}{b^*}c^* - c\frac{a^*}{b} = 2aa^* - ac^* - ca^*,$$

so that

$$aba^*b^* - abc^* - ca^*b^* + aa^* = bb^*(2aa^* - ac^* - ca^*)$$

and so, provided $bb^* \neq 1$, it must be the case that $2aa^* - ac^* - ca^* = 0$, and this shows that OAC is a right angle and hence that OA is a tangent to the circle.

STEP Mathematics III 2005: Hints and Answers

Section B: Mechanics

9   Let the speeds of A and B after the first collision be u_1, u_2,
then conservation of momentum gives

$$4eu_1 + (1-e)^2 u_2 = 4e(1-e)v - (1-e)^2 \cdot 2ev = 2ev(1-e^2)$$

and the restitution equation gives

$$u_2 - u_1 = (1-e)v + 2ev = e(1+e)v.$$

To find u_1, multiply the second equation by $(1-e)^2$ and subtract it from the first:

$$u_1 = \frac{2ev(1-e^2) - (1-e)^2 e(1+e)v}{4e + (1-e)^2} = \frac{ev(1-e^2)}{(1+e)^2}\,(2 - (1-e)) = e(1-e)v$$

To find u_2, multiply the second equation by $4e$ and add it to the first:

$$u_2 = \frac{2ev(1-e^2) + 4e^2(1+e)v}{4e + (1-e)^2} = \frac{2ev(1+e)}{(1+e)^2}\,((1-e) + 2e) = 2ev.$$

After B strikes the vertical wall, it rebounds with speed $2e^2v$, so if x is the distance from the wall at which the second collision occurs, the total time between collisions is

$$\frac{d-x}{e(1-e)v} = \frac{d}{2ev} + \frac{x}{2e^2v},$$

so that $2e(d-x) = (1-e)(ed+x)$ or $x = ed$.
Note that the situation is now that given initially, with all distances and speeds scaled by e. Thus the nth collision occurs a distance de^{n-1} from the wall, and the speed of A between the nth and the $(n+1)^{\text{th}}$ collisions is $(1-e)ve^n$, so the time between collisions is

$$\frac{de^{n-1} - de^n}{(1-e)ve^n} = \frac{d}{ev},$$

which is independent of n.

STEP Mathematics III 2005: Hints and Answers

10   When the discs are a distance $2x$ apart, their centres are $2(x+r)$ apart and the length of the band is $4(x+r) + 2\pi r$. Therefore the tension in the band is

$$T(x) = 2\left(\frac{\pi mg}{12}\right)\frac{4x+4r}{2\pi r} = \frac{mg}{6r}(x+r),$$

and hence the force on each disc is $F(x) = 2T(x) = \frac{mg}{3r}(x+r)$; the elastic potential energy stored in the band is

$$E(x) = \tfrac{1}{2}\left(\frac{\pi mg}{12}\right)\frac{(4x+4r)^2}{2\pi r} = \frac{mg}{3r}(x+r)^2.$$

(i)   The maximum frictional resistance to the motion of a disc is μmg, so for the disc to start sliding requires $F(2r) > \mu mg$, that is $1 > \mu$. For the disc then to come to rest before a collision occurs, the elastic energy released by the band as x decreases from $2r$ to 0 must be insufficient to do the work against friction required to bring the discs into contact. This required work is $2r\mu mg$ for each disc, so $4r\mu mg$ in total, so the condition you need is $E(2r) - E(0) < 4r\mu mg$; that is $4r\mu > 3r - \frac{1}{3}r$ or $\mu > \frac{2}{3}$.

(ii)   By the argument in (i), $E(2r) = E(0) + K + 4r\mu mg$, where K is the kinetic energy just before collision, so

$$K = 3rmg - \frac{1}{3}rmg - 4\mu rmg = mgr\left(\frac{8}{3} - 4\mu\right).$$

(iii)   Notice first that for the discs to come to rest after the first collision, it is necessary that the discs collide, so $\mu^2 < \frac{4}{9}$, by part (i).

In order that the discs do not begin to move again, once they have come to rest for the first time after collision, each must come to rest at a point where $F(x) < \mu mg$, that is $x < (3\mu - 1)r$.

The value of x at which the particles do come to rest is given by the requirement that the loss in elastic and kinetic energy from the point of collision to the point where the discs are $2x$ apart is equal to the work done against friction on both particles in moving from 0 to $2x$ separation, that is $E(0) + \frac{1}{2}K - E(x) = 2x\mu mg$ or

$$0 = \frac{mgr}{3} + mgr\left(\frac{4}{3} - 2\mu\right) - \frac{mg}{3r}(x+r)^2 - 2x\mu$$

$$> \frac{mgr}{3} + mgr\left(\frac{4}{3} - 2\mu\right) - \frac{mg}{3r}(3\mu)^2 - 2mg\mu(3\mu - 1)r$$

using the inequality on x, so

$$0 > \frac{mgr}{3}\left(5 - 27\mu^2\right)$$

or $\mu^2 > \frac{5}{27}$.

STEP Mathematics III 2005: Hints and Answers

11   The energy of the system is the sum of the gravitational potential energy (GPE) and the kinetic energy (KE), with KE $= \frac{1}{2}m\left(a^2 + b^2 + c^2\right)\dot{\theta}^2$ and, taking the zero of GPE to be at the height of the spindle,

$$\text{GPE} = mg\left(a\cos\theta - b\cos\left(\frac{\pi}{3} - \theta\right) - c\cos\left(\frac{\pi}{3} + \theta\right)\right)$$
$$= \tfrac{1}{2}mg\left((2a - b - c)\cos\theta - (b - c)\sqrt{3}\sin\theta\right)$$

simplifying using the $\cos(A \pm B)$ identities and the exact values of $\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{3}$.

Equilibrium occurs at a stationary point of the GPE:

$$\frac{\text{dGPE}}{\text{d}\theta} = \tfrac{1}{2}mg\left(-(2a - b - c)\sin\theta - (b - c)\sqrt{3}\cos\theta\right) = 0$$

or when the total moment about the spindle of the gravitational forces on the particles is zero:

$$mg\left(a\sin\theta + b\sin\left(\frac{\pi}{3} - \theta\right) - c\sin\left(\frac{\pi}{3} + \theta\right)\right) = 0$$

which simplifies to the same equation using the $\sin(A \pm B)$ identities.

This equation is satisfied if

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = -\frac{(b - c)\sqrt{3}}{2a - b - c} < 0$$

which has two solutions between 0 and 2π unless $a = b = c = 0$. It is useful to realise here that $\tan\theta = -\frac{p}{q} \Rightarrow \sin\theta = \mp\frac{p}{h}$ and $\cos\theta = \pm\frac{q}{h}$, where h is the positive number with $h^2 = p^2 + q^2$. In this case, $\sin\theta = = -\frac{(b - c)\sqrt{3}}{2R}$ and $\cos\theta = \frac{(2a - b - c)}{2R}$ give one equilibrium and $\sin\theta = \frac{(b - c)\sqrt{3}}{2R}$ and $\cos\theta = -\frac{(2a - b - c)}{2R}$ the other, where

$$R^2 = \frac{1}{4}\left((2a - b - c)^2 + 3(c - b)^2\right) = \frac{1}{2}\left((a - b)^2 + (b - c)^2 + (c - a)^2\right)$$

The equilibrium is stable at a minimum of the GPE and unstable at a maximum. Since

$$\frac{\text{d}^2\text{GPE}}{\text{d}\theta^2} = \tfrac{1}{2}mg\left(-(2a - b - c)\cos\theta + (b - c)\sqrt{3}\sin\theta\right).$$

the first equilibrium position given above is unstable and the second is stable.

For the system to make complete revolutions, you need the KE at the point with maximum GPE to be greater than zero: that is, the difference in GPE between the two equilibria (which is twice the maximum GPE) is less than the KE at the point with minimum GPE.

If $\omega =$ angular velocity at stable equilibrium, you therefore require

$$\tfrac{1}{2}m\left(a^2 + b^2 + c^2\right)\omega^2 > mg\left((2a - b - c)\frac{(2a - b - c)}{2R} + (c - b)\sqrt{3}\frac{(c - b)\sqrt{3}}{2R}\right)$$

That is, $\left(a^2 + b^2 + c^2\right)\omega^2 > 2g\left(\frac{4R^2}{2R}\right) = 4gR.$

STEP Mathematics III 2005: Hints and Answers

Section C: Probability and Statistics

12   If $X = aT + b(T_1 + T_2 + T_3 + T_4)$ then

$$\mathrm{E}[X] = at + b(t_1 + t_2 + t_3 + t_4) = (a + b)t,$$

since $t_1 + t_2 + t_3 + t_4 = t$, by definition. You require $\mathrm{E}[X] = t$ which gives $a + b = 1$.

$$\mathrm{Var}[X] = a^2\mathrm{Var}[T] + b^2\mathrm{Var}[T_1] + b^2\mathrm{Var}[T_2] + b^2\mathrm{Var}[T_3] + b^2\mathrm{Var}[T_4],$$

assuming the errors made by the five timers are independent,

$$= (a^2 + 4b^2)\sigma^2 = (a^2 + 4(1 - a)^2)\sigma^2 = (5a^2 - 8a + 4)\sigma^2 = \left(5\left(a - \frac{4}{5}\right)^2 + \frac{4}{5}\right)\sigma^2$$

which has a minimum value of $\frac{4}{5}\sigma^2$ when $a = \frac{4}{5}$; that is, when $X = \frac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4))$.

Rearranging the identity $\mathrm{Var}[Y] = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2$ gives $\mathrm{E}[Y^2] = \mathrm{Var}[Y] + \mathrm{E}[Y]^2$,
so if $Y = cT + d(T_1 + T_2 + T_3 + T_4)$ then

$$\mathrm{E}[Y^2] = (c^2 + 4d^2)\sigma^2 + (ct + d(t_1 + t_2 + t_3 + t_4))^2 = (c^2 + 4d^2)\sigma^2 + (c + d)^2t^2,$$

which is equal to σ^2 regardless of the true lap times if $c + d = 0$ and $1 = c^2 + 4d^2 = 5c^2$,
so that $c = \dfrac{1}{\sqrt{5}}$ and $Y^2 = \frac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2$.

The timers could reasonably expect the true time for the race to lie within k estimated standard
errors of the estimated value where, for instance, $k = 2$ or 3; so between

$$\tfrac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4)) + k\sqrt{\tfrac{4}{5} \times \tfrac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2}$$
and
$$\tfrac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4)) - k\sqrt{\tfrac{4}{5} \times \tfrac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2};$$

that is, between $220.1 + \dfrac{k}{5}$ and $220.1 - \dfrac{k}{5}$. For $k = 2$, this is the interval $[219 \cdot 7 \, , \, 220 \cdot 5]$.

STEP Mathematics III 2005: Hints and Answers

13 The probability that the player wins exactly £3 is equal to the probability that the next 3 scores which lie in the range 0 to w are non zero, and the fourth score which lies in the range 0 to w is zero as the occurrence of outcomes which lead to the game continuing does not affect the final result.

Hence the probability that the player wins exactly £3 is equal to $\left(\dfrac{w}{w+1}\right)^3 \dfrac{1}{w+1}$.

Similarly, the probability that he wins exactly £r is $\left(\dfrac{w}{w+1}\right)^r \dfrac{1}{w+1}$ and so his expected winnings are

$$\sum_{r=0}^{\infty} r \left(\frac{w}{w+1}\right)^r \frac{1}{w+1} = \frac{w}{(w+1)^2} \sum_{r=1}^{\infty} r \left(\frac{w}{w+1}\right)^{r-1} = \frac{w}{(w+1)^2} \frac{1}{\left(1-\dfrac{w}{w+1}\right)^2} = w.$$

This calculation uses the result $\displaystyle\sum_{r=1}^{\infty} rp^{r-1} = \sum_{r=0}^{\infty} rp^{r-1} = \frac{1}{(1-p)^2}$, which you may know, or can be derived by noticing that $\displaystyle\sum_{r=0}^{\infty} rp^{r-1} = \frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{r=0}^{\infty} p^r\right)$ and that $\displaystyle\sum_{r=0}^{\infty} p^r = \frac{1}{1-p}$, recognising an infinite geometric series.

In a second game, consider the cards set out in the order in which they will be drawn. Then only $w+1$ cards are relevant, and the zero card is equally likely to be any of these, so that the probability that he wins exactly £r is $\dfrac{1}{w+1}$ (for $r = 1, 2, \ldots w$) and so his expected winnings are now

$$\sum_{r=0}^{w} r \frac{1}{w+1} = \frac{1}{w+1} \sum_{r=0}^{w} r = \frac{1}{w+1} \tfrac{1}{2}w(w+1) = \tfrac{1}{2}w.$$

STEP Mathematics III 2005: Hints and Answers

14   The integral of the density function from 0 to infinity must equal 1:

$$1 = \int_0^\infty \frac{Ck^{a+1}x^a}{(x+k)^{2a+2}}\, \mathrm{d}x = Ck^{a+1}\frac{a!(2a-a)!}{(2a+1)!k^{2a-a+1}},$$

using the given result with $m = a$ and $n = 2a$

$$= C\frac{a!a!}{(2a+1)!} \text{ so } C = \frac{(2a+1)!}{a!a!}.$$

Use the substitution $x = \dfrac{k^2}{u}$:

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}}\, \mathrm{d}x = \int_\infty^{\frac{k^2}{v}} \frac{k^{2a}}{u^a\left(\dfrac{k}{u}(u+k)\right)^{2a+2}} \frac{-k^2\mathrm{d}u}{u^2} = \int_{\frac{k^2}{v}}^\infty \frac{u^a}{(u+k)^{2a+2}}\, \mathrm{d}u$$

Choosing $v = k$ gives $\displaystyle\int_0^k \mathrm{f}(x)\, \mathrm{d}x = \int_k^\infty \mathrm{f}(x)\, \mathrm{d}x$, so k is the median, and

$$\mathrm{E}[V] = \int_0^\infty \frac{C\,k^{a+1}\,x^{a+1}}{(x+k)^{2a+2}}\, \mathrm{d}x = \frac{(2a+1)!}{a!\,a!}k^{a+1}\frac{(a+1)!\,(a-1)!}{(2a+1)!\,k^a} = k\left(\frac{a+1}{a}\right).$$

Notice that $T < t$ if and only if $V > \dfrac{s}{t}$, so that

$$\mathrm{P}(T < t) = \mathrm{P}\left(V > \frac{s}{t}\right) = \int_{\frac{s}{t}}^\infty \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}}\, \mathrm{d}x$$

and making the substitution $x = \dfrac{s}{u}$:

$$= \int_t^0 \frac{C\,k^{a+1}\,s^a}{u^a\left(\dfrac{k}{u}\left(\dfrac{s}{k}+u\right)\right)^{2a+2}} \frac{-s\,\mathrm{d}u}{u^2} = \int_0^t \frac{C\,u^a\left(\dfrac{s}{k}\right)^{a+1}}{\left(\dfrac{s}{k}+u\right)^{2a+2}}\, \mathrm{d}u$$

so the density function is $\dfrac{C\,u^a\left(\dfrac{s}{k}\right)^{a+1}}{\left(\dfrac{s}{k}+u\right)^{2a+2}}$, which is the same as that of V with k replaced by $\dfrac{s}{k}$.

This means that the median time is $\dfrac{s}{k}$ and that the expected time is $\dfrac{s}{k}\left(\dfrac{a+1}{a}\right)$ and hence

median velocity $\times$ median time $= s$, but $\mathrm{E}[V]\,\mathrm{E}[T] = s\left(\dfrac{a+1}{a}\right)^2$, which is greater than s.

Report on the Components
June 2005

9465 - Mathematics 1

General comments

This paper was found to be more straightforward than last year's, with the exception of the questions on Probability and Statistics. The Mechanics questions (in particular, 9 and 10) were more popular than previously. Inaccurate algebraic manipulation remains the biggest obstacle to candidates' success: at this level, the fluent, confident and correct handling of mathematical symbols is necessary and is expected. Many good starts to questions soon became unstuck after a simple slip. There was little evidence that candidates were prepared to check their working, doing so would have improved many candidates' overall mark.

The weakness of many candidates' integration was striking, and somewhat alarming.

Comments on specific questions

1   This was a popular question, and most candidates were familiar with the underlying principle that, if there are n symbols of which a are of one type and b are of another type etc, then there are $n! \div (a! \times b! \times \ldots)$ distinct rearrangements of the n symbols. It was important to enumerate systematically the combinations totalling 39, to avoid counting possibilities more than once.

2   This was a popular question, but was one which required careful algebraic manipulation. It was pleasing that most candidates saw how to use the statement that "(1, 0) lies on the line *PQ*" to deduce that $pq = -1$. Proving that *PSQR* was a rectangle was rarely done in full: most candidates proved necessary conditions (e.g. there were two interior right angles) rather than sufficient conditions (e.g. there were three interior right angles in a quadrilateral, hence there were four). Considering the lengths of the sides without considering at least one interior angle did not remove the possibility that the quadrilateral was a (non-rectangular) parallelogram.

3   Most candidates who tackled this question knew what to do, but did not express clearly the necessary reasoning. In part (i) the solution "$x = \sqrt{ab}$ or $x = -\sqrt{ab}$" did not show that the given equation had "two distinct real solutions"; there was needed an explicit statement that since a and b were either both positive or both negative then $ab > 0$, hence $\sqrt{ab}$ was real. Similarly, in part (ii) most candidates did not explain why $c^2 \neq 0$.

In such questions, candidates are reminded of the need to explain clearly each component of the result they have been given.

4   Part (a) was usually well done, though quite a few candidates did not justify the negative value of sin θ. Arithmetical errors marred many evaluations of cos 3θ. The given identity was not found difficult to prove, and most candidates saw that in part (b) they were being asked to solve $2x^3 - 33x^2 - 6x + 11 = 0$. Unfortunately, very few were able to make further progress: substituting (correctly) $x = \frac{1}{2}$ was rarely seen. Most of those who made progress remembered to explain which of the three values of x was the value of tan θ.

5   Neither integral in this question was at all difficult, so it caused some concern to see poor implementation of routine techniques such as integration by substitution or by parts. In part (i) not every candidate linked the two cases together. In part (ii) $m = 0$ was often asserted to be a special case before the integration had been performed. Even after that, the terms $m +$

1 and $m + 2$ in the denominator of the answer did not always prompt candidates to consider $m = -1$ and also $m = -2$ as special cases.

6 Part (i) was well done. Those who attempted part (ii) derived the appropriate equations, but then found it hard to proceed: a common error was to assert that if $ax^2 + bx + cy^2 = d$ was to be the same as $x^2 + 14x + y^2 = 51$, then $a = 1$, $b = 14$, $c = 1$ and $d = 51$, rather than the correct deduction that $b \div a = 14$, $c \div a = 1$ and $d \div a = 51$.

7 This was not a popular question, though part (i) was usually well done. Part (ii) required the factorisation of $r^2 - 1$, and those who saw this usually simplified the product. Very few solutions to part (iii) were seen: the replacing of cot θ with cos $\theta \div$ sin θ and the simplification of the two terms into a single fraction was very rarely seen. Those who reached this stage did not all recognise that cos A sin B + cos B sin A can be simplified further.

8 Most candidates who attempted this question did so confidently and largely successfully. It was pleasing to see that they understood how to use the hint implicit in the first result they derived. However, a lot of solutions were flawed by the omission of the constant of integration.

9 Many candidates attempted this question, and made good progress. Commonly seen was the incorrect statement that ½ T sin $\theta = \mu R$ (derived by resolving horizontally on the rod). There was no statement in the question that the rod was about to slip, hence it was wrong to assert that the frictional force equalled μR. Full marks were not awarded unless the candidate was careful to state that $F \leq \mu R$.

10 This was the most popular Mechanics question, and was often well done. Clearly labelled diagrams would have helped both candidates and examiners. Candidates are encouraged to simplify answers as fully as possible: the results in parts (i) and (ii) were not always reduced to their simplest forms. It was not necessary to do so to achieve full marks, but at this level candidates should expect to give answers as neatly as possible.

11 The Mechanics tested by this question was not demanding, but candidates found that solving the resulting equations was taxing. Great care was needed in part (iii). Many solutions began "**v** = k**r** so 2 cos 2t = k $\times$ sin 2t, and –2 sin t = k $\times$ 2 cos t". The subsequent deduction that 2 cot 2t = k = – tan t should not have been made, without considering whether sin 2t or cos t equalled zero.

12 This question was very poorly answered. In part (a) almost every candidate assumed that hat-wearing and pipe-smoking were independent, and so multiplied together the given probabilities. A handful of attempts at part (b) were seen. It was intended that this question be tackled with Venn diagrams rather than tree diagrams: candidates seemed utterly unfamiliar with these.

13 Very few attempts at this question were seen, which was surprising since, with the aid of sketch graphs and a tree diagram, it was probably a lot more straightforward than question 12.

14 No successful attempts at this question were seen. Those who started it usually failed to realise that they had been given the cumulative distribution function rather than the probability density function.

UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE

STEP MATHEMATICS PAPER 2: 9470: JULY 2005

REPORT FOR CENTRES

General Remarks

Almost all levels of achievement in this examination were represented significantly. Thus as much as a third of all candidates failed to complete more than two questions, whereas about a fifth completed at least four questions. In particular, the best candidates showed considerable mathematical potential.

Again, it is good to be able to record that the practice of contravening the rubric by handing in responses to more than six questions is still in decline and this trend continues to benefit candidates generally. In this respect, it was good to see that some candidates made quality their first priority and so achieved an excellent overall script total from fewer than six questions.

The distribution of question choice was extremely non-uniform, as was the case last year, and the proportion of candidates who concentrate exclusively on Section A appears to be on the increase. In any case, it is clear that the best work usually comes from Section A and the worst from Section C.

The standard of presentation of work continues to decline to the extent that some material is unreadable. The idea that what is marked is necessarilly restricted to what can be read does not appear to be universally understood. Most of the candidature has probably had little experience of an examination of this type and so found it difficult to set out well structured solutions. In passing therefore, it seems reasonable to suggest that future candidates should acquire mathematical communication skills as a necessary preliminary.

An inevitable consequence of chaotic working was the proliferation of elementary errors in consequence of which some candidates wandered into intractable mathematical situations. Here checking

and perhaps a fresh start is needed, not a mindless drive forward into a dead end.

Three particular skills were substandard. They were, the approximation of a function $f(x)$ for small x by a polynomial, the correct working of inequalities (these featured in all sections) and the presentation of a clear, well annotated force diagram in a mechanical situation.

Nevertheless, despite the above criticisms, one can say that much impressive work was in evidence. Many candidates showed courage and determination in the face of an onslaught of difficult mathematics.

Comments on responses to individual questions

SECTION A: PURE MATHEMATICS

Q1 This was the most popular question of the paper and the majority of candidates made some progress with it. Many responses were undermined by elementary errors.

At the outset, the preliminary result $d[x^2 e^{-x^2}]/dx = 2xe^{-x^2} - 2x^3 e^{-x^2}$ appeared in almost all responses. Remarkably, however, not all candidates were able to identify all the roots of $x - x^3 = 0$.

A small minority of candidates thought that $P(x) \equiv x(x^2 - a^2)(x^2 - b^2)$ whereas the correct starting point is $P'(x) - 2xP(x) \equiv x(x^2 - a^2)(x^2 - b^2)$ $(*)$. Actually, some wrote $x(x^2 - a)(x^2 - b)$ for $x(x^2 - a^2)(x^2 - b^2)$.

Beyond this beginning, the best candidates could see immediately that $P(x)$ can take the form $-x^4/2 + px^2 + q$ and this approach certainly simplified later working. In contrast, some started with $P(x) \equiv \sum_{i=0}^{4} c_i x^i$, but this more complicated strategy generated many erroneous solutions.

There were also those who attempted to obtain $P(x)$ from $e^{x^2} \int x(x^2 - a^2)(x^2 - b^2)e^{-x^2} dx$. The systematic and correct use of the integration by parts rule will readily show how for $I_n = \int x^{2n+1} e^{-x^2} dx$, I_1 and I_2 relate to $I_0 = -(1/2)e^{-x^2}$ in a simple way. In this context, few responses were at all clear and generally notational confusions led to inaccuracy.

$x = -1, 0, 1 : P(x) =$ any non-zero scaling of $-x^4/2+(a^2/2+b^2/2-1)x^2+-1+a^2/2+b^2/2-a^2b^2/2$

Q2 This question was also popular and most responses made significant progress with the majority of the sections available.

(a) (i) The correct results for $f(12)$ and $f(180)$ appeared in almost all responses.

(ii) The working here was often hazy and protracted. Notational confusions led to poor reasoning. What was required was an argument based on

$$N = p_1^{\alpha_1}\ldots p_k^{\alpha_k} \Rightarrow \ldots \Rightarrow f(N) = p_1^{\alpha_1-1}\ldots p_k^{\alpha_k-1}(p_1-1)\ldots(p_k-1).0$$

(b) In all three parts of this section, it was expected that, at least, the conclusion would be made clear. However, there were many instances where this did not happen.

(i) Most responses showed a suitable counterexample, e.g., $f(3)f(9) = 2 \times 6 = 12 \neq f(27) = 18$, and thus proved that the displayed result lacks generality.

(ii) There were few failures here. The correct conclusion was usually supported by a simple argument such as $f(p)f(q) = p(1-1/p)q(1-1/q) = pq(1-1/p)(1-1/q) = f(pq)$.

(iii) Most responses groped there way through working to show, e.g., $f(5) = 4$, $f(6) = 2$, $f(30) = 2 \times 4 = 8$, from which the required conclusion can be made. However, some candidates were unable to do this. In this respect there were erroneous statements of the form $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$ and especially, $(P' \Rightarrow Q') \Rightarrow (P \Rightarrow Q)$.

(c) Responses generally started with $p^{m-1}(p-1) = 146410$, but some led on to incorrect results, or simply faded out. A popular incorrect conclusion was $p = 11, m = 4$.

(a) (i) $f(12) = 4$, $f(180) = 48$: (b) (i) not always true, (ii) true, (iii) false: (c) $p = 11$, $m = 5$.

Q3 This very popular question provided an opportunity for candidates to show their competence with basic A-level mathematics. Only the last part was at all unusual. Unexpectedly, most responses showed at least one error in (ii), but contained sound mathematics to establish the final inequality.

For the introductory inequality it is only necessary to state that $y(0) = 0$, $y(\pi/2) = 1$ and to show that $dy/dx = 0$ at the origin and that $dy/dx > 0$ for $0 < x \leq \pi/2$. The majority of responses proceeded in this way, though the layout of the working was unclear in some instances. The majority of sketch graphs were incorrect and/or incomplete. Common errors were a non-zero gradient at the origin and/or a zero gradient at the point $(\pi/2, 1)$.

(i) Most responses showed accurate and complete working to establish the displayed result.

(ii) The working in almost all responses soon led to the preliminary result that $J = \int_0^{\pi/2} y^2 \, dx = I_1 - I_2 + I_3$, where $I_1 = \int_0^{\pi/2} \sin^2 x \, dx$, $I_2 = \int_0^{\pi/2} x \sin 2x \, dx$, $I_3 = \int_0^{\pi/2} x^2 \cos^2 x \, dx$.

The results $I_1 = \pi/4$, $I_2 = \pi/4$ were usually in evidence, though sometimes these followed from erroneous working. In contrast, many candidates were unable to supply the extensive technical detail needed for the determination of I_3. Thus a complete and correct evaluation of J appeared in only a minority of responses.

Q4 Though less popular than its predecessors it nevertheless generated some good work. The very best solutions sailed through the first part, used the displayed result there to establish the second displayed result and then produced a carefully worked solution for the final part. However, such quality mathematics came only from a minority.

In contrast the majority did not understand how the second part relates to the first and so started all over again. Moreover, it was common for an unchecked solution to appear. Even if this was correct, it could not be awarded full credit for the simple reason that in this context there are several plausible possibilities.

Q5 This was not a popular question and some candidates made absolutely no progress.

Some responses at least got as far as establishing that $2r = b + c - a$ and so went on to get a result for R in terms of a, b, c. The transposition of the expression obtained into a quadratic function, Q, of q proved to be beyond most.

The calculus based responses to the final part were usually incomplete in that the nature of the stationary point of Q was not considered. Those who used an exclusively algebraic method fared better. Generally, this strategy was worked with impressive accuracy and the interpretation of the result obtained for Q in terms of the displayed inequality was usually satisfactory.

Q6 This too was an unpopular question and very few responses were complete and correct. It appeared that some candidates were unfamiliar with the concept of a general term and yet others did not work easily with the summation operator Σ.

(i) Those who were familiar with the basic terminology appertaining to infinite series got through the first sentence of the question though, in some cases, with a lot of unnecessary labour.

Most responses showed an attempt to uses some, or all, of $S_j = (1 - x)^{-j}$, $j = 1, 2, 3$ in order to evaluate $\sum_{n=1}^{\infty} n2^{-n}$ and $\sum_{n=1}^{\infty} n^2 2^{-n}$.

(ii) Almost all candidates who got this far produced sufficient working to show that the general term of $(1 - x)^{-1/2}$ can be put into the form displayed in the question.

They usually went on to put $x = 1/3$ in order to evaluate the first of the 2 series in the final part of the question and generally worked accurately. However, the evaluation of the second series proved to be much more difficult. The majority of responses at least hinted at a correct overall strategy, namely differentiation followed by setting $x = 1/3$, but lack of technical expertise undermined much of the working. Nevertheless, it must be said, this evident lack of mathematical competence was not a feature of the responses to *Q6* alone.

(i) x^n, $(n + 1)x^n$, $(1/2)(n + 1)(n + 2)x^n$: 2, 6.

(ii) $\sqrt{3/2}$, $(1/4)\sqrt{3/2}$.

Q7 This was the least well answered question of Section A and this was due mainly to a lack of understanding of the geometry supplemented by defective expertise in dealing with inequalities involving trigonometric functions.

(i) The description of the locus of P was usually correct as far as it went. Basically, what was required was the identification of it as a circle, the radius of this circle, the location of the centre and the plane in which it is contained. The same is required for the locus of Q, but here, some candidates thought that it is an ellipse and were usually unable to specify the plane in any geometrically intelligible way.

(ii) Most responses showed the correct formation of a relevant scalar product in terms of t. Thereafter there was some illegal working of the trigonometry based on the incorrect resolution of $\cos(.)\cos(..)$ into a sum of cosines and $\sin(.)\sin(..)$ into a difference of cosines. Moreover much energy was expended on finding OQ even when the candidate had already concluded in (i) that it is constant and equal to 3, and even more remarkably, the proving that $OQ = 3$ did not lead to any correction of the statement in (i) that Q describes an ellipse.

(iii) Generally, failure in (ii) did not inhibit sensible attempts at this concluding section of the question. Nevertheless, few candidates seemed to realise that 1 cycle in the $t-$ domain corresponds to 2 cycles in the $\theta-$ domain. Certainly candidates would have helped themselves considerably if they had worked from an accurate sketch graph of the cosine function. As it was, very few did this and for the most part fell back on hazy inequality arguments which, more often than not, were inconsistent with basic properties such as $\cos\theta$ is decreasing over the open interval $(0,\pi)$ and is increasing over the open interval $(\pi, 2\pi)$.

(i) P describes a circle centre O and radius 1 in the $x - y$ plane.

Q describes circle centre O and radius 3 in the plane $\sqrt{3}x - z = 0$.

$Q8$ This question led to application of essentially correct methodologies. However, at the technical level there many inaccuracies.

Most responses soon arrived at something like $A - 1/y = \int x^3(1 + x^2)^{-5/2}\, dx$.

At the integration stage, at least six methods were in evidence. The most popular of these was based

on the integration by parts rule. This obvious and simple strategy was generally applied accurately. Also in evidence were the use of substitutions such as $u = x^2$, $v = 1 + x^2$, $x = \tan t$, $x = \sinh w$ and a consideration of the derivative of $1/y = (2 + 3x^2)/[3(1 + x^2)^{3/2}]$.

This last method would have been completely acceptable had the solution not been displayed in the question. In this context, however, it must necessarily be regarded as verification and, as such, did not merit full credit.

Application of the given initial condition was usually accurate if only because the displayed result was at hand.

The large positive x approximation was established in a rigorous way by only a minority. Following the expansion of $(1 + x^2)^{-3/2}$, terms were prematurely discarded and then brought back into the working again in some illegal way so as to produce the displayed approximation.

The sketch of C was often deficient in some way in that the gradient at (0,1) was non-zero and the location of the horizontal asymptote was not identified.

A substantial minority of candidates failed to observe that the second differential equation can be obtained immediately from the first by the substitution $y = z^2$. Such a deduction makes the drawing of the second diagram, involving 2 branches, a simple matter.

SECTION B: MECHANICS

Q9 Few responses showed a clear and complete diagram for both parts of this question and no doubt this deficiency was the main cause of bad modelling. Candidates generally needed to think through the application of Newton's laws of motion in this context in order to ensure that all terms were present and this is what some failed to do.

(i) A particular direction of P, usually parallel to the slope, was assumed in about half of all responses. Such a presupposition trivialised the question and so very little credit could be given for this strategy.

In contrast, the best responses went very rapidly to the fundamental equation

$$P \cos\theta = mg + mg/2 + (1/2\sqrt{3})(mg\sqrt{3}/2) + (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta),$$

where $\theta + \pi/6$ is the angle which P makes with the horizontal. Here, calculus methods were often used in order to determine the optimal direction of P, in the sense of the question, and very often such arguments were incomplete and almost incoherent.

The superior method of first writing

$$P \sin(\theta + \pi/3) = (11\sqrt{3}/8)mg$$

followed by the use of $\sin(.) \le 1$ was not to be seen to any great extent and this would suggest that lack of facility in the working of trigonometric expressions was a partial cause of failure in this question.

(ii) Much the same comments carry over to the situation here where the critical equation is

$$P \cos\theta = mg + mg/2 - (1/2\sqrt{3})(mg\sqrt{3}/2) - (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta).$$

However, few responses at this stage were complete and correct.

(i) Direction for least magnitude of P is parallel to the slope.

(ii) Least magnitude of P is $(\sqrt{3}/8)mg$.

Q10 This turned out to be the most popular and the best answered question of Section B.

The key to the obtaining of a completely correct response is the setting up of displacement-time equations which take into account both the different time origins and the different displacement origins of the two missiles. Without a correct form of these equations, or some equivalent, it is impossible to make significant progress with this question. With these, accurate working will speedily take the candidate to a complete and correct solution. As it was, most responses made

almost no progress, though on the other hand, a few were complete in every respect and well presented.

The four basic equations which describe the horizontal and vertical components of the motion of the particles are essentially:

$$x_1 = 80t, \; y_1 = 60t - 5t^2, \; x_2 = 180 - 120(t - T), \; y_2 = 160(t - T) - 5(t - T)^2 \; (t \geq T) \; (*)$$

In this context, some candidates used $t + T$ and t in place of t and $T - t$, respectively, but then left themselves with problems with regard to the interpretation of results at the end of their analysis.

Almost all candidates who got as far as (*), or some equivalent, worked from $x_1 = x_2$ and $y_1 = y_2$ to establish (eventually) a quadratic equation in T alone. It was in this process that many inaccuracies, such as one would not expect at this level, occurred. Beyond that, a small minority went on to obtain the roots $T = 1, 90$, but hardly anyone could produce an effective argument as to why, in fact, $T = 1$.

Q11 This question generated mainly incomplete responses. As with *Q9*, there was a dearth of useful and properly annotated diagrams. Streamlined versions of Newton's second law of motion appeared in some solutions.

For the obtaining of the first result there usually appeared a correct equation of motion for each particle. For the particle on the slope this included an accurate specification of the frictional force opposing the motion. After some algebra the first result appeared. Candidates would have found it helpful, both here and later in the question, to have denoted the pervasive constant $(m_2 - m_1)/(m_2 + m_1)$ by a single letter, say λ.

The particle on the slope begins the second phase of the motion with speed $u = \lambda gT$ and so the total time to the highest point is $(1 + \lambda)T$.

For the final part of the question, the information given leads to the key equation

$$(g/10)(1 + \lambda)^2 T^2 = (\lambda g/2)T^2 + (\lambda^2 g/2)T^2. \ (*)$$

Provided g is set to 10 (ms^{-2}), as directed by the question, it is easy to solve $(*)$ for λ and hence for m_1/m_2.

$m_1/m_2 = 3/5$

SECTION C: PROBABILITY AND STATISTICS

Q12 This was the most popular question of Section C. Attempts generally lacked coherence and this was true whether responses were exclusively symbolic and/or a tree diagram was employed. Very few candidates produced complete and correct working for all sections.

In the first part, most responses produced a valid argument to establish the meaning of the basic quantity $G = ap + bq$ and from there could establish the first result.

Both (ii) and (iii) can be worked easily in terms of G, or by use of a tree diagram, but progress was usually negated by lack of a clear meaningful notation and also by the suposition that it was the same twin who responded to both e-mails.

Q13 Few candidates made significant progress with this question. Responses generally began with an explanation as to why $p = e^{-\lambda}(1 + \lambda)$, but generally did not include any serious attempt to show that Y is a binomial variate.

(i) It was expected that candidates would get as far as

$$q \approx 1 - (1 + \lambda)[1 - \lambda + \lambda^2/2)]$$

when λ is small. However, as in *Q8*, candidates seemed to be unfamiliar with the concept of approximating a function by use of its power series expansion and so there were few satisfactory solutions here.

(ii) Similarly, few candidates got beyond

$$P(Y = n) = p^n > 1 - \lambda \Rightarrow e^{-n\lambda}(1 + \lambda)^n > 1 - \lambda$$

to write something like

$$-n\lambda + n\lambda - n\lambda^2/2 > -\lambda - \lambda^2/2 + O(\lambda^3)$$

and so go on to establish the required result.

(iii) Some responses began (correctly) with

$$P(Y > 1 | Y > 0) = P(Y > 1)/P(Y > 0) = (1 - q^n - npq^{n-1})/(1 - q^n)$$

but again, most candidates lacked the technical expertise needed for further progress.

Q14 This was the least popular question of the paper. Less than half of all responses got as far as establishing the displayed result for σ^2.

Some approximately sensible sketch graphs appeared for (i), but the number of those candidates who made further progress was almost zero.

9475 - MATHEMATICS III

Report for Publication to Centres

General Comments

Almost all candidates this year chose to answer questions 2, 3, 5, 6 and 7, together with question 1 or 4. Many, usually the weaker candidates, tackled more than six questions, though most of these were usually incomplete, or even barely started. Some good candidates concentrated on very full and thorough answers to fewer than six questions. Few attempts were made at the Mechanics questions and even fewer at the Probability and Statistics questions.

Some very impressive work was again seen from the best candidates, and this year there were far fewer candidates who were essentially unable to answer any of the questions. Those doing less well are typically candidates who can do effectively what they are explicitly told to but are unable to make progress where the method is not provided by the question or those who can see what do, but who are hampered by poor technical skills, especially in algebra, but also in trigonometry or calculus.

Lack of clarity about the direction of implications is often a weakness, even of good candidates: for instance (in question 1) $\cos B = \sin\left(\dfrac{\pi}{2} - B\right)$ so $\sin A = \cos B \;\Rightarrow\; A = \dfrac{\pi}{2} - B$, but cos is an even function so $A = \dfrac{\pi}{2} \pm B$. Many are not clear what is required to "show" a result: either sufficient working to indicate the steps taken or a written explanation of what is going on are essential. For instance, it was common to see (in question 2)

$$\frac{d}{dx}\left(x^2 + y^2\right) = 2x\left(1 - \frac{c^2}{\left(x^2 + a^2\right)^2}\right) \;\; \text{so} \;\; \frac{d^2}{dx^2}\left(x^2 + y^2\right) = 2\left(1 - \frac{c^2}{\left(x^2 + a^2\right)^2}\right) + \frac{8c^2x^2}{\left(x^2 + a^2\right)^3}$$

where candidates were asked to show the latter result, which is on the question paper: of course, it is easy to reconstruct the steps in the differentiation, but that is what the question is asking for.

Few candidates seem prepared to check their work, or to go back to look for obvious errors. It was common to see the plaintive remark "I must have made a slip somewhere" at the end of a derivation that failed to give the expected answer, where the error was a simple as the mistranscribing from one line to the next of a negative sign as a positive sign.

Comments on Specific Questions

1   This question was surprisingly unpopular, given that it was the first question on the paper – perhaps the sight of unfamiliar trigonometric graphs put candidates off. The first two parts were very poorly done: for the first result almost all only showed that $A = \left(4n + 1\right)\dfrac{\pi}{2} \pm B$ was sufficient for sin A = cos B, while in the second many looked for turning points but assumed without comment that the maximum of the modulus of a function must occur at the maximum of the function. Almost all could then use these results to show that $\sin\left(\sin x\right) = \cos\left(\cos x\right)$ had no solutions. The first two graphs were usually correctly sketched, though many had cusps on one or other curve, but the graph of $y = \sin(2\sin x)$ was very often attempted

without further calculation and almost always then had a maximum at $x = \dfrac{\pi}{2}$ instead of a minimum there and maxima either side of this. Candidates should be aware that graph sketches in STEP papers will often require considerable working, such as determining turning points and their nature, even if this is not explicitly indicated in the question.

2 This question was attempted by almost all candidates and most managed the early parts successfully, though many used the expression $2 + 2(y')^2 + 2yy''$ for the second derivative of $x^2 + y^2$, which made this part much harder than necessary. Very few than achieved full marks for determining the closest points to the origin on the curve, noticing correctly the existence or otherwise of two turning points of $x^2 + y^2$ other than at $x = 0$, and showing clearly which points were minima, under the two conditions on a and c.

3 Almost all candidates attempted this question, and could complete the algebra correctly; most could also solve the quartic equation in the last part using one of the ideas from the earlier parts. Relatively few, however, understood what was meant by a necessary condition on a, b and c and simply gave formulae for these in terms of p, r and s, while virtually no-one, even among those who found that $a^3 + 8c = 4ab$ was necessary, could establish the sufficiency of this condition.

4 This question was one of the less popular Pure questions. There seemed to be a fairly sharp divide between the many candidates who dealt very effectively with the induction and the many who were unsure what the induction hypothesis was or what was required for the base case or who went round in circles with the recurrences. In the last part, most were able to find the required conditions successfully.

5 Almost all attempted this question and found the first two parts straightforward. Most (apart from the significant number who did not understand the phrase 'common tangent') could also identify the discriminant condition for there to be only one tangent. Disappointingly few, however, could link this accurately to the condition for the two curves to touch. The last part was almost always poorly done, with candidates either continuing to use the discriminant, which is not appropriate in this case, since the equation is linear, or being unable to state clearly what the condition is for a linear equation to have exactly one solution.

6 This question was tackled by most candidates. Almost all could do the first part; most could show that $u + \dfrac{b}{u}$ is a root of the equation, by a variety of methods, and relate their results to the final equation. Fewer could convincingly establish the quadratic satisfied by the other two roots of the cubic, and startlingly few could accurately solve this quadratic to get roots in terms of ω with, in particular, very many sign errors.

7 This question was popular and almost all could obtain the general result quoted, with most being able to go on to use this result on the given integrals. Many could then complete the integration for (i), but very few knew how to tackle the integral $\displaystyle\int \dfrac{du}{u\sqrt{u+1}}$ in (ii). Candidates (perhaps because this was the most recent technique they had learnt) almost all saw this as an opportunity to substitute $u = \sinh^2 t$ or $u = \tan^2 t$, which will work in principle, but are not as

simple to execute correctly as $u = t^2 - 1$. Many perfectly satisfactory methods of integration not based on the general result given were also seen.

8 This was easily the least popular Pure question, though still attempted more frequently than any of the applied questions. Solutions getting beyond the first couple of parts successfully were extremely rare, with many not recognising $|a - c|$ as the radius of the circle, and hardly any being able to use the result $2aa^* = ac^* + ca^*$ to show that the conditions for B and B' to lie on the circle were equivalent, let alone the converse.

9 This was the most popular Mechanics question and most of those who attempted it seemed to know what to do, though those who had a formulaic approach to Newton's Law of restitution, writing it as $-e = \dfrac{\text{difference of final velocities}}{\text{difference of initial velocities}}$, frequently made at least one sign error in using this equation, and there were many who did not make a consistent decision about the sign convention for the final velocities, for instance by drawing a diagram. However, virtually all were defeated by the algebra required – it was very common indeed, for instance, to see the expression $(1-e)^2$ miscopied as $(1-e^2)$ at some point in the calculation, or vice versa. Unfortunately, the problem was unforgiving about this, and incorrect early results made it very difficult to complete the question successfully.

10 There were a reasonable number of attempts at this question, with some good efficient solutions, but with most candidates giving up when they did not get the required result in (i). This was usually because they had included the work done by friction on only one disc, or because they had not realised that the extension in the band is twice the distance apart of the centres, or because they assumed that there would be no elastic potential energy stored in the band if the two discs were in contact, or some combination of these.

11 This question was rarely attempted. There were a few good solutions, but most could only tackle the first part – distinguishing the two equilibria by their stability (for example by finding the value of $\sin\theta$ and $\cos\theta$ at each) was found difficult, and consequently the last part was inaccessible.

12 There were very few attempts at this question and few of these progressed beyond the first part, the result $E\left[Y^2\right] = \text{Var}\left[Y\right] - E\left[Y\right]^2$ not being recognised as useful.

13 This was the only Probability and Statistics question attempted by more than a handful of candidates, but was still only tackled by a small minority. In the first part, most could derive a correct expression for the probability that the player wins exactly £r, but hardly any could either sum the series for the expectation or, alternatively, spot the connection with a Geometric random variable. Many did not then proceed to the second part, but those who did often found it more straightforward.

14 There were hardly any attempts at this question: most of these could successfully get to the expected value of V, but were unable to use the asterisked result to find the density function.

STEP Mathematics (9465/9470/9475)
June 2005 Assessment Session

Unit Threshold Marks

Unit	Maximum Mark	S	1	2	3	U
9465	120	96	80	62	45	0
9470	120	89	64	49	31	0
9475	120	80	59	47	35	0

The cumulative percentage of candidates achieving each grade was as follows:

Unit	S	1	2	3	U
9465	11.3	27.5	47.9	70.3	100.0
9470	15.0	45.0	65.5	86.9	100.0
9475	14.7	40.5	61.9	81.6	100.0

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