APM 4663/5663—Fall 2024
Assignment #2
Due date: Sunday, October 13, 2024

100% for graduate students: 34 pts. 100% for undergraduate students: 30 pts.

1. (8 pts.) Prove that if G is a simple graph on n vertices such that $\deg_G(v) \geq \lceil \frac{n+1}{2} \rceil$ for every vertex $v \in V(G)$, then deleting any two vertices of G results in a connected graph.

   Prove that this result is sharp for $n \geq 4$ by showing that if the lower bound $\lceil \frac{n+1}{2} \rceil$ on the degree is replaced by $\lceil \frac{n+1}{2} \rceil - 1$ in the assumption, then the conclusion of the claim does not necessarily hold. Give a counterexample for each n.

2. (8 pts.) Let $\delta \geq 2$. Prove that every simple graph G satisfying $\delta_{\min}(G) \geq \delta$ and containing no triangles contains a cycle of length at least 2δ.

   Prove that this result is sharp by showing that we cannot guarantee the existence of a cycle of length at least $2\delta + 1$. Give a counterexample for each δ.

3. (8 pts.) Show that for every simple connected graph G, if G has an Eulerian tour, then its *line graph* $L(G)$ also has an Eulerian tour. *(The* line graph *of G is defined as follows: $L(G)$ is the simple graph with vertex set $E(G)$ such that any two distinct vertices $e, f \in E(G)$ of $L(G)$ are joined by an edge in $L(G)$ if and only if e and f have a common endpoint in G.)*
   Find a simple connected graph G with at least three edges that has no Eulerian tour, but for which $L(G)$ has an Eulerian tour.

4. (6 pts.) Prove that if the complete graph K_n can be decomposed into 5-cycles (i.e., each edge of K_n appears in exactly one of the 5-cycles of the decomposition), then $n-1$ or $n-5$ is divisible by 10.

5. (6 pts.) Show that every loopless graph G contains a tripartite spanning subgraph H such that $\deg_H(v) \geq \frac{2\deg_G(v)}{3}$ for every vertex $v \in V(G)$. *(A tripartite graph is a graph whose vertices can be partitioned into three disjoint sets such that every edge joins two vertices in different parts.)*

6. (4 pts.) (Optional) Make an appointment with me to discuss which problem from Assignment #1 to generalize and in what direction. Make a conjecture by generalizing the chosen problem. Try to prove or disprove your conjecture (explain how far you got, and whether you believe that the conjecture is true or not).

Students choosing the Computer Science Option should do problems 1–3 and the following:

1. (12 pts.) Write a computer program that finds a set of binary codewords corresponding to a set of symbols occurring with a given frequency by Huffman's algorithm. Specifications of the program:

   Input: The input symbols and frequencies are given in a file in the following format: the first line contains just the number of symbols, and then the symbols and frequencies are listed, each line containing first the symbol then its frequency. Sample files can be found in Moodle.

   Output: The program should print into a file the binary codewords corresponding to each symbol in the same order; first line should explain the content of each line; then each line should contain a symbol then a space, then its codeword. The last line should say what the average number of bits used is per symbol.

   E-mail me the source code of the program as a simple text file and the resulting outputs for the given sample inputs. In addition, e-mail me or hand in documentation of the program (this may partially be in the source code), explaining the data structures and the main variables.