# Regular Assignment 1 Clean

Sunday, September 29, 2024 7:51 PM

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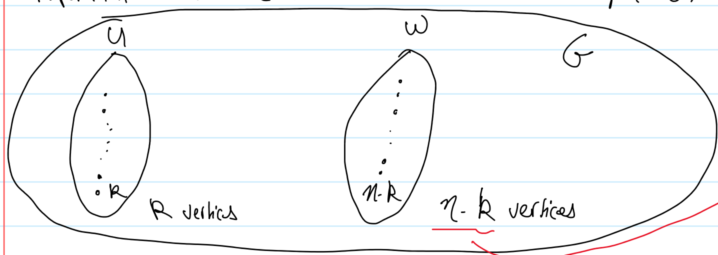
(1)

Claim: Prove the Number of edges in every simple Bipartite graph on $n \geq 2$ vertices is at Most $\lfloor n^2/4 \rfloor$

Pf: Assume we have $k$ vertices on 1 side of Bipartition with $n$ total vertices in our Graph $G$.




Co-Pilot Tip/hint

The Maximum # of edges is when $v \in |U|$ connects to every $v \in |W|$. This would be equal to $k \cdot (n-k)$. Now, Maximize this expression

$$\frac{d}{dk}\left(k \cdot (n-k)\right) \rightarrow kn - k^2 \rightarrow \frac{d}{dk} \rightarrow n - 2k$$

Statisticshowto.com to help with how to maximize a fn

Now, we set this equal to 0, and find the Critical Points

$$n - 2k = 0 \rightarrow n = 2k \rightarrow k = n/2$$

Sub this back into our original expression

$$k \cdot (n-k) \rightarrow n/2 \cdot \underbrace{(n - n/2)}_{\rightarrow n/2} \rightarrow n/2 \cdot n/2 \rightarrow \boxed{n^2/4}$$

Now, why do we round down?

$\rightarrow$ edge # must be a whole number, so we must Round

$\rightarrow$ we Round down because it is "at most."

thus, $n^2/4 \rightarrow \lfloor n^2/4 \rfloor$ $\square$

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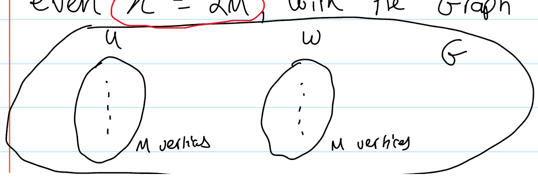
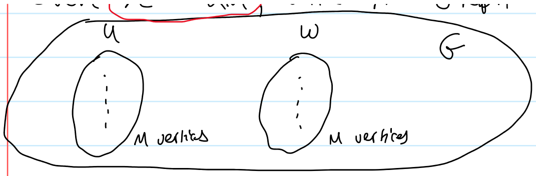
Is this Sharp?

Well, lets show an even and odd $n$ that test this upper bound.

Co-Pilot Hint

even $n = 2m$, with the Graph $G$:



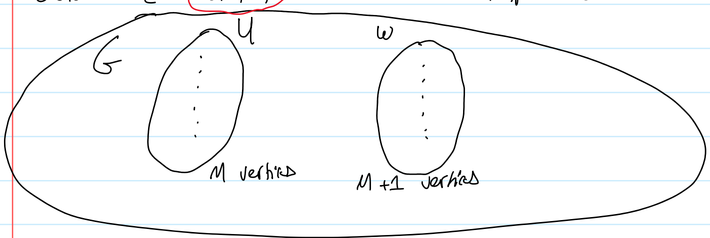



The # of edges must be $m \cdot m = \underline{m^2}$

Using the Constraint of $\lfloor n^2/4 \rfloor$, and with $n = 2m$, we get $(2m)^2/4 = 4m^2/4 = m^2/1 = \underline{m^2}$, which is good for our G graph above ✓

Now, check odd → Co-Pilot Hint

odd: $n = 2m+1$ vertices in Graph G




# of edges would be $m \cdot m+1 \to \underline{m^2+m}$.

Using the Constrain $\lfloor n^2/4 \rfloor$, with $n = 2m+1$, we get

$$\left\lfloor \frac{(2m+1)^2}{2} \right\rfloor \to \left\lfloor \frac{(4m^2+4m+2)}{2} \right\rfloor \to \left\lfloor m^2 + m + \frac{1}{4} \right\rfloor$$

$\to 4m^2+4m+2$

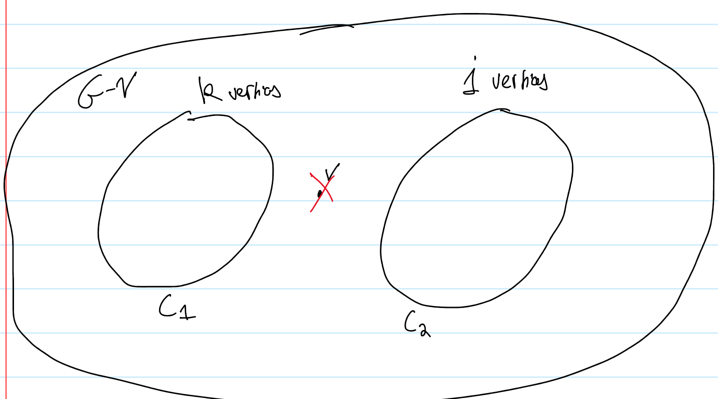
→ Now, this round down means we can drop the fraction $1/4$, leaving us with $\underline{m^2+m}$, which matches our above graph G ✓

(2) Claim: Prove that if $G$ is a simple graph on at least $n \geq 3$ vertices s.t. $\deg_G(v) \geq \lceil n/2 \rceil$ for every vertex $v \in V(G)$, then the Graph $G-v$ is connected for every vertex $v$ of $G$.

Pf: Using Contradiction

Assume: $G$ is simple with $n \geq 3$ vertices, and $\deg_G(v) \geq \lceil \frac{n}{2} \rceil$ for every $v \in V(G)$

W.T.S. $\exists v \in V(G)$ s.t. $G-v$ is <u>Not</u> Connected.




our $G-v$ graph must have 2 components, $C_1, C_2$ to be not connected. We also know we have $-1$ vertices, so lets say $n-1$ total vertices.

$n-1 = k+j$ vertices, and $\deg(v) \geq \lceil n/2 \rceil$ in $G$

WLOG: Assume $k \geq j$. $\rightarrow$ $k \geq \lceil n/2 \rceil$ GPT Really helped to remind me about WLOG Principle!

$n-1 = k+j \rightarrow j = (n-1)-k \rightarrow j \leq (n-1) - \lceil n/2 \rceil$

$n-1 = k+j \rightarrow j = (n-1)-k \rightarrow j \le (n-1) - \lceil n/2 \rceil$

$\hookrightarrow$ If $k \ge \lceil n/2 \rceil$, then $k$ is at least half or more of $n-1$ GPT Hint

But, $j \le (n-1)-k$ means that $j$ is strictly half or nearly half of $n-1$, So,

$k+j = n-1$ cannot work if $k$ is sufficiently large ↯

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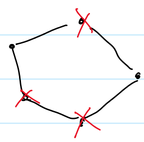
Is it Sharp?

Need to Test lower bound, likely with $n$ for odd + even with $n$ s.t. $\deg_G(v) \ge \lceil n/2 \rceil - 1$.

odd $n = 5$ vertices, our deg is $\lceil n/2 \rceil = \frac{5}{2}\uparrow = 3-1 = \underline{2}$

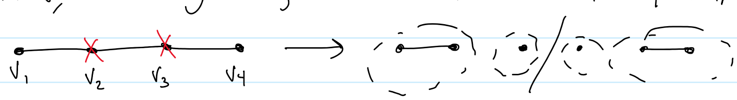
5 vertices and deg 2 $\rightarrow$ $C_5$ graph



, we can delete any vertex and we are still connected. ✗

Even $n=4 \rightarrow$ deg constraint is $\lceil \frac{4}{2} \rceil - 1 \rightarrow 2-1 = \underline{1}$

4 vertices, and deg $\ge 1$, we can look at a $P_4$ Graph




If we cut out either of the $v_2, v_3$ vertices, we no longer have a connected graph, so the $-1$ does not work. So, $\lceil n/2 \rceil$ is Sharp. ✓

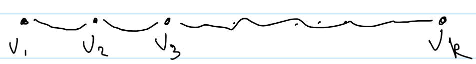
(3) Claim: let $\delta \geq 2$ be any Integer. Prove that every Simple Graph $G$, Satisfying $\delta_{min}(G) \geq \delta$ has a Cycle of length at least $\delta + 1$.

Assume the Path $P_1 = v_1, v_2, v_3, \ldots v_k$ to be the longest Path in our Simple Graph $G$.

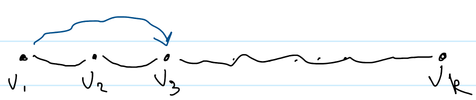
GPT asked me "what can you assume about the neighbors?" which helped!

$$P_1 = v_1 \; v_2 \; v_3 \; v_4 \; \ldots \; v_{k-1} \; v_k$$

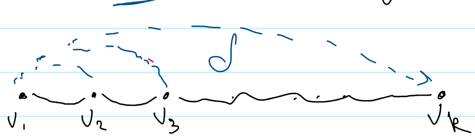
All of the endpoints $(v_1\, v_k)$ neighbors Must already reside on the given path, otherwise, we could extend this Path from the endpoint to its neighbor. This new Path would be called $P_2$, but it would have a length longer than $P_1$, so this can't be. So, any neighbors of $v_1$ must be on the Path $P_1$.



From $\delta_{min}(G) \geq \delta$, and $\delta \geq 2$, so $v_1$ Must have at least 2 neighbors, so, there must be at least A cycle of length 2.



But, we know $v_1$ Must have $\delta$ neighbors from Constraints, and if is Simple G (No Parallel edges) so, $v_1$ Must have an edge At least $\delta$ vertices away.



Thus, At worst, If the longest Path away was $\delta$ vertices, it would be a Cycle of $\delta + 1$ length

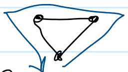
$\square$

Is it Sharp?

Show this lower bound of length, at least $\delta+1$ for cycle length.

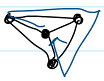
Well, lets start with some specific cases for $\delta$.

$\delta = 2$, min degree $= 2$



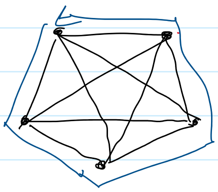
$\rightarrow$ A Cycle length $= \underline{3} = \delta+1$

$\delta = 3$



$\rightarrow$ A Cycle length is $\underline{4} = \delta+1$

$\delta = 4$



$\rightarrow$ A cycle of length $5 = \delta+1$

GPT helped!

Now, in General, and Complete Graph $K_\delta$ will have a cycle of at least length $\delta+1$ ✓

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④ Claim: Prove that in every connected graph, every two paths of maximum length have a common vertex.

Pf: Contradiction

Assume $\Rightarrow$ Connected graph, so there is a path between any 2 vertices.

(Assume ⇒ Connected graph, so there is a path between any 2 vertices.

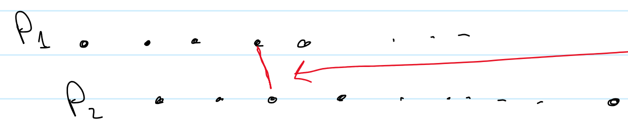
Assume ⇒ we have connected graph, and two paths of maximum length

Q: Contradiction ⇒ They do not share a common vertex

↳ Pf: Two paths, longest, $P_1$, $P_2$

It's a connected graph, so every vertex in $P_1$ must have a path to every vertex in $P_2$, and vice versa.

So, $P_1$ and $P_2$ must be connected via some edge (From Assumptions)




However, now, you have a path that is the length of $P_1$ and $P_2$ $\boxed{+1}$, thus $P_1$ and $P_2$ must not have been the maximum length paths ↯