Cambridge Assessment Admissions Testing

Sixth Term Examination Paper [STEP]

Mathematics 3 [9475]

2018

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Sixth Term Examination Paper

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2018

Examiner's Report

General Comments

The total entry was a record number, an increase of over 6% on 2017. Only question 1 was attempted by more than 90%, although question 2 was attempted by very nearly 90%. Every question was attempted by a significant number of candidates with even the two least popular questions being attempted by 9%. More than 92% restricted themselves to attempting no more than 7 questions with very few indeed attempting more than 8. As has been normal in the past, apart from a handful of very strong candidates, those attempting six questions scored better than those attempting more than six.

Question 1

As usual, this was the most popular question to be attempted with more than 93% of candidates doing so. However, scoring for it was only moderately good with a mean below 9/20. Most successfully differentiated and obtained a value for β from the cubic but ignored considering whether this was the only stationary point. Sketches frequently did not display the asymptote; some that did showed the negative branch of the curve touching rather than intersecting the asymptote at the maximum. Many did not appreciate that to sketch the second curve in part (i) it was not sufficient to just offer a drawing without the working; the horizontal point of inflection and asymptote were frequent casualties. Part (ii) was straightforward for most. Many recognised that part (iii) made use of the first function $f(\beta)$, provided that they used the condition to substitute for α . However, their justification suffered from ignoring the reality condition and using specious arguments, as a consequence. Part (iv) followed a similar trend to part (iii), except using (β) , and only differing in that of those that did apply the reality condition, quite a few overlooked $\beta = 1$ as a solution of the cubic inequality, and so their final answer was wrong.

Question 2

With about 89% attempting this, it was the second most popular question, and with a mean score of nearly 11/20, the second most successful. Part (i) was generally well-handled, and although most scored some marks on the proof by induction in part (ii), candidates often struggled to complete it, many of them because they attempted to use the original definition of the functions, which rarely led to success, rather than employ the result of part (i). Some candidates noticed that the proof by induction in part (ii) was equivalent to proving $\frac{dy_n}{dx} = 2ny_{n-1}$ by induction. This gave them a simpler base case but did not significantly simplify the inductive step. For the deduction in part (ii), it was very common for the first result of part (ii) to be squared, leading to pages of algebra, although they were often then successful. Part (iii) was surprisingly poorly attempted as few candidates realised that a proof by induction (or equivalent) was required.

Question 3

Only 65% attempted this, and it was the second weakest of the Pure questions with a mean score of about 7/20. It was imperative for candidates to demonstrate high levels of algebraic accuracy to score highly. Most successfully differentiated the initial expression, and equated coefficients but then failed to solve explicitly for a, b and c in terms of p and q (or demonstrate that such a, b and c existed). Candidates without these explicit expressions then often failed to spot one of the main strands of the question, integration of x^n in the two cases $n = -1$, and $n \neq -1$; some considered superfluous other cases. Some candidates fell at the final hurdle having used correct methods but then did not express their solutions in terms of p and q. Also, some were thrown by the two possible sets of solutions for a, b and c, successful candidates realising that these gave the same solutions to the differential equations.

Question 4

The third most popular question being attempted by just short of three quarters of the candidature, it was however the most successfully attempted with a mean score of not quite 12/20. The stem was usually correctly attempted either using parametric or implicit differentiation. Simultaneous equations were sensibly attempted for part (i), but sometimes they confused the two pairs of equations and as a result got the wrong answer. Some solutions elegantly achieved the correct result having found just one of the coordinates and arguing that as it lay on the tangent, it had to be the point P. Part (ii) was quite often abandoned partway through, giving up after obtaining x^2 and y^2 in the face of the algebra, although some forgot to answer the question at this point even though they had employed simultaneous equations to obtain x and y. Few managed to conclude the question, and it was very rare indeed that the non-zero nature of the denominator $(\sin\varphi - \sin\theta)$ was justified.

Question 5

A little under half the candidates attempted this, scoring marginally less on average than on question 1. Some candidates used the arithmetic mean/geometric mean inequality which was what the question was proving in part (iii), and so were heavily penalised as their arguments were thus circular. Part (i) was generally well done, with most justifying that their steps were reversible to obtain 'if and only if'. Marks were sometimes lost when justifying that $G_k > 0$ was not used to legitimise division by G_k and also that inequalities did not change sign. Part (ii) was well done too, though many candidates did not justify the 'only if', although some tried to use part (i) to prove (ii), which could not succeed. Part (iii) expressly required deduction, and only deduction, so those who had learned another proof of the inequality and just copied it out could not be rewarded. Many just used the results of (i) and (ii) without justifying why they could be used and some were imprecise with their induction arguments.

Question 6

The least popular of the Pure questions being attempted by under 40% of the candidates, it was the second least successfully attempted question in the whole paper with a mean score of only just better than 5/20. Many alternative solutions were successfully offered for the very first result, but then it was not an uncommon pitfall that q and c were treated as scalar multiples of a, and moreover, the * notation appeared to confuse some candidates. Those who moved onto part (ii), got started fairly well, but then found difficulties; those that used the fact that the points were on the unit circle, however, derived the final equation with ease, earning full marks at that stage. The few candidates who had the stamina to see the question through to part (iii) were generally very successful, though odd marks were lost when division by various factors was not justified as valid by their being non-zero.

Question 7

Marginally more popular than question 6, the mean score was 8/20. The first part of (i) created no problems, but far fewer got further, although those that expanded $(\cot\theta + i)^{2n+1}$ almost always managed to succeed, even if they did not always quite deal with the $\sin\theta \neq 0$ condition. Most that attempted part (ii) scored all the marks, though some forgot to divide $\binom{2n+1}{3}$ by $\binom{2n+1}{1}$.
Almost all that tried part (iii) proved the first result but applying the results to obtain the conclusion proved harder. A common mistake was to sum over θ rather than m and there were quite often mistakes in the algebra for the last part.

Question 8

The fifth most popular question, this was attempted by five eighths of the candidates, and was the fourth most successfully attempted with a mean score of a little over 9/20. The first result in (i) was generally very well done, but success in the second result was usually restricted to those that spotted the possibility of using partial fractions. The first integral in (ii) was usually done correctly. However the second integral attracted a range of mistakes such as failing to see that or justify how the first part could be used and not simplifying their answers. The nature of the periodicity and the need to integrate different expressions over the differing ranges were frequent stumbling blocks.

Question 9

Whilst being the most popular of the applied questions, it was much less popular than any of the Pure questions being attempted by only just over a quarter of the candidates. It was fairly weakly attempted with a mean score of 5.5/20, only slightly better than question 6. The low scores were mainly caused by sign errors in the velocities of P and Q, and candidates finding it difficult to know which variables to eliminate. However, most did attempt to express the momentum conservation and Newton's Experimental Law of Impact equations which were the starting points to the question. Most students considered the very first and second collisions, rather than the n^{th} and $(n-1)^{\text{th}}$; algebraically, these expressions were the same, but they did then need to justify the generalisation. The few candidates that got as far as part (ii), scored higher overall. In this case, often students showed that the given formula was a solution, which was not asked for, rather than finding u_0 and u_1 . Those who found u_1 in terms of u_0 and v_0 generally managed to solve the question well. Very few attempted the last part, but those that did scored well, taking the limit correctly at the end.

Question 10

9% of the candidates attempted this question, exactly as many as attempted question 13, which meant that they were the joint least popular questions. With a mean score of just over 8/20, it was slightly more successfully done than question 7. Most candidates managed to draw the correct diagram, those that did not performed poorly in the rest of the question. The majority of candidates managed to find $\sin\beta$ and a wide variety of techniques were used to find AP, some of which led them astray due to their algebraic nature. They often struggled to justify the forms of each of the terms in the energy expression, in particular, the kinetic energy of the disc and the potential energy. In the latter case, often the zero potential energy level was not defined, too. Few were put off if their energy expression was incorrect and continued to attempt the question using either their expression or that given. Some candidates attempted to find the equation of motion by taking moments, but these solutions tended to be poor. In general, candidates struggled with the algebra and would often find their way to the solution erroneously from incorrect working.

Question 11

20% of the candidates attempted this, but it was the least successfully attempted question with a mean score of under 5/20. The vast majority managed only to express V correctly. Often, candidates failed to resolve forces correctly, and even those that did frequently abandoned the question at this point. Of those that did continue, roughly half then failed to obtain correct expressions for the initial coordinates of the particle in freefall, which led to incorrect expressions for the general freefall coordinates; candidates that did find these typically progressed well apart from any algebraic errors. Candidates that reached the third part often had a reasonable attempt at it; a significant minority confused displacements and velocities indicating a lack of physical understanding of the question. Although not many attempts were made at the last part, those that understood the conditions did well whilst those that did not could not complete it.

Question 12

Although it was only attempted by 14% of the candidates, it was moderately successfully done, just slightly less so than question 8 but still with a mean over 9/20. Part (i) was frequently poorly justified with candidates often attempting to describe the given expression without connecting it to the actual problem. However, binomial coefficients and factorials were generally successfully manipulated in part (ii), although care had to be taken to choose a form for the second expression which would be useful later. Most realised that it was necessary to differentiate the cdf and apply the previous result, though some failed to take care of the details. The deduction of the integral at the end of part (ii) was generally well done, and the majority correctly spotted that this a constant multiple of the integral of the pdf. Part (iii) was largely well done by those reaching this stage, either by recognising that this was of the form of the previous part or by direct integration by parts. A common mistake in this last part was to forget the constant term of the pdf when calculating the expectation.

Question 13

As already mentioned, this was the joint least popular question, although its success rate was only very marginally less than that for question 2. The first result of the stem was generally well done with clear explanation. Likewise, the pgf for the Poisson distribution was generally well calculated, although a few candidates merely quoted the result from the formula booklet and then performed some trivial rearrangements which, of course, did not satisfy what was required. In part (i), candidates recognised how to apply the stem to calculate k, and then either applied it further to express the pgf as $\frac{1}{2}\left(G_X(t) + G_X(-t)\right)$ or directly identified the cosh power series. Similarly, most justified the result for the expectation correctly. For part (ii), $G_Z(t) = \frac{1}{2}\left(G_Y(t) + G_Y(it)\right)$ neatly gave the pgf of Z (not asked but required to progress) from the previous result, although frequently, candidates worked from $G_X(t)$ or directly from power series again. Occasional attempts to use the same method again (i.e. considering $\frac{1}{2}\left(G_Y(t) + G_Y(-t)\right)$) failed. Candidates generally failed to complete the last stage, either not attempting it, or providing poor justifications of the existence of counterexamples as $\lambda \to \infty$ rather than simply furnishing an explicit counterexample.

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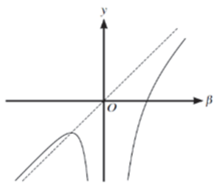
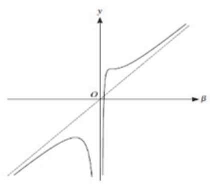
Hints and Solutions

Question 1

Differentiating $f(\beta)$ and setting equal to zero yields a cubic equation with a single real root, the quadratic factor being demonstrated to be non-zero either by completing the square or considering the discriminant. The stationary point is thus $(-1,-1)$ and with the asymptote $y=\beta$ the sketch results.



The same strategy for $g(\beta)$ yields a cubic equation with three real roots, two being coincident, and thus a maximum $\left(-2,-\frac{15}{4}\right)$ and a point of inflection $(1,3)$.



Employing Vieta's formulae gives $u+v+\frac{1}{uv}=-\alpha+\frac{1}{\beta}$ and $\frac{1}{u}+\frac{1}{v}+uv=\frac{-\alpha}{\beta}+\beta$. The first condition of part (iii) enables an expression for α to be obtained in terms of β, and so the subject of the required inequality is $f(\beta)$. When the discriminant condition is employed to impose the reality of u and v , the resulting cubic inequality has a quadratic factor which should be demonstrated to be positive (similarly to part (i)) and hence $\beta \leq 1$ which by reference to part (i), gives the requested result. Part (iv) follows the same strategy as (iii) except that it makes use of $g(\beta)$ and the reality condition cubic inequality has a squared linear factor which allows $\beta=1$ as well as $\beta \leq \frac{1}{4}$, and so by reference to the sketch, the greatest value is 3. The reader might like to consider what effect it would have on parts (iii) and (iv) if the extra condition $u \neq v$ were to be imposed.

Question 2

(i) is obtained by differentiating the defined function y_n using the product rule and a little tidying up of z and its differential. The inductive step in the proof of part (ii) can be established by differentiating the assumed case and then removing the three differentials using the result of part (i); the base case is established by obtaining $y_1 = 2x$ and $y_2 = -2 + 4x^2$ from the original definition. The deduction in (ii) is most simply obtained by eliminating x between the result for y_{n+1} just obtained and the similar one for y_{n+2}. Part (iii) is obtained by using the deduction of part (ii) to establish the desired induction, the base case being obtained using the same results used for the base case in part (ii)'s induction.

Question 3

Differentiating and equating coefficients generates three simultaneous equations, two of which are in b and c only, and can be solved by substituting for one of these variables, then giving a from the third; $a = 1 + p \mp q$, $= 1 \pm 2q$, $c = \mp q - p$. Part (i) makes use of the form obtained in the stem which can be then integrated twice, with a minor algebraic rearrangement after each integration, with different cases arising as $b = 1$ or not; the solutions are

$y = Ax^{p+q} + Bx^{p-q}$ if $q \neq 0$, and $y = Ax^p \ln x + Bx^p$ if $q = 0$. Part (ii) proceeds similarly, except that a, b, and c are simplified as $q = 0$. However, two cases arise again and so

$y = \frac{x^n}{(n-p)^2} + A\,x^p \ln x + Bx^p$ if $\neq p$, and $y = x^n \frac{(\ln x)^2}{2} + Ax^n \ln x + Bx^n$ if $n = p$.

Question 4

The equation of the tangent to the hyperbola at P is found in the usual way, having obtained the gradient through differentiation. In part (i) the points S and T can be found by solving simultaneously the equations for each of the given lines with that for the tangent and their midpoint is found to be P, using a Pythagorean trigonometry result to simplify the common denominators. Solving simultaneous equations using the equations of the two tangents gives $x^2 = \left[a \frac{(\sin\varphi\cos\theta - \sin\theta\cos\varphi)}{(\sin\varphi - \sin\theta)}\right]^2$ and $y^2 = -a^2 \sin\theta\ \sin\varphi \left[\frac{(\cos\theta - \cos\varphi)}{(\sin\varphi - \sin\theta)}\right]^2$, having eliminated b from the latter expression using the condition that the tangents are perpendicular; that same condition and the knowledge that $a > b$ not only justifies that such tangents exist, but also that the denominator of the two results found is non-zero. Adding the two results, expanding brackets in the numerator, and then removing any cos squared terms in favour of $\sin$ squared terms leaves an expression which cancels with the denominator, and the resulting simple expression can then be seen to yield the desired result.

Whilst not within the remit of the question posed, an elegant method of obtaining the result of part (ii) is to consider the solution of the hyperbola equation with the equation of a general straight line through a point on the hyperbola and another point. Solving simultaneously for (say) the x-coordinate of the meeting of the curve and line and imposing that the solutions to the quadratic must be coincident for a tangent yields a quadratic equation for the gradients, and as the tangents are perpendicular, the product of those gradients is -1 giving the desired result without recourse to trigonometry.

Question 5

As with many inequalities, rather than proving that one expression is greater than or equal to another, it is easier to subtract and produce a single expression that one requires to show is greater than or equal to zero. Substituting for the As, simplifying and dividing by $\mathrm{G_k}$ $(\mathrm{G_k} > 0)$ gets most of the way to obtaining the first result, having demonstrated that $\frac{\mathrm{G_{k+1}}}{\mathrm{G_k}} = \lambda_\mathrm{k}$ en route, and observing the reversibility of the argument throughout. Part (ii) can be simply shown using differentiation to find and justify that there is a single stationary point for $\mathrm{f(x)}$, that it is a minimum (zero) and that it occurs for $\mathrm{f(x)}$. Part (iii) (a) can be deduced using part (i), observing that the condition for (i) is met by use of part (ii). There are several different but equivalent arguments that can be used for part (iii) (b). Assuming $\mathrm{A_k} = \mathrm{G_k}$ for some k , then by part (a), and (i), it can be shown that $\mathrm{A_{k-1}} = \mathrm{G_{k-1}}$, but also that $\lambda_{\mathrm{k-1}} = 1$ and so $\mathrm{a_k} = \mathrm{G_{k-1}}$. Thus if $\mathrm{A_n} = \mathrm{G_n}$, the argument just employed obtains $\mathrm{A_k} = \mathrm{G_k}$ and $\mathrm{a_k} = \mathrm{G_{k-1}}$ for all k , for $\mathrm{k} = 1$ to n . The final step of the argument follows simply.

Question 6

The very first result is very simple bookwork, and easily justified in numerous ways. Equating the expression to its conjugate invoking its reality, employing the properties of the conjugates of sums, products etc. and algebraic rearrangement yields the second result of part (i). Substitution for the conjugates of a and c using the unit circle property gives the final result of (i) after some algebraic rearrangement including division by $(\mathrm{c}-\mathrm{a})$ which can be justified as non-zero. Use of Q lying on AC and part (i)'s result and then similarly Q lying on BD gives two equations which can be combined linearly to give the required equation for q^* in (ii). Combining the two equations linearly can also give an equation for q , which can be added and rearranged to that already found to give the desired second result of (ii). The first result of (iii) employs the result from (i) with P lying on AB and bearing in mind the reality of p. Repeating this for P lying on CD gives a similar result. Multiplying the final result of (ii) by p gives two expressions which can be simplified by the two just written down, and a little simplifying algebra allied to the legitimate division by $(\mathrm{ac}-\mathrm{bd})$ finishes the question. As some candidates recognised, but then did not score particularly well as they quoted results without justifying them, this question is about the topic of pole and polar.

Question 7

Expressing $\cot\theta$ as $\frac{\cos\theta}{\sin\theta}$ and employing De Moivre's theorem gives the first result. Expanding the left-hand side of the first equation binomially and collecting like terms for the odd powered terms and dividing by 2i then simplifies to the left-hand side of the second result, which then equals zero if $(2n+1)\theta = m\pi$ where $m = 1, 2, \dots, n$ and satisfies the initial condition. Part (ii) is obtained by considering the sum of roots of the equation just obtained in (i). The first result of (iii) can be derived from the given small angle inequality taking a little care with positivity when reciprocating and squaring, and then using the appropriate Pythagorean trigonometrical identity. Summing the inequalities for $m = 1$ to n, rearranging to give the required object of the summation and using the value of θ as defined for part (i), gives bounds of $\frac{n(2n-1)}{3(2n+1)^2} \times \pi^2$ and $\frac{n(2n+2)}{3(2n+1)^2} \times \pi^2$, both of which tend to the desired limit as $n \to \infty$.

Question 8

The first result in part (i) can be achieved by interpreting the sum of integrals as a single integral with limits of 1 and ∞ , and by making the change of variable, $\mathrm{y} = \mathrm{x}^{-1}$. The deduction can be made by splitting the integral into partial fractions, changing the variable using $\mathrm{y} = \mathrm{x} + 1$ in the integral of the first fraction, employing the periodicity condition and then telescoping the sums to leave the single required integral. The first integral of (ii) uses the result just obtained in (i) followed by observing that $\{\mathrm{x}\} = \mathrm{x}$ for the range of the integral, and then integrating normally to obtain the answer $1 - \ln 2$. For the second integral, it needs to be observed that

$\{2(\mathrm{x} + 1)\} = \{2\mathrm{x} + 2\} = \{2\mathrm{x}\}$ and the integral making use of the result for part (i) can be split into two integrals with limits 0 to $\frac{1}{2}$, and $\frac{1}{2}$ to 1 to deal with the two different explicit formulae for $\{2\mathrm{x}\}$; the result can be simplified to $2 + \ln\frac{3}{16}$ or a couple of equally simple equivalents.

Question 9

Newton's Experimental Law of Impact gives $\mathrm{v_{n-1} - u_{n-1} = e(v_{n-2} + u_{n-2})}$ and conserving momentum (once a factor of m has been cancelled)

$\mathrm{kv_{n-1} + u_{n-1} = u_{n-2} - kv_{n-2}}$ for the $(n-1)^{\text{th}}$ collision between P and Q; $\mathrm{v_{n-2}}$ can be eliminated between these two equations. Similarly, $\mathrm{v_n}$ can be eliminated between the corresponding equations for the n^{th} such collision. Then, the two equations that have been obtained can be manipulated to eliminate $\mathrm{v_{n-1}}$ and produce the desired result for (i). The derived equation from the n^{th} impact can be used to express $\mathrm{u_1}$ in terms of $\mathrm{u_0}$ and $\mathrm{v_0}$, after which, using the given solution in the cases $\mathrm{n} = 0$ and $\mathrm{n} = 1$ yields

$\mathrm{A = -17u_0 + 3v_0}$ and $\mathrm{B = 18u_0 - 3v_0}$. Expressing $\mathrm{u_n}$ as

$\left(\frac{5}{7}\right)^{\mathrm{n}} \left[(-17\mathrm{u_0} + 3\mathrm{v_0})\left(\frac{49}{50}\right)^{\mathrm{n}} + (18\mathrm{u_0} - 3\mathrm{v_0})\right]$, the final result of the question follows.

Question 10

That $\sin\beta = \frac{m}{M+m}$ can be found by taking moments about O, A, or G (say, the centre of mass of the combined disc and particle). Applying the cosine rule to triangle OAP and using the first result obtained leads directly to the first displayed result. The constant expression is obtained by conserving energy, the first term being the kinetic energy of the disc, the second term being the kinetic energy of the particle and the last term being the potential energy of the combined centre of mass relative to a zero energy level defined by the equilibrium position of G. Differentiating the energy expression with respect to time, substituting for m, I, $\sin\beta$, and $\cos\beta$, the derived equation simplifies to that of an approximate simple harmonic motion, which with the small angle approximation leads to SHM with the required period.

Question 11

Resolving radially for the particle and then setting the tension to zero (as the string becomes slack) yields $\mathrm{V}^2 = \mathrm{bg}\cos\alpha$. Expressing the horizontal and vertical displacements of the particle from O at a time T from the instant the projectile flight commences, and then expressing the sum of the squares of these displacements as the length of the string squared gives an equation which when simplified and the first result is used to eliminate $\mathrm{bg}\cos\alpha$ does give the required equation. $\tan\beta$ can be found by dividing the components of the projectile velocity, employing the result just found to simplify to obtain the desired result. The condition that the particle reaches instantaneous rest is that it is moving radially at the point the string becomes taut, and so $\tan\beta$ is also equal to the division of the displacements found when the time is T. Substituting for V^2 and T gives an equation solely in terms of $\sin\alpha$ and $\cos\alpha$ which can be simplified to a biquadratic in $\sin\alpha$, which in turn can be solved but only the required result is positive.

Question 12

Part (i) relies on appreciating that the desired probability is that at least k numbers are less than or equal to y, and so is the sum of the probabilities that a certain number of numbers are less than or equal to y, each of which is a binomial probability. The first result of (ii) relies on the definition of the binomial coefficient and manipulation of the factorials. The similar expression for $(\mathrm{n}-\mathrm{m})\binom{\mathrm{n}}{\mathrm{m}}$

is $\mathrm{n}\binom{\mathrm{n}-1}{\mathrm{m}}$ which can be arrived at in the same way; an alternative nice method relies on the symmetry of binomial coefficients, namely

$(\mathrm{n}-\mathrm{m})\binom{\mathrm{n}}{\mathrm{m}} = (\mathrm{n}-\mathrm{m})\binom{\mathrm{n}}{\mathrm{n}-\mathrm{m}} = \mathrm{n}\binom{\mathrm{n}-1}{\mathrm{n}-\mathrm{m}-1} = \mathrm{n}\binom{\mathrm{n}-1}{\mathrm{m}}$. The expression (*) is a cdf so differentiating it (by the product rule), appreciating that one term in one of the sums is zero and using the first two results of the part (one in each sum), the two sums are then the same, when one sum is re-indexed, bar one term which is the required answer. As the integral of a pdf is 1, the deduced value of the pdf without the constant is $\frac{1}{\mathrm{n}\binom{\mathrm{n}-1}{\mathrm{k}-1}}$.

Integrating in the usual way to find the expectation in part (iii) gives an expression which is a multiple of the pdf for the case of $\mathrm{n}+1$ numbers and the variable defined as the $\mathrm{k}+1$ th smallest, and the two constant expressions cancel to give $\frac{\mathrm{k}}{\mathrm{n}+1}$.

Question 13

The first result of the question is most easily shown by explicitly expressing $\mathrm{G}(1)$ and $\mathrm{G}(-1)$ in terms of probabilities and then evaluating the RHS of the equation. The pgf of the Poisson distribution is standard bookwork, expressing it as an infinite sum using the Poisson probabilities, and recognising that it is an exponential function. The pgf of Y may be found either by finding k first using the results of the stem and then recognising the power series working from the pgf definition or alternatively $\frac{1}{2}\big(\mathrm{G_X(t)} + \mathrm{G_X(-t)}\big)$. The expectation of Y can be found using the pgf as $\mathrm{G'_Y}(1)$, at which point it only remains to demonstrate that $\tanh\lambda < 1$. In part (ii), it is sensible to obtain the pgf of Z first, which can be done directly and recognising the sum of a cos and cosh power series or via $\mathrm{G_Z(t)} = \frac{1}{4}\big(\mathrm{G_X(t)} + \mathrm{G_X(-t)} + \mathrm{G_X(it)} + \mathrm{G_X(-it)}\big)$, or even more neatly

$\mathrm{G_Z(t)} = \frac{1}{2}\big(\mathrm{G_Y(t)} + \mathrm{G_Y(it)}\big)$. $\mathrm{E(Z)} = \mathrm{G'_Z}(1) = \frac{\lambda(\sinh\lambda - \sin\lambda)}{(\cosh\lambda + \cos\lambda)}$ and so, for example, choosing $\lambda = \frac{3\pi}{2}$ it can be shown that this is greater than λ by expressing the hyperbolic functions in terms of exponential functions.

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2018

Mark Scheme

1. (i) $f(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2}$

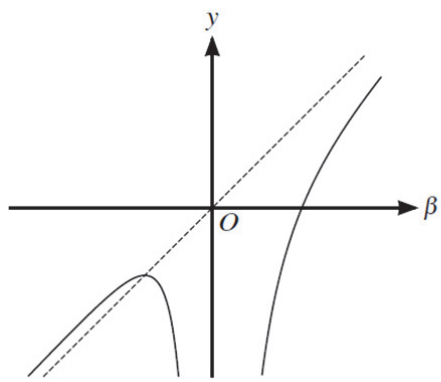
$f'(\beta) = 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3} = \frac{\beta^3+\beta+2}{\beta^3}$

$f'(\beta) = 0 \Rightarrow \beta^3 + \beta + 2 = 0$ **M1**

$\beta^3 + \beta + 2 = (\beta + 1)(\beta^2 - \beta + 2)$

$\beta^2 - \beta + 2 \neq 0$ as $\beta^2 - \beta + 2 = \left(\beta - \frac{1}{2}\right)^2 + \frac{7}{4} \geq \frac{7}{4} > 0$ or discriminant $= -7$ **E1**

So the only stationary point is $(-1, -1)$ **A1**



G2 (5)

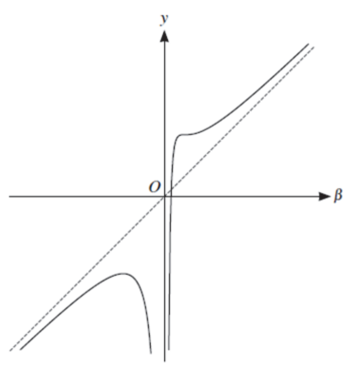
$g(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2}$

Then $g'(\beta) = 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3}$

$$1 - \frac{3}{\beta^2} + \frac{2}{\beta^3} = \frac{\beta^3 - 3\beta + 2}{\beta^3} = \frac{(\beta - 1)^2(\beta + 2)}{\beta^3}$$

M1

So the stationary points are $\left(-2, -\frac{15}{4}\right)$ and $(1,3)$ **A1**



G2 (4)

(ii) $u+v=-\alpha$ and $uv=\beta$ so $u+v+\frac{1}{uv}=-\alpha+\frac{1}{\beta}$

and $\frac{1}{u}+\frac{1}{v}+uv=\frac{u+v}{uv}+uv=\frac{-\alpha}{\beta}+\beta$ **B1 (1)**

(iii)

$u+v+\frac{1}{uv}=-1 \;\Rightarrow\; -\alpha+\frac{1}{\beta}=-1$ so $\alpha=1+\frac{1}{\beta}$

Thus $\frac{1}{u}+\frac{1}{v}+uv=\frac{-\alpha}{\beta}+\beta=\beta-\frac{1}{\beta}-\frac{1}{\beta^2}$ **M1**

u,v real $\Leftrightarrow \alpha^2-4\beta\geq 0$

As $\alpha^2-4\beta\geq 0$, $\left(1+\frac{1}{\beta}\right)^2-4\beta\geq 0$ and thus $4\beta^3-\beta^2-2\beta-1\leq 0$

$(\beta-1)(4\beta^2+3\beta+1)\leq 0$ **M1 A1**

$4\beta^2+3\beta+1=\left(2\beta+\frac{3}{4}\right)^2+\frac{7}{16}\geq\frac{7}{16}>0$ or discriminant $=-7$ and positive quadratic so

$4\beta^2+3\beta+1>0$, **E1**

and so $\beta\leq 1$ **B1**

Hence, from the sketch in part (i), $f(\beta)=\beta-\frac{1}{\beta}-\frac{1}{\beta^2}=\frac{1}{u}+\frac{1}{v}+uv\leq -1$ as required. **E1 (6)**

(iv) If $u+v+\frac{1}{uv}=3 \;\Rightarrow\; -\alpha+\frac{1}{\beta}=3$ so $\alpha=\frac{1}{\beta}-3$

Thus $\frac{1}{u}+\frac{1}{v}+uv=\frac{-\alpha}{\beta}+\beta=\beta+\frac{3}{\beta}-\frac{1}{\beta^2}$ **M1**

u,v real $\Leftrightarrow \alpha^2-4\beta\geq 0$, so $\left(\frac{1}{\beta}-3\right)^2-4\beta\geq 0$ and thus $4\beta^3-9\beta^2+6\beta-1\leq 0$

Therefore $(\beta-1)^2(4\beta-1)\leq 0$ and so $\beta=1$ or $\beta\leq\frac{1}{4}$ **M1 A1**

From the graph of $g(\beta)$ we can deduce that the greatest value of $\frac{1}{u}+\frac{1}{v}+uv$ is 3 **E1 (4)**

2. (i)

$$\frac{dy_n}{dx} = \frac{d}{dx}\left((-1)^n \frac{1}{z}\frac{d^n z}{dx^n}\right) = (-1)^n \left[2xe^{x^2}\frac{d^n z}{dx^n} + \frac{1}{z}\frac{d^{n+1}z}{dx^{n+1}}\right]$$

M1

$$= 2x(-1)^n \frac{1}{z}\frac{d^n z}{dx^n} - (-1)^{n+1}\frac{1}{z}\frac{d^{n+1}z}{dx^{n+1}}$$

M1

$$= 2xy_n - y_{n+1}$$

as required. **A1* (3)**

(ii) Suppose $y_{k+1} = 2xy_k - 2ky_{k-1}$ for some k **B1**

$$\frac{dy_{k+1}}{dx} = 2x\frac{dy_k}{dx} + 2y_k - 2k\frac{dy_{k-1}}{dx}$$

M1

So using (i)

$$2xy_{k+1} - y_{k+2} = 2x(2xy_k - y_{k+1}) + 2y_k - 2k(2xy_{k-1} - y_k)$$

M1

$$= 4x^2y_k - 2xy_{k+1} + 2y_k + 2ky_k - 2x(2xy_k - y_{k+1})$$

M1

$$= 4x^2y_k - 2xy_{k+1} + 2y_k + 2ky_k - 4x^2y_k + 2xy_{k+1}$$

$$= 2y_k + 2ky_k$$

Thus $y_{k+2} = 2xy_{k+1} - 2(k+1)y_k$ which is the required result for $k+1$ **A1**

$$y_0 = 1$$

$$y_1 = (-1)\frac{1}{z}\frac{dz}{dx} = -1e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right) = -e^{x^2}.-2xe^{-x^2} = 2x$$

B1

$$y_2 = (-1)^2\frac{1}{z}\frac{d^2z}{dx^2} = e^{x^2}\frac{d^2\left(e^{-x^2}\right)}{dx^2} = e^{x^2}\left[\frac{d}{dx}\left(-2xe^{-x^2}\right)\right] = e^{x^2}\left(-2e^{-x^2} + 4x^2e^{-x^2}\right)$$

$$= -2 + 4x^2$$

M1A1

$2xy_1 - 2 \times 1y_0 = 2x(2x) - 2 = 4x^2 - 2$ so result true for $n = 1$ **A1**

Hence $y_{n+1} = 2xy_n - 2ny_{n-1}$ for $n \geq 1$ by induction. **A1 (10)**

As $y_{n+1} = 2xy_n - 2ny_{n-1}$, $y_{n+2} = 2xy_{n+1} - 2(n+1)y_n$ **M1**

Eliminating x,

$$(y_{n+1})^2 - y_n y_{n+2} = 2(n+1)(y_n)^2 - 2ny_{n-1}y_{n+1}$$

M1

Thus

$$y_{n+1}(y_{n+1} + 2ny_{n-1}) = y_n(y_{n+2} + 2(n+1)y_n)$$

So

$$y_{n+1}{}^2 - y_n y_{n+2} = 2n(y_n{}^2 - y_{n-1}y_{n+1}) + 2y_n{}^2$$

A1 (3)

(iii) Suppose $y_k{}^2 - y_{k-1}y_{k+1} > 0$ for some $k \geq 1$ **B1**

Then, as $2y_k{}^2 \geq 0$, $2k(y_k{}^2 - y_{k-1}y_{k+1}) + 2y_k{}^2 > 0$, i.e. by (ii) $y_{k+1}{}^2 - y_k y_{k+2} > 0$ **E1**

Consider $k = 1$

$$y_1{}^2 - y_0 y_2 = (2x)^2 - 1 \times (-2 + 4x^2) = 4x^2 + 2 - 4x^2 = 2 > 0$$

B1

Hence the result $y_n{}^2 - y_{n-1}y_{n+1} > 0$ for $n \geq 1$

B1 (4)

3.

$$x^a\left(x^b(x^c y)'\right)' = x^a[x^b(x^c y' + cx^{c-1}y)]'$$

M1

$$= x^a[x^{b+c}y' + cx^{b+c-1}y]'$$

$$= x^a[x^{b+c}y'' + cx^{b+c-1}y' + (b+c)x^{b+c-1}y' + c(b+c-1)x^{b+c-2}y]$$

$$= x^{a+b+c}y'' + (b+2c)x^{a+b+c-1}y' \ + c(b+c-1)x^{a+b+c-2}y$$

M1A1

This is of the required form if

$$a+b+c = 2$$

$$b+2c = 1-2p$$

$$c(b+c-1) = p^2 - q^2$$

M1

Thus

$$c(1-2p-2c+c-1) = p^2 - q^2$$

$$c^2 + 2pc + p^2 = q^2$$

$$c + p = \pm q$$

M1

Thus it is possible if

$c = q - p$, $b = 1 - 2q$, $a = 1 + p + q$

or

$c = -q - p$, $b = 1 + 2q$, $a = 1 + p - q$

A1 (6)

(i) $x^2y'' + (1-2p)xy' + (p^2 - q^2)y = 0$

So

$$x^a\left(x^b(x^c y)'\right)' = 0$$

$$\left(x^b(x^c y)'\right)' = 0$$

$$x^b(x^c y)' = A$$

M1

$$(x^c y)' = Ax^{-b}$$

$x^c y = \frac{Ax^{-b+1}}{-b+1} + B$ unless $b = 1$ **B1**

$b \neq 1 \Rightarrow q \neq 0$ in which case

$$y = \frac{Ax^{-b-c+1}}{-b+1} + Bx^{-c}$$

$$y = \frac{Ax^{p\pm q}}{\pm 2q} + Bx^{p\pm q}$$

That is

$$y = Cx^{p+q} + Dx^{p-q}$$

M1A1

However, if $b = 1$, $x^c y = A\ln x + B$ **M1**

so $y = Ax^{-c}\ln x + Bx^{-c}$

So if $q = 0$, $y = Ax^p \ln x + Bx^p$ **M1A1 (7)**

(ii) $x^2 y'' + (1 - 2p)xy' + p^2 y = x^n$

Thus $q = 0$, and $c = -p$, $b = 1$, $a = 1 + p$ **B1**

$$x^a(x(x^c y)')' = x^n$$

$$(x(x^c y)')' = x^{n-a}$$

$x(x^c y)' = \frac{x^{n+1-a}}{n+1-a} + A$ for $n + 1 - a \neq 0$ or $x(x^c y)' = \ln x + A$ for $n + 1 - a = 0$ **M1 B1**

$$n + 1 - a = 0 \Rightarrow n = p$$

Thus $(x^c y)' = \frac{x^{n-a}}{n+1-a} + Ax^{-1}$ or $(x^c y)' = x^{-1}\ln x + Ax^{-1}$

So $x^c y = \frac{x^{n+1-a}}{(n+1-a)^2} + A\ln x + B$ or $x^c y = \frac{(\ln x)^2}{2} + A\ln x + B$ **M1 M1**

So for $n \neq p$, $y = \frac{x^n}{(n-p)^2} + A\,x^p \ln x + Bx^p$ **A1**

and for $n = p$, $y = x^n \frac{(\ln x)^2}{2} + Ax^n \ln x + Bx^n$ **A1 (7)**

4.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

M1

(Alternatively,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

also earns **M1**)

Thus at P

$$\frac{dy}{dx} = \frac{a\sec\theta}{a^2}\frac{b^2}{b\tan\theta} = \frac{b}{a\sin\theta}$$

A1

So the tangent at P is

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

M1

Hence

$$ay\sin\theta - ab\frac{\sin^2\theta}{\cos\theta} = bx - ab\frac{1}{\cos\theta}$$

So

$$bx - ay\sin\theta = ab\left(\frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta}\right) = ab\frac{\cos^2\theta}{\cos\theta} = ab\cos\theta$$

as required. **A1* (4)**

(i) S is the intersection of $bx - ay\sin\theta = ab\cos\theta$ and $\frac{x}{a} = \frac{y}{b}$

So $bx - bx\sin\theta = ab\cos\theta$ **M1**

Thus, S is $\left(\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}\right)$ **A1**

Similarly, T is the intersection of $bx - ay\sin\theta = ab\cos\theta$ and $\frac{x}{a} = -\frac{y}{b}$

So T is $\left(\frac{a\cos\theta}{1+\sin\theta}, -\frac{b\cos\theta}{1+\sin\theta}\right)$ **M1A1**

The midpoint of ST is therefore $\left(\frac{1}{2}\left(\frac{a\cos\theta}{1-\sin\theta} + \frac{a\cos\theta}{1+\sin\theta}\right), \frac{1}{2}\left(\frac{b\cos\theta}{1-\sin\theta} - \frac{b\cos\theta}{1+\sin\theta}\right)\right)$ **M1**

$$\frac{1}{2}\left(\frac{a\cos\theta}{1-\sin\theta} + \frac{a\cos\theta}{1+\sin\theta}\right) = \frac{1}{2}a\cos\theta\frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = a\sec\theta$$

$$\frac{1}{2}\left(\frac{b\cos\theta}{1-\sin\theta}-\frac{b\cos\theta}{1+\sin\theta}\right)=\frac{1}{2}b\cos\theta\frac{1+\sin\theta-1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)}=b\tan\theta$$

M1

which means it is P. **A1 (7)**

(ii) As the tangents at P and Q are perpendicular,

$$\frac{b}{a\sin\theta}\times\frac{b}{a\sin\varphi}=-1$$

B1

(This is possible because $a > b$)

That is

$$a^2\sin\theta\ \sin\varphi+b^2=0$$

The intersection of the tangents is given by the solution of

$$bx-ay\sin\theta=ab\cos\theta$$

$$bx-ay\sin\varphi=ab\cos\varphi$$

Thus

$$x=a\frac{(\sin\varphi\cos\theta-\sin\theta\cos\varphi)}{(\sin\varphi-\sin\theta)}$$

M1

and

$$y=b\frac{(\cos\theta-\cos\varphi)}{(\sin\varphi-\sin\theta)}$$

$$x^2=\left[a\frac{(\sin\varphi\cos\theta-\sin\theta\cos\varphi)}{(\sin\varphi-\sin\theta)}\right]^2$$

A1

$$y^2=-a^2\sin\theta\ \sin\varphi\left[\frac{(\cos\theta-\cos\varphi)}{(\sin\varphi-\sin\theta)}\right]^2$$

A1

Note $\sin\theta\ \sin\varphi=-\frac{b^2}{a^2}<0$ so $\sin\varphi\neq\sin\theta$ and so $\sin\varphi-\sin\theta\neq0$ **E1**

So

$$x^2+y^2=\frac{a^2}{(\sin\varphi-\sin\theta)^2}[(\sin\varphi\cos\theta-\sin\theta\cos\varphi)^2-\sin\theta\ \sin\varphi\,(\cos\theta-\cos\varphi)^2]$$

$$=\frac{a^2}{(\sin\varphi-\sin\theta)^2}[\sin^2\varphi\cos^2\theta+\sin^2\theta\cos^2\varphi-\sin\theta\ \sin\varphi\cos^2\theta-\sin\theta\ \sin\varphi\ \cos^2\varphi]$$

M1

$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[\sin^2\varphi\,(1-\sin^2\theta) + \sin^2\theta\,(1-\sin^2\varphi) - 2\sin\varphi\sin\theta + 2\sin\varphi\sin\theta -$
$\sin\theta\ \sin\varphi\cos^2\theta - \sin\theta\ \sin\varphi\ \cos^2\varphi]$ **M1**

$$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[(\sin\varphi - \sin\theta)^2 + \sin\varphi\sin\theta(2 - 2\sin\varphi\sin\theta - \cos^2\theta - \cos^2\varphi)]$$

$$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[(\sin\varphi - \sin\theta)^2 + \sin\varphi\sin\theta\ (\sin^2\varphi - 2\sin\varphi\sin\theta + \sin^2\theta)]$$

$$= a^2 + a^2\sin\theta\ \sin\varphi = a^2 - b^2$$

as required. **M1A1* (9)**

5. (i)

$(k+1)(\ A_{k+1}-G_{k+1})-k(A_k-G_k)\geq 0$

$\Leftrightarrow\ (a_1+a_2+\cdots+a_k+a_{k+1})-(k+1)G_{k+1}-(a_1+a_2+\cdots+a_k)+kG_k\geq 0$ **M1**

$\Leftrightarrow\ a_{k+1}+kG_k\geq(k+1)G_{k+1}$

$\Leftrightarrow\ \frac{a_{k+1}}{G_k}+k\geq(k+1)\frac{G_{k+1}}{G_k}$ as $G_k>0$ **M1 E1**

$\frac{G_{k+1}}{G_k}=\frac{\left({G_k}^k a_{k+1}\right)^{1/k+1}}{G_k}=\left(\frac{a_{k+1}}{G_k}\right)^{1/k+1}=\lambda_k$ **B1**

So

$\frac{a_{k+1}}{G_k}+k\geq(k+1)\frac{G_{k+1}}{G_k}\ \Leftrightarrow\ {\lambda_k}^{k+1}+k\geq(k+1)\lambda_k$

$\Leftrightarrow\ {\lambda_k}^{k+1}-(k+1)\lambda_k+k\geq 0$ as required. **M1A1 (6)**

(ii) $f(x)=x^{k+1}-(k+1)x+k$

So $f'(x)=(k+1)x^k-(k+1)=(k+1)\left(x^k-1\right)$ and $f''(x)=(k+1)kx^{k-1}$ **M1M1**

Thus, if x is positive, there is a single stationary point for $x=1$ and it is a minimum. **E1**

$f(1)=0$

and so $f(x)=x^{k+1}-(k+1)x+k\geq 0$ **E1* (4)**

(iii) (a) Assume $A_k\geq G_k$ for some particular k **B1**

then by (ii), the condition for (i) is met and so $A_{k+1}-G_{k+1}\geq\frac{k}{k+1}(A_k-G_k)$ **E1**

and thus $A_{k+1}\geq G_{k+1}$

$A_1=a_1$ and $G_1=a_1$ and so $A_1\geq G_1$ (in fact $A_1=G_1$) **B1**

Thus, by the principle of mathematical induction, $A_n\geq G_n$ for all n **B1 (4)**

(b) If $A_k=G_k$ for some k, then as $A_n\geq G_n$ for all n, $A_{k-1}\geq G_{k-1}$ **E1**

and by (i) and (ii) $A_{k-1}=G_{k-1}$ and **E1**

$$\left(\frac{a_k}{G_{k-1}}\right)^{1/k}=1$$

in which case $a_k=G_{k-1}$. **B1**

But as $A_n = G_n$, $A_k = G_k$ and $a_k = G_{k-1}$ for all k , for $k = 1$ to n **E1**

But, $A_1 = G_1 = a_1$ **B1** and so $a_2 = G_1 = a_1$ and thus $A_2 = G_2 = a_1$ and $a_3 = G_2 = a_1$ and so on up to $A_n = G_n = a_1$

Hence $a_1 = a_2 = a_3 = \cdots = a_n$ **E1 (6)**

6. (i)

$\overrightarrow{AQ}$ is parallel to $\overrightarrow{AC}$

So $q - a = \lambda(c - a)$ where λ is real.

Therefore $\frac{q-a}{c-a} = \lambda$ which is real as required. **E1 (1)**

Hence,

$$\frac{q-a}{c-a} = \left(\frac{q-a}{c-a}\right)^*$$

M1

$$\left(\frac{q-a}{c-a}\right)^* = \frac{(q-a)^*}{(c-a)^*} = \frac{q^* - a^*}{c^* - a^*}$$

M1

So as

$$\frac{q-a}{c-a} = \frac{q^* - a^*}{c^* - a^*}$$

$$(c-a)(q^* - a^*) = (c^* - a^*)(q - a)$$

A1* (3)

$$(c-a)\left(q^* - \frac{1}{a}\right) = \left(\frac{1}{c} - \frac{1}{a}\right)(q - a)$$

M1

Multiplying by ac ,

$$(c-a)(acq^* - c) = -(c-a)(q-a)$$

M1

Thus

$$acq^* - c = -(q - a)$$

as $c - a \neq 0$ **E1**

and so

$$q + acq^* = a + c$$

A1* (4)

(ii) Q lies on AC, so from (i)

$$q + acq^* = a + c$$

Also Q lies on BD, so similarly

$$q + bdq^* = b + d$$

M1

Subtracting $(ac - bd)q^* = (a + c) - (b + d)$ **M1***

Multiplying the AC equation by bd and the BD one by ac and subtracting,

$$(ac - bd)q = ac(b + d) - bd(a + c)$$

M1

So adding these two equations

$$(ac - bd)(q + q^*) = ac(b + d) - bd(a + c) + (a + c) - (b + d)$$

M1

Rearranging

$$(ac - bd)(q + q^*) = (a - b)(1 + cd) + (c - d)(1 + ab)$$

as required. **A1* (5)**

(iii) P lies on AB, so from (i)

$$p + abp^* = a + b$$

M1

But as p is real, $p = p^*$, and so

$$p + abp = a + b$$

M1*

That is

$$p(1 + ab) = a + b$$

Similarly, as P lies on CD

$$p(1 + cd) = c + d$$

M1

Multiplying the final result of (ii) by p we have

$$(ac - bd)p(q + q^*) = (a - b)p(1 + cd) + (c - d)p(1 + ab)$$

M1

Thus

$$(ac - bd)p(q + q^*) = (a - b)(c + d) + (c - d)(a + b)$$

M1

So, simplifying

$$(ac - bd)p(q + q^*) = 2ac - 2bd = 2(ac - bd)$$

And as $ac - bd \neq 0$, $p(q + q^*) = 2$ **E1 A1* (7)**

7. (i)

$$\frac{(\cot\theta + i)^{2n+1} - (\cot\theta - 1)^{2n+1}}{2i} = \frac{(\cos\theta + i\sin\theta)^{2n+1} - (\cos\theta - i\sin\theta)^{2n+1}}{2i\sin^{2n+1}\theta}$$

M1

$$= \frac{(\cos(2n+1)\theta + i\sin(2n+1)\theta) - (\cos(2n+1)\theta - i\sin(2n+1)\theta)}{2i\sin^{2n+1}\theta}$$

M1

$$= \frac{2i\sin(2n+1)\theta}{2i\sin^{2n+1}\theta} = \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}$$

A1* (3)

Let $y = \cot\theta$, then

$$\frac{(y+i)^{2n+1} - (y-i)^{2n+1}}{2i}$$

M1

$$= \frac{\left(y^{2n+1} + \binom{2n+1}{1} y^{2n} i + \binom{2n+1}{2} y^{2n-1} i^2 + \cdots\right) - \left(y^{2n+1} - \binom{2n+1}{1} y^{2n} i + \binom{2n+1}{2} y^{2n-1} i^2 - \cdots\right)}{2i}$$

M1

$$= \binom{2n+1}{1} y^{2n} - \binom{2n+1}{3} y^{2n-2} + \cdots + (i)^{2n}$$

M1

So if $(2n+1)\theta = m\pi$ where $m = 1, 2, \ldots, n$, $\sin\theta \neq 0$ and $\sin(2n+1)\theta = 0$ **E1**

So

$$\binom{2n+1}{1}\cot^{2n}\theta - \binom{2n+1}{3}\cot^{2n-2}\theta + \cdots + (-1)^n = 0$$

Thus if $x = \cot^2\theta$, $\binom{2n+1}{1} x^n - \binom{2n+1}{3} x^{n-1} + \cdots + (-1)^n = 0$ which gives the required result. **A1* (5)**

(ii)

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right)$$

is the sum of the roots of the equation in part (i) , and so is equal to **M1**

$$\frac{\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{(2n+1)!}{(2n-2)!\,3!} \times \frac{(2n)!\,1!}{(2n+1)!} = \frac{(2n)!}{(2n-2)!\,3!} = \frac{2n \times (2n-1)}{3 \times 2} = \frac{n(2n-1)}{3}$$

A1 **A1* (3)**

(iii) As $0 < \sin\theta < \theta < \tan\theta$, $\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{1}{\tan\theta}$ as these are all positive, and

$$\frac{1}{\sin^2\theta} > \frac{1}{\theta^2} > \frac{1}{\tan^2\theta}$$

M1

Thus

$$\csc^2\theta > \frac{1}{\theta^2} > \cot^2\theta$$

M1

That is

$$1 + \cot^2\theta > \frac{1}{\theta^2} > \cot^2\theta$$

M1

or as required

$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta$$

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) < \sum_{m=1}^{n} \left(\frac{2n+1}{m\pi}\right)^2 < \sum_{m=1}^{n} \left(1 + \cot^2\left(\frac{m\pi}{2n+1}\right)\right)$$

M1

$$\frac{n(2n-1)}{3} < \left(\frac{2n+1}{\pi}\right)^2 \sum_{m=1}^{n} \frac{1}{m^2} < n + \frac{n(2n-1)}{3}$$

M1

$$\frac{n(2n-1)}{3(2n+1)^2} \times \pi^2 < \sum_{m=1}^{n} \frac{1}{m^2} < \frac{n(2n+2)}{3(2n+1)^2} \times \pi^2$$

M1

Letting $n \to \infty$,

$$\frac{n(2n-1)}{3(2n+1)^2} \times \pi^2 \to \frac{\pi^2}{6}$$

M1

and

$$\frac{n(2n+2)}{3(2n+1)^2} \times \pi^2 \to \frac{\pi^2}{6}$$

M1

and so $\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$ **A1* (9)**

8. (i)

$$\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{y(1+y)}\,dy=\int_{1}^{\infty}\frac{f(y)}{y(1+y)}\,dy$$

M1

Making a change of variable, $y=x^{-1}$, $\frac{dy}{dx}=-x^{-2}$ **M1**

so

$$\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{y(1+y)}\,dy=\int_{1}^{\infty}\frac{f(y)}{y(1+y)}\,dy=\int_{1}^{0}\frac{f(x^{-1})}{x^{-1}(1+x^{-1})}.-x^{-2}\,dx$$

M1

$$=\int_{1}^{0}\frac{-f(x^{-1})}{(x+1)}.dx=\int_{0}^{1}\frac{f(x^{-1})}{(1+x)}.dx=I$$

A1* (4)

$$I=\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{y(1+y)}\,dy=\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{y}-\frac{f(y)}{1+y}\,dy$$

M1

$$=\sum_{n=1}^{\infty}\int_{x=n-1}^{n}\frac{f(x+1)}{x+1}dx-\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{1+y}\,dy$$

M1

$$=\sum_{n=1}^{\infty}\int_{x=n-1}^{n}\frac{f(x)}{x+1}dx-\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{1+y}\,dy$$

M1

$$=\int_{x=0}^{\infty}\frac{f(x)}{x+1}dx-\int_{x=1}^{\infty}\frac{f(x)}{x+1}dx$$

M1

$$=\int_{x=0}^{1}\frac{f(x)}{1+x}dx$$

as required. **A1* (5)**

(ii) From (i)

$$\int_{x=0}^{1} \frac{\{x^{-1}\}}{x+1}\,dx = \int_{x=0}^{1} \frac{\{x\}}{x+1}\,dx$$

M1

$$= \int_{x=0}^{1} \frac{x}{x+1}\,dx = \int_{x=0}^{1} 1 - \frac{1}{x+1}\,dx = [x - \ln(x+1)]_0^1 = 1 - \ln 2$$

M1 **M1** **A1 (4)**

$$\int_{x=0}^{1} \frac{\{2x^{-1}\}}{x+1}\,dx$$

$\{2(x+1)\} = \{2x+2\} = \{2x\}$ **E1**

and so we can once again use the result from part (i), and thus

$$\int_{x=0}^{1} \frac{\{2x^{-1}\}}{x+1}\,dx = \int_{x=0}^{1} \frac{\{2x\}}{x+1}\,dx = \int_{x=0}^{\frac{1}{2}} \frac{2x}{x+1}\,dx + \int_{x=\frac{1}{2}}^{1} \frac{2x-1}{x+1}\,dx$$

M1 **M1A1**

$$= 2[x - \ln(x+1)]_0^{\frac{1}{2}} + [2x - 3\ln(x+1)]_{\frac{1}{2}}^{1}$$

dM1

$$= 1 - 2\ln\frac{3}{2} + 2 - 3\ln 2 - 1 + 3\ln\frac{3}{2}$$

$$= 2 + \ln\frac{3}{16} = 2 - \ln\frac{16}{3}$$

M1A1 (7)

9. (i) NELI for the n-1 th collision between P and Q gives

$$v_{n-1} - u_{n-1} = e(v_{n-2} + u_{n-2})$$

M1

and conserving momentum

$$kmv_{n-1} + mu_{n-1} = mu_{n-2} - kmv_{n-2}$$

M1

which simplifies to

$$kv_{n-1} + u_{n-1} = u_{n-2} - kv_{n-2}$$

Eliminating v_{n-2} between the two equations by multiplying the first by k and the second by e and adding gives **M1**

$$k(1+e)v_{n-1} + (e-k)u_{n-1} = e(1+k)u_{n-2}$$

A1

Similarly, the nth collision gives

$$v_n - u_n = e(v_{n-1} + u_{n-1})$$

and

$$kv_n + u_n = u_{n-1} - kv_{n-1}$$

M1

Eliminating v_n between these two equations by multiplying the first by k and subtracting from the second **M1**

$(1+k)u_n = (1-ke)u_{n-1} - k(1+e)v_{n-1}$ (**) **A1**

Adding the left hand side of the equation from the n-1 th collision to the right hand side of that just obtained (and vice versa)

$$(1+k)u_n + e(1+k)u_{n-2} = (1-ke)u_{n-1} + (e-k)u_{n-1}$$

M1

Thus

$$(1+k)u_n + e(1+k)u_{n-2} = (1+e-k-ke)u_{n-1} = (1-k)(1+e)u_{n-1}$$

M1

Giving

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0$$

A1* (10)

(ii) The first impact gives using (**)

$$\left(1+\frac{1}{34}\right)u_1 = \left(1-\frac{1}{34}\times\frac{1}{2}\right)u_0 - \frac{1}{34}\left(1+\frac{1}{2}\right)v_0$$

M1

Thus

$$70u_1 = 67u_0 - 3v_0$$

A1

Letting $n = 0$

$$u_0 = A + B$$

M1

and letting $n = 1$

$$A\left(\frac{7}{10}\right) + B\left(\frac{5}{7}\right) = u_1 = \frac{67u_0 - 3v_0}{70}$$

M1

So $49A + 50B = 67u_0 - 3v_0$

Thus

$$A = -17u_0 + 3v_0$$

and

$$B = 18u_0 - 3v_0$$

M1 A1 (6)

Thus

$$u_n = (-17u_0 + 3v_0)\left(\frac{7}{10}\right)^n + (18u_0 - 3v_0)\left(\frac{5}{7}\right)^n$$

M1

$$= \left(\frac{5}{7}\right)^n \left[(-17u_0 + 3v_0)\left(\frac{49}{50}\right)^n + (18u_0 - 3v_0)\right]$$

If $v_0 > 6u_0$, $(-17u_0 + 3v_0 > u_0)$ and $18u_0 - 3v_0 < 0$ **M1**

For large n, the term $\left(\frac{49}{50}\right)^n \to 0$

E1

$u_n \to \left(\frac{5}{7}\right)^n (18u_0 - 3v_0) < 0$

E1* (4)

10. If G is the centre of mass of the combined disc and particle, then

$$(M+m)\times OG = M\times 0 + m\times a$$

M1

In equilibrium, G is vertically below A, so $\sin\beta = \frac{OG}{OA} = \frac{m}{M+m}$ **A1 (2)**

Applying the cosine rule to triangle OAP,

$$AP^2 = a^2 + a^2 - 2a^2\cos\left(\frac{\pi}{2}-\beta\right) = 2a^2(1-\sin\beta)$$

M1 **M1**

Thus

$$\frac{AP}{a} = \sqrt{2(1-\sin\beta)} = \sqrt{2\left(1-\frac{m}{M+m}\right)} = \sqrt{\frac{2M}{M+m}}$$

M1 **A1* (4)**

The kinetic energy of the disc about L is $\frac{1}{2}I\dot{\theta}^2$ **B1** and the kinetic energy of the particle about L is $\frac{1}{2}m\left(AP\dot{\theta}\right)^2 = \frac{1}{2}m2a^2(1-\sin\beta)\dot{\theta}^2 = (1-\sin\beta)ma^2\dot{\theta}^2$

B1 **M1**

The potential energy of the system relative to the zero level of the point G in equilibrium is $(M+m)gAG(1-\cos\theta) = (M+m)ga\cos\beta(1-\cos\theta)$

B1 **M1**

So, during the motion, conserving energy

$$\frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (M+m)ga\cos\beta(1-\cos\theta)$$

Is constant. **E1 (6)**

$$\frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (M+m)ga\cos\beta(1-\cos\theta) = c$$

Differentiating with respect to time,

$$I\dot{\theta}\ddot{\theta} + 2(1-\sin\beta)ma^2\dot{\theta}\ddot{\theta} + (M+m)ga\cos\beta\sin\theta\,\dot{\theta} = 0$$

M1 A1

Thus, as $m = \frac{3}{2}M$, $\sin\beta = \frac{m}{M+m} = \frac{3}{5}$ and $\cos\beta = \frac{4}{5}$ ($\cos\beta$ is positive as β is acute) **B1**

and because $I = \frac{3}{2}Ma^2$,

$$\left(\frac{3}{2}Ma^2 + 2\times\frac{2}{5}\times\frac{3}{2}Ma^2\right)\ddot{\theta} + \left(M+\frac{3}{2}M\right)ga\times\frac{4}{5}\sin\theta = 0$$

M1

For small oscillations, $\sin\theta \approx \theta$, **M1**

so

$$\frac{27}{10}a\ddot{\theta} + 2g\theta \approx 0$$

That is

$$\ddot{\theta} \approx -\frac{20}{27}\frac{g}{a}\theta$$

A1

and hence the period of small oscillations is

$$2\pi\sqrt{\frac{27a}{20g}} = 3\pi\sqrt{\frac{3a}{5g}}$$

M1 **A1* (8)**

11. At a general moment in the motion, when the acute angle between the string and the upward vertical is θ , and the speed of the particle is v , resolving towards O

$$T' + mg\cos\theta = m\frac{v^2}{b}$$

where T' is the tension in the string and m is the mass of the particle. **M1**

So at the point when the string becomes slack,

$$g\cos\alpha = \frac{V^2}{b}$$

M1

i.e. $V^2 = bg\cos\alpha$ **A1 (3)**

If x is the horizontal displacement of the particle from O at time t, and y the vertical. Then

$$x = b\sin\alpha - Vt\cos\alpha$$

M1

and

$$y = b\cos\alpha + Vt\sin\alpha - \frac{1}{2}gt^2$$

M1

The string is taut when $x^2 + y^2 = b^2$ **M1**

So

$$(b\sin\alpha - VT\cos\alpha)^2 + \left(b\cos\alpha + VT\sin\alpha - \frac{1}{2}gT^2\right)^2 = b^2$$

A1

Thus

$$\begin{aligned} b^2\sin^2\alpha - 2b\sin\alpha VT\cos\alpha + V^2T^2\cos^2\alpha \\ + b^2\cos^2\alpha + 2b\sin\alpha VT\cos\alpha + V^2T^2\sin^2\alpha - bgT^2\cos\alpha - gVT^3\sin\alpha \\ + \frac{1}{4}g^2T^4 = b^2 \end{aligned}$$

M1

So

$$V^2T^2 - bgT^2\cos\alpha - gVT^3\sin\alpha + \frac{1}{4}g^2T^4 = 0$$

But as $V^2 = bg\cos\alpha$,

$$V^2T^2 - V^2T^2 - gVT^3\sin\alpha + \frac{1}{4}g^2T^4 = 0$$

M1

So

$$g^2T^4 = 4gVT^3 \sin\alpha$$

and as $T \neq 0$,

$$gT = 4V \sin\alpha$$

A1* (7)

$$\dot{x} = -V\cos\alpha$$

B1

$$\dot{y} = V\sin\alpha - gt$$

B1

Thus

$$\tan\beta = \frac{V\sin\alpha - gT}{-V\cos\alpha} = \frac{V\sin\alpha - 4V\sin\alpha}{-V\cos\alpha} = 3\tan\alpha$$

M1 A1* (4)

The particle comes instantaneously to rest if and only if its motion is radial at the point of impact.

In other words,

$$\frac{y}{x} = \tan\beta$$

when $t = T$. **M1**

Thus

$$x = b\sin\alpha - VT\cos\alpha = b\sin\alpha - V\frac{4V\sin\alpha}{g}\cos\alpha = b\sin\alpha - 4b\sin\alpha\cos^2\alpha$$

M1

and

$$y = b\cos\alpha + VT\sin\alpha - \frac{1}{2}gT^2 = b\cos\alpha + V\frac{4V\sin\alpha}{g}\sin\alpha - \frac{1}{2}g\left(\frac{4V\sin\alpha}{g}\right)^2$$

$$= b\cos\alpha + 4b\cos\alpha\sin^2\alpha - 8\,b\cos\alpha\sin^2\alpha = b\cos\alpha - 4b\cos\alpha\sin^2\alpha$$

M1

So

$$\tan\beta = 3\tan\alpha = \frac{b\cos\alpha - 4b\cos\alpha\sin^2\alpha}{b\sin\alpha - 4b\sin\alpha\cos^2\alpha}$$

M1

Rewritten. this is

$$\frac{3\sin\alpha}{\cos\alpha} = \frac{\cos\alpha\,(1 - 4\sin^2\alpha)}{\sin\alpha\,(1 - 4\cos^2\alpha)}$$

$$3\sin^2\alpha\,\big(1 - 4(1 - \sin^2\alpha)\big) = (1 - \sin^2\alpha)(1 - 4\sin^2\alpha)$$

M1

Thus

$$8(\sin^2\alpha)^2 - 4\sin^2\alpha - 1 = 0$$

So

$$\sin^2\alpha = \frac{4 \pm 4\sqrt{3}}{16} = \frac{1 \pm \sqrt{3}}{4}$$

However, $\sin^2\alpha > 0$, so $\sin^2\alpha = \frac{1+\sqrt{3}}{4}$ **A1* (6)**

12. (i) $P(Y_k \leq y)$ is the probability that at least k numbers are less than or equal to y **E1**

The probability that exactly k are smaller than or equal to y is given by

$$\binom{n}{k} y^k (1-y)^{n-k}$$

M1

So

$$P(Y_k \leq y)$$

$$= \binom{n}{k} y^k (1-y)^{n-k} + \binom{n}{k+1} y^{k+1}(1-y)^{n-k-1} + \binom{n}{k+2} y^{k+2}(1-y)^{n-k-2} + \cdots \binom{n}{n} y^n$$

$$= \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}$$

as required. **A1* (3)**

(ii)

$$m\binom{n}{m} = \frac{m \times n!}{(n-m)!\, m!} = \frac{n!}{(n-m)!\,(m-1)!} = \frac{n \times (n-1)!}{\big((n-1)-(m-1)\big)!\,(m-1)!} = n\binom{n-1}{m-1}$$

B1 **M1** **A1***

$$(n-m)\binom{n}{m} = \frac{(n-m) \times n!}{(n-m)!\, m!} = \frac{n!}{(n-m-1)!\, m!} = \frac{n \times (n-1)!}{\big((n-1)-m\big)!\, m!} = n\binom{n-1}{m}$$

M1 **A1 (5)**

$$F(y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}$$

so

$$f(y) = \frac{d}{dy} \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}$$

M1

$$= \sum_{m=k}^{n} m\binom{n}{m} y^{m-1}(1-y)^{n-m} + \sum_{m=k}^{n} -(n-m)\binom{n}{m} y^m (1-y)^{n-m-1}$$

M1

$$= \sum_{m=k}^{n} m\binom{n}{m} y^{m-1}(1-y)^{n-m} + \sum_{m=k}^{n-1} -(n-m)\binom{n}{m} y^m (1-y)^{n-m-1}$$

M1

$$= \sum_{m=k}^{n} n\binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} + \sum_{m=k}^{n-1} -n\binom{n-1}{m} y^m (1-y)^{n-m-1}$$

M1

$$= \sum_{m=k}^{n} n \binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} + \sum_{m=k+1}^{n} -n \binom{n-1}{m-1} y^{m-1}(1-y)^{n-m}$$

M1

$$= n \binom{n-1}{k-1} y^{k-1}(1-y)^{n-k}$$

as required. **A1* (6)**

Because $f(y)$ is a probability density function, $\int_0^1 f(y)dy = 1$ **E1**

Thus

$$\int_0^1 y^{k-1}(1-y)^{n-k} dy = \frac{1}{n\binom{n-1}{k-1}}$$

B1 (2)

(iii)

$$E(Y_k) = n\binom{n-1}{k-1} \int_0^1 y \times y^{k-1}(1-y)^{n-k} dy = n\binom{n-1}{k-1} \int_0^1 y^k (1-y)^{n-k} dy$$

M1

$$= n\binom{n-1}{k-1} \int_0^1 y^{k+1-1}(1-y)^{n+1-(k+1)} dy = n\binom{n-1}{k-1} \frac{1}{(n+1)\binom{n+1-1}{k+1-1}}$$

M1 **M1**

$$= n\binom{n-1}{k-1} \frac{1}{(n+1)\binom{n}{k}} = \frac{n(n-1)!}{(n-k)!\,(k-1)!} \frac{(n-k)!\,k!}{(n+1)n!} = \frac{k}{n+1}$$

A1 (4)

13.

$$G(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \cdots$$

$$G(1) = p_0 + p_1 + p_2 + p_3 + \cdots$$

$$G(-1) = p_0 - p_1 + p_2 - p_3 + \cdots$$

M1

$$G(1) + G(-1) = 2p_0 + 2p_2 + \cdots = 2(p_0 + p_2 + p_4 + \cdots) = 2P(X = 0 \text{ or } 2 \text{ or } 4 \cdots)$$

M1

Thus

$$P(X = 0 \text{ or } 2 \text{ or } 4 \cdots) = \frac{1}{2}\big(G(1) + G(-1)\big)$$

as required. **A1* (3)**

$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!}$$

$$G(t) = \sum_{r=0}^{\infty} t^r e^{-\lambda}\frac{\lambda^r}{r!} = e^{-\lambda}\sum_{r=0}^{\infty}\frac{(\lambda t)^r}{r!} = e^{-\lambda}e^{\lambda t} = e^{-\lambda(1-t)}$$

M1 **A1* (2)**

(i)

$$\sum_{r=0}^{\infty} P(Y = r) = k\,P(X = 0 \text{ or } 2 \text{ or } 4 \cdots) = k \times \frac{1}{2}\big(G(1) + G(-1)\big)$$

M1

$$= \frac{k}{2}\left(1 + e^{-2\lambda}\right) = k\frac{e^{\lambda} + e^{-\lambda}}{2e^{\lambda}} = k\frac{\cosh\lambda}{e^{\lambda}}$$

As

$$\sum_{r=0}^{\infty} P(Y = r) = 1$$

$$k = \frac{e^{\lambda}}{\cosh\lambda}$$

A1

$$G_Y(t) = \sum_{r=0}^{\infty} P(Y = r)\,t^r = k\left(e^{-\lambda} + e^{-\lambda}\frac{\lambda^2}{2!}t^2 + e^{-\lambda}\frac{\lambda^4}{4!}t^4 + \cdots\right)$$

M1

$$= ke^{-\lambda}\left(1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right)$$

$$= \frac{ke^{-\lambda}}{2}\left[\left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^4}{4!} + \cdots\right) + \left(1 - \lambda t + \frac{(\lambda t)^2}{2!} - \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^4}{4!} - \cdots\right)\right]$$

$$= \frac{e^{\lambda} e^{-\lambda}}{\cosh \lambda} \frac{e^{\lambda t} + e^{-\lambda t}}{2} = \frac{\cosh \lambda t}{\cosh \lambda}$$

M1 **A1* (5)**

as required.

$$E(Y) = G'_Y(1)$$

$$G'_Y(t) = \frac{\lambda \sinh \lambda t}{\cosh \lambda}$$

M1

Thus

$$E(Y) = \frac{\lambda \sinh \lambda}{\cosh \lambda} = \lambda \tanh \lambda < \lambda$$

M1 **A1* (3)**

for $\lambda > 0$

Alternatively,

$$E(Y) = \frac{\lambda \sinh \lambda}{\cosh \lambda} = \lambda \frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}} = \lambda \frac{1 - e^{-2\lambda}}{1 + e^{-2\lambda}} < \lambda$$

(ii)

$$G(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \cdots$$

$$G(1) = p_0 + p_1 + p_2 + p_3 + \cdots$$

$$G(-1) = p_0 - p_1 + p_2 - p_3 + \cdots$$

$$G(i) = p_0 + ip_1 - p_2 - ip_3 + \cdots$$

$$G(-i) = p_0 - ip_1 - p_2 + ip_3 + \cdots$$

$$G(1) + G(-1) + G(i) + G(-i) = 4p_0 + 4p_4 + \cdots = 4P(X = 0 \text{ } or \text{ } 4 \cdots)$$

$$\sum_{r=0}^{\infty} P(Z = r) = c\, P(X = 0 \text{ } or \text{ } 4 \cdots) = c \times \frac{1}{4}\big(G(1) + G(-1) + G(i) + G(-i)\big)$$

$$= \frac{c}{4}\big(1 + e^{-2\lambda} + e^{-\lambda(1-i)} + e^{-\lambda(1+i)}\big)$$

$$= \frac{c}{4}\big(1 + e^{-2\lambda} + e^{-\lambda}(\cos \lambda + i \sin \lambda + \cos \lambda - i \sin \lambda)\big)$$

$$= \frac{c}{4} \frac{\left(e^{\lambda} + e^{-\lambda} + 2\cos \lambda\right)}{e^{\lambda}} = \frac{c(\cosh \lambda + \cos \lambda)}{2e^{\lambda}}$$

M1

As $\sum_{r=0}^{\infty} P(Z = r) = 1$, $c = \frac{2e^{\lambda}}{\cosh \lambda + \cos \lambda}$ **A1**

$$G_Z(t) = \sum_{r=0}^{\infty} P(Z = r)\, t^r = c\left(e^{-\lambda} + e^{-\lambda}\frac{\lambda^4}{4!}t^2 + e^{-\lambda}\frac{\lambda^8}{8!}t^8 + \cdots\right)$$

$$= ce^{-\lambda}\left(1 + \frac{(\lambda t)^4}{4!} + \frac{(\lambda t)^8}{8!} + \cdots\right)$$

$$= \frac{ce^{-\lambda}}{4}\left[\left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots\right) + \left(1 - \lambda t + \frac{(\lambda t)^2}{2!} - \cdots\right) + \left(1 + \lambda i t - \frac{(\lambda t)^2}{2!} + \cdots\right) + \left(1 - \lambda i t - \frac{(\lambda t)^2}{2!} + \cdots\right)\right]$$

$$= \frac{ce^{-\lambda}}{4}\left(e^{\lambda t} + e^{-\lambda t} + e^{i\lambda t} + e^{-i\lambda t}\right)$$

$$= \frac{ce^{-\lambda}}{2}(\cosh \lambda t + \cos \lambda t)$$

M1

As $G_Z(1) = 1$, $c = \frac{2e^{\lambda}}{\cosh \lambda + \cos \lambda}$

So

$$G_Z(t) = \frac{(\cosh \lambda t + \cos \lambda t)}{(\cosh \lambda + \cos \lambda)}$$

A1ft

$$G'_Z(t) = \frac{\lambda(\sinh \lambda t - \sin \lambda t)}{(\cosh \lambda + \cos \lambda)}$$

And thus

$$E(Z) = G'_Z(1) = \frac{\lambda(\sinh \lambda - \sin \lambda)}{(\cosh \lambda + \cos \lambda)}$$

M1

If $\lambda = \frac{3\pi}{2}$, **M1**

$$E(Z) = \frac{3\pi}{2}\frac{\left(\sinh \frac{3\pi}{2} + 1\right)}{\cosh \frac{3\pi}{2}}$$

$$= \frac{3\pi}{2} \times \frac{\frac{e^{\frac{3\pi}{2}} - e^{\frac{-3\pi}{2}}}{2} + 1}{\frac{e^{\frac{3\pi}{2}} + e^{\frac{-3\pi}{2}}}{2}} = \frac{3\pi}{2} \times \frac{\frac{e^{\frac{3\pi}{2}} + e^{\frac{-3\pi}{2}}}{2} + \left(1 - e^{\frac{-3\pi}{2}}\right)}{\frac{e^{\frac{3\pi}{2}} + e^{\frac{-3\pi}{2}}}{2}}$$

As $e^{\frac{-3\pi}{2}} < 1$, in this case, $E(Z) > \lambda$ and so no, $E(Z)$ is not less than λ for all positive values of λ

A1 (7)

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