STEP Mathematics I 2007: Report

General comments

There were significantly more candidates attempting this paper this year (an increase of nearly 50%), but many found it to be very difficult and only achieved low scores. In particular, the level of algebraic skill required by the questions was often lacking. The examiners' express their concern that this was the case despite a conscious effort to make the paper more accessible than last year's. At this level, the fluent, confident and correct handling of mathematical symbols (and numbers) is necessary and is expected; many good starts to questions soon became unstuck after a simple slip. Graph sketching was usually poor: if future candidates wanted to improve one particular skill, they would be well advised to develop this.

There were of course some excellent scripts, full of logical clarity and perceptive insight. It was pleasing to note that the applied questions were more popular this year, and many candidates scored well on at least one of these. It was however surprising how rarely answers to questions such as 5, 9, 10, 11 and 12 began with a diagram.

However, the examiners were left with the overall feeling that some candidates had not prepared themselves well for the examination. The use of past papers to ensure adequate preparation is strongly recommended. A student's first exposure to STEP questions can be a daunting, demanding experience; it is a shame if that takes place during a public examination on which so much rides.

Further, and fuller, discussion of the solutions to these questions can be found in the *Hints and Answers* document.

Comments on specific questions

1 This question required little more than a clear head and some persistence: candidates had either ample or very little of both, and thus most scores were either high or very low. The examiners would like to stress that a solution to a question such as this must be written out methodically and coherently: many answers which began promisingly were soon hopelessly fragmented and incoherent, leaving the candidate unable to regain his or her train of thought. This was especially true when deriving the final expression given on the exam paper. Examiners follow closely a candidate's line of reasoning, and they have to be certain that the candidate has constructed a complete argument, and that he or she has not arrived at a printed result without full justification.

2 This was a popular question, and was usually well done. The argument at the end was often incomplete, though: many candidates simply stated that $t = 1$ or $t = 2$ without explaining why no other values were possible. To do so, use had to be made of the fact that s and t have no common factor other than 1.

3 This was the most popular question on the paper, and many different methods were seen. The intended method was to use the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \tfrac{1}{2}\sin^2 2\theta$ to evaluate the integrals of $\cos^4\theta - \sin^4\theta$ and $\cos^4\theta + \sin^4\theta$, and hence be able to write down

separately the values of the integrals of $\cos^4 \theta$ and $\sin^4 \theta$. A similar approach works well for $\cos^6 \theta - \sin^6 \theta$ and $\cos^6 \theta + \sin^6 \theta$. Other methods were, of course, acceptable, and many candidates received high marks for this question.

4 This question was found to be very difficult. The initial factorisation was beyond most candidates, even given the linear factor $x + b + c$. Anyone who wants to read Mathematics at university must be able to factorise quickly cubic expressions such as this one, and also $x^3 \pm y^3$. The *Hints and Answers* document discusses this in more detail.

Candidates who progressed to the second part of the question often deduced that $ak^2 + bk + c = 0$ and $bk^2 + ck + a = 0$, but then tried to eliminate k; given that the result they were asked to derive was still in terms of k, this was an unwise strategy.

5 Only a few candidates made much progress with this question, even though it only required GCSE Mathematics. Basic properties of triangles (for example, the sine and cosine rule, and the location of the centroid, the circumcentre and the incentre) are assumed knowledge at this level. It was surprising how many candidates tried to answer this question without a diagram.

6 This was a popular, straightforward question, which was often answered well. However, algebraic errors still occurred, for example when expanding $(x - y)^3$.

7 Part (i) was well done by most of those who attempted this question, but many then found it difficult to develop the strategy in part (ii). A certain amount of trial and error is needed to complete the squares in an expression in terms of both α and β, but the coefficients (in particular, $\mathbf{1}\alpha^2$, $\mathbf{1}\beta^2$ and $\mathbf{26}\beta^2k^2$) do not permit many possibilities. This question demanded some stamina, as Mathematics at university level also does.

8 This question was answered poorly; many candidates were unable to sketch the graphs correctly, even given the results derived earlier in the question. For example, many graphs did not touch at $(2, 8)$. Also, many graphs were drawn with turning points, when a simple check of the derivative would have revealed that there were none. In part (iii), the effect of the negative coefficient of x^3 was often ignored.

Graph sketching is a very important skill in all mathematical subjects – from Economics to Engineering. STEP candidates are strongly advised to practise this skill as much as possible.

9 This was a popular question, and was usually well done. Not many candidates recognised that $\sin \theta \cos \theta \equiv \frac{1}{2} \sin 2\theta$, which makes the final inequality easier to obtain. Knowing identities "both ways" is important.

10 Only a few attempts at this question were seen, and those that did rarely made much headway; worryingly, the accurate simplification of the solutions of simple linear equations was found to be very difficult.

11 Hardly any attempts at this question were seen. It was remarkable how few diagrams were seen; it is always much easier for both the candidate and the examiner if answers begin with a labelled diagram.

12 Very few tree diagrams were seen here, and hence very few correct solutions were constructed; a clear tree diagram is invaluable when attempting a complicated probability question such as part (ii). Most candidates identified some (if not all) of the possible outcomes, but many mistakes were made (for example, writing a denominator of N rather than $N + 1$ or $N - 1$).

The subsequent algebraic simplification was found to be very demanding. Candidates would have probably made more progress if they had been more willing to factorise groups of terms which had obvious common factors, rather than (for example) attempting to write all the fractions with a common denominator.

13 A lot of attempts at this question were seen, but conceptual errors undermined many solutions. In particular, a lot of candidates seemed not to realise that they were being asked to calculate conditional probabilities in parts (ii) to (vi).

14 Only a few attempts at this question were seen. Poor graph sketching limited many candidates' progress; the importance of the ability to sketch accurately standard graphs such as $y = xe^{-x}$ cannot be overstated.

General Remarks

Although the paper was by no means an easy one, it was generally found a more accessible paper than last year's, with most questions clearly offering candidates an attackable starting-point. The candidature represented the usual range of mathematical talents, with a pleasingly high number of truly outstanding students; many more who were able to demonstrate a thorough grasp of the material in at least three questions; and the few whose three-hour long experience was unlikely to have been a particularly pleasant one. However, even for these candidates, many were able to make some progress on at least two of the questions chosen.

Really able candidates generally produced solid attempts at five or six questions, and quite a few produced outstanding efforts at up to eight questions. In general, it would be best if centres persuaded candidates not to spend valuable time needlessly in this way – it is a practice that is not to be encouraged, as it uses valuable examination time to little or no avail. Weaker brethren were often to be found scratching around at bits and pieces of several questions, with little of substance being produced on more than a couple. It is an important examination skill – now more so than ever, with most candidates now not having to employ such a skill on the modular papers which constitute the bulk of their examination experience – for candidates to spend a few minutes at some stage of the examination deciding upon their optimal selection of questions to attempt.

As a rule, question 1 is intended to be accessible to all takers, with question 2 usually similarly constructed. In the event, at least one – and usually both – of these two questions were among candidates' chosen questions. These, along with questions 3 and 6, were by far the most popularly chosen questions to attempt. The majority of candidates only attempted questions in Section A (Pure Maths), and there were relatively few attempts at the Applied Maths questions in Sections B & C, with Mechanics proving the more popular of the two options.

It struck me that, generally, the working produced on the scripts this year was rather better set-out, with a greater logical coherence to it, and this certainly helps the markers identify what each candidate thinks they are doing. Sadly, this general remark doesn't apply to the working produced on the Mechanics questions, such as they were. As last year, the presentation was usually appalling, with poorly labelled diagrams, often with forces missing from them altogether, and little or no attempt to state the principles that the candidates were attempting to apply.

Comments on responses to individual questions

SECTION A: PURE MATHEMATICS

Q1 Most candidates attempted this question and the majority coped fairly well with the algebraic demands. Surprisingly, it was when the work went numerical that candidates tended to let themselves down; poor arithmetic providing the main difficulty. The final three marks available in (i) parts (a) and (b) were the marks most frequently scorned, generally being lost by candidates' unwillingness or inability to simplify fractions and/or turn them into decimals. In many cases, candidates had difficulty deciding on a suitable

value for *k* in (i) (b) and (ii). In (b), the value k = 50 was often selected, rather than the intended value of − 4. Although this does lead to a similar set of working, the ultimate approximation is relatively poor, and they lost the final mark here. It is, rather more tellingly, indicative of the way in which many modern A-level mathematicians have great difficulty in thinking only in terms of positive integers! A small, but significant, number of candidates offered a value of *k* that exceeded 100 (the denominator of the "*x*" term), and these were penalised all four of the available marks for this part of the question, on the not unreasonable grounds that they really should have appreciated the general convergence condition $\mid \frac{k}{100} \mid < 1$ for binomial series of this kind.

Q2 This question was also a very popular one, although many candidates gave up their attempt when the algebra started to get a little too tough for them, which generally happened later if not sooner. With this in mind, it has to be said that when candidates *did* get stuck at some stage of this question, the principal cause was (again!) an unwillingness or inability to simplify algebraic expressions before attempting to work with them. This was particularly important when factorising otherwise lengthy expressions with lots of ($q − p$)s involved in them.

The sketch required in (ii) was intended to be a gentle prod in the right direction for later use in the question, and should have been four easy marks for the taking. Strangely, however, it was often not very well attempted at all. A surprising number of candidates couldn't even manage to draw their cubic through *O*; and many others seemed unable to make good use of the given conditions, which – despite looking complicated – actually just ensured that all the fun was going on in the first quadrant in an attempt to make life easy. Even more surprising still was the number of sketches that had non-cubic-like kinks, bumps and extra inflection points in them. I was particularly baffled by this widespread lack of grasp as to what a cubic should actually look like! I was equally baffled by the extraordinarily large body of candidates who failed to do what the question explicitly told them they were required to do, by not marking the point of inflection on their sketch and, in many cases, not even attempting to describe the symmetry of it either.

An apology has to be made at this point, since the region *R* in the question was insufficiently clearly defined and there were, in fact, two possibilities. Candidates were not penalised for choosing the "wrong" one at any stage of the proceedings, although the choice of the "left-hand" *R* would have prevented such candidates from using the short-cut for the following attempt at the area. ALL scripts where candidates made the "wrong" choice were passed to the Principal Examiner and given careful individual consideration. Only about 25 candidates made such a choice: of these, over half had failed to make any attempt at all at the area, and most of the rest had started work on the area and, to all intents and purposes, given up immediately. Two more had found the intended area anyway, despite their previous working (and were not penalised for having switched regions), and (I think) only three had pursued the "left-hand" area almost to a conclusion. Of course, they were unable to get the given answer, but they did get 7 of the 8 marks available. In each of these cases, it was fortunate (for us and them) that this was their last question, so it was safe to say that they hadn't been unduly penalised for time in any way. It is, of course, impossible to say whether they might have seen the intended short-cut approach. In this respect, however, it has to be said that remarkably few candidates saw the symmetry approach anyhow. Partly, I

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suspect, due to not having picked up the hint at the diagram stage (see earlier)! On the plus side, for us, I imagine that the reference to the point of inflection on the diagram had at least ensured that most candidates chose the intended region **R**. Only 2 of the 25 or so candidates scored an overall mark that fell just below a grade boundary, and both of these were given the benefit of the doubt by the Chief Examiner.

Q3 This was another popular question, and was usually a good source of marks for those candidates who attempted it. The first two parts were usually successfully completed. In part (i) (b), candidates had to employ the t = tan ½ *x* substitution which seems to have fallen into disuse in recent years (due to modularity!). Having said that, most candidates were able to make some progress and, where they did fall down, it was generally due to a lack of confidence in handling trigonometric identities. One of the advantages of these last two parts to the question was that they could be done in one of two directions, and many candidates were able to spot the connections and exploit them satisfactorily. When errors arose, they were frequently due to a lack of care with constants, and a correct final answer was not often to be found as a result.

Q4 This question was a popular one for partial attempts; with most candidates giving up towards the end of the introductory part and going elsewhere. It was slightly surprising to see candidates being put off in this way, since the given result made it perfectly possible to move successfully into the three following cases. For those who did press on, many lost a mark for not verifying (somewhere) that the chosen values of α, β and γ actually satisfied the required condition. Then, in (ii), one of the two brackets was identically zero, the significance of which was largely overlooked, with many candidates offering again the same two solutions as had been found in part (i). In (iii), it was important to note first *A* and *B* in terms of *x*, although some candidates adopted a valid alternative approach by first collecting up the two 3*x* terms.

Q5 Although this was not a popular choice of question, those who attempted it generally did rather well on it. Finding f^2 and f^3 was a routine algebraic slog, and most attempts coped successfully with it. Spotting, and then exploiting, the periodicity of the function was then a relatively easy matter. Pretty much everyone used x = tan θ appropriately in (ii), with formal and informal induction approaches evenly mixed. Some shrewder candidates identified the two forms for the cases *n* = 1, 2, and 3 and then noted that the periodicity of the tan function accounted for everything thereafter.

The final part of the question had intended to be a simple take on part (ii), but with t = sin θ this time, so that $\sqrt{1-t^2}$ = cos θ, and attempts at this part of the question generally fell evenly into one of the two following camps: those who gave up, and those who proceeded as intended. In all, I think there were just three candidates who noticed the extra complication that can arise in this case, with just two or three more following a separate line of enquiry without realising the inherent dichotomy in the "powers" of the function g. A full inspection of the function exposes the fact that g^n takes different forms depending upon which part of the domain of g is employed. This is because the $\sqrt{1-t^2}$ bit should actually be | cos θ |, and this leads to different answers for g^2 in the range $\frac{1}{2} \le t \le 1$ than in the rest of g's domain, so that candidates could get different answers from slightly different approaches.

With so few candidates expected to attempt this last part of the question, and with the alternate route leading to a much easier answer (where the sequence g^n turns out to be periodic with period 2), it was considered to be a suitable final part to the question. Candidates were not expected to take more than one route, nor to comment on the potential for different answers. In the event, none did the former, although a few gave a mention of the latter property.

Q6   This was one of the most popular questions on the paper, although the number of completely successful attempts could be counted without having to resort to toes! Part (i) was reasonably routine, although attempts at simplification were often not very well done, and left many candidates having to resort to "otherwise" approaches for integrating $\sqrt{3+x^2}$, which was a great shame as they got no marks for ignoring the "hence" instruction in the question. Treating the differential equation in part (ii) as a quadratic in dy/dx proved an obstacle for many, but a lot of candidates seemed quite happy to work with it as such and made good progress in the rest of the question. The biggest hurdle to completely successful progress, however, once again lay in candidates' inability to simplify expressions at various stages, and sign and/or constant errors proliferated.

Q7   Not very many candidates attempted this question, but those who did usually found it to be relatively straightforward. It was only the very last part that required much thought, and this was where most attempts lost a few marks. A small number of efforts failed to get beyond part (ii); this was due to not finding a suitable function to work with that gave what turns out to be the *Arithmetic Mean-Geometric Mean Inequality*. This was a bit of a shame, since the question actually gives the log. function at the very beginning, along with the sine function, which is used in (i).

Q8   This is really just half of the ($\Leftrightarrow$) proof of Ceva's Theorem. Several candidates even recognised it as such. Of the remarkably small number of attempts submitted, most fell down at some stage (again) by failing to be sufficiently careful with signs/arithmetic/the modest amounts of algebra involved. It often didn't help those candidates who chose completely different symbols each time they did a stage of the working.

SECTION B: MECHANICS

Q9   This was the least popular of the Mechanics questions, perhaps because it commenced with a request for an explanation. As mentioned already, a clearly labelled diagram or two would have been enormously helpful here! The fact that there are only the two mechanical principles being employed here should have made it an easy question, but efforts were generally very poor.

Q10   This was the most popular of the three Mechanics questions, although most efforts failed to get very far into it. The routine opening part, finding the position of a centre of mass, probably accounts for its initial (relative) popularity, but progress beyond this point was pitifully weak in most cases. Resolving and taking moments frequently appeared, but often had to be searched-for in amidst a sea of other statements, many of which were incorrect, repetitive or just nonsensical. Very few candidates indeed grasped the fact that the horizontal force *P* could be in either direction, and the given

answer was mostly fiddled, usually by simply placing modulus signs around the answer.

Q11 This was almost as popular a question on Section B as Q10, being (in principle, at least) a reasonably straightforward projectiles question. Whilst many efforts were successful up to the final part, an awful lot of the attempts foundered at the very outset by failing to do the simplest of tasks: namely, noting exact values for $\sin \theta$ and $\cos \theta$ from $\tan \theta = \frac{1}{2}$. It simply beggars belief that serious candidates can proceed through quite a large part of a question like this with expressions such as sin(arc tan ½) still in there! They may as well just hang out a flag which says "I'm an incompetent mathematician" on it! The three-dimensional aspect of the introduction was enough to confound most candidates attempting this question, and they were forced to resort to fiddling the given answer for the distance *OP*. Many attempts picked up several marks here and there throughout the question without producing anything particularly coherent, and few coped with the hazards of the last part – largely, I suspect, due to the fact that they were required to do some approximating!

SECTION C: STATISTICS

Q12 This was the least popular of the Statistics questions, with very few attempts seen on it. Most of these tended to consist of muddled or unexplained reasoning which led to the fiddling of (i)'s given result. Progress into (ii) was either non-existent or sketchy as a consequence.

Q13 This is a lovely approach to a well-known problem, and employs a very handy rational approximation to ln 2. Although it drew a small number of attempts, many of these were partial attempts at best, and few were seen of a good standard throughout. Disappointingly, several candidates arriving at the correct quadratic equation in the third part didn't seem to know how to go about solving it. As was mentioned earlier, regarding the end of Q11, working with approximations proved to be a particular obstacle for most candidates who made it to the last part here.

Q14 This was the most popular of the Statistics questions, probably due to the high pure mathematical content. The sketch introduction was intended to ensure that candidates drew something which would remind them what integrals they should be working with later on. As with Q2, it presented more problems than should have been the case, with many candidates losing marks for fairly trivial things which would have cost them dearly even on an ordinary AS/A-level module paper. The integration for total probability was generally done very well, although several candidates had often failed to gain *a* and *b* in terms of *k* in a simplified form, or at all, and this rather hindered them. In (iii), most candidates didn't seem to feel that it was necessary to justify which region of the function that the median lay in, often doing one calculation after making an assumption about the matter. In general, it is always best if candidates can justify their choices.

Section A: Pure Mathematics

1.   This question was popular. Many candidates did not simplify their first expression into the symmetrical form which made it harder for them to spot the use of the sums and products of roots results. A common slip was to make a 1 by default which also obscured what was going on. Most struggled to take the given equation requiring solution and produce a quartic equation in t ($\tan \vartheta$), some producing a quartic equation in $\cos \vartheta$, and somehow expecting to use the earlier results.

2.   This question was popular though not well answered. Solutions to part (i) were frequently unconvincing, though to part (ii) were quite good if they avoided elementary errors in working. Part (iii) was less well attempted with some not spotting to use integration, some stumbling over "+ *c*" and some not spotting the value of *x* to substitute.

3.   This question was popular. Many solutions to part (ii) were rambling and lacked a sense of direction, even if correct. The induction in (iii) was frequently incorrectly handled and a common error was to replace *n* by *k*/2. Part (iv) caused difficulties.

4.   This question was quite popular. A lot of attempts involved rambling trigonometrical manipulations, and few spotted the standard differential of $\ln \tan \dfrac{t}{2}$.

The curve sketch was often omitted or incorrect, and there was a lot of complicated working using e.g. the equation of the normal etc. to find the centre of curvature.

5.   This was frequently attempted, though lack of facility with hyperbolic functions meant that few progressed beyond the first two differentials, and for those going further, the working was not methodical enough to spot the factorial that would emerge in the general result.

6.   This was the least popular Pure question and very little success was achieved by the few that attempted it. The first result was often obtained correctly by expressing each of the four complex numbers in modulus-exponential form, but then the perpendicularity was the stumbling block.

7.   This was a very popular question. As the question led the candidates through there were a number of unconvincing solutions to parts of the question, but overall it was reasonably well handled.

8.   This ranked alongside question 5 in popularity and success. Frequently, it was calculation errors that obscured the path through part (i) and the two differences between part (i) and part (ii) were enough to put most off the track for part (ii), even if they had completed (i) successfully.

Section B: Mechanics

9.   This was little attempted. Some did struggle through to the solution of the differential equation, but the appreciation of the three possible cases eluded them.

10.   This was the most popular of the Mechanics questions, but less so than any but question 6 of the Pure. Most managed to obtain the first two results correctly, but then struggled to find the further result. The deduction for the largest R was rarely spotted leading to some unnecessarily unwieldy calculus.

11.   There were very few attempts at this question.

Section C: Probability and Statistics

12.   There were some attempts at this question but they faltered when trying to find the expectation of Y, even though some may have believed that they had obtained the required result through false logic.

13.   This was the most popular of the Probability and Statistics questions, ranking alongside questions 5 and 8. The first two parts were competently handled, but most got bogged down in the algebra of part (iii) through not having a clear strategy to solve the equations.

14.   There were few answers of any substance to this question.