Regular Assignment 2 Clean

Sunday, October 13, 2024 7:54 PM

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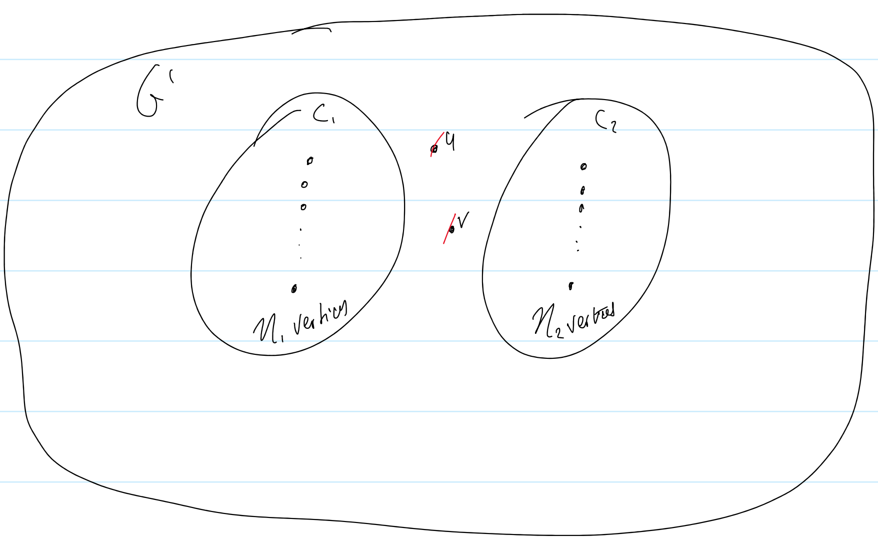
1) Prove that if G is a simple Graph on n vertices that $\deg_G(v) \geq \lceil \frac{n+1}{2} \rceil$ for every vertex $v \in V(G)$, then deleting any 2 vertices in G results in a Connected Graph.

Pf: Contradiction

Assume we have a simple Graph G, on n vertices where every vertex $v \in V(G)$ has $\deg_G(v) \geq \lceil \frac{n+1}{2} \rceil$.
Assume there exists 2 vertices u, v s.t. when we delete these from $V(G)$, the resulting Graph, G', is disconnected

W.T.S. $\Rightarrow$ Contradiction

Start with disconnected G'



G' must have $n-2$ total vertices, so, $n_1 + n_2 = n-2$. With deg constraint of $\deg_G(v) \geq \lceil \frac{n+1}{2} \rceil$, then each $v \in V(C_1)$ must have $\lceil \frac{n+1}{2} \rceil$ neighbors. GPT Hint → The total degree in C_1 must be $|C_1| \cdot \lceil \frac{n+1}{2} \rceil$, so, we have $n_1 \cdot \lceil \frac{n+1}{2} \rceil$ degree in C_1, edges # are $n(n-1)/2$, via property,

So, $n_1 \cdot \lceil \frac{n+1}{2} \rceil \leq \cancel{2} \cdot \frac{n_1(n_1-1)}{\cancel{2}}$

↳ edges count twice for each endpoint.

↳ Combining with n_1 we get

$n_1 \cdot \lceil \frac{n+1}{2} \rceil \leq n_1(n_1-1)$, GPT Hint → Divide by n_1 (Assume $n_1 \geq 0$) → we have now

↳ $\lceil \frac{n+1}{2} \rceil \leq n_1 - 1$

$n-2 = n_1 + n_2 \rightarrow n_2 = n-2-n_1$

but, $n_2 \geq 0$, so → $0 \leq n-2-n_1 \rightarrow n_1 \leq n-2$

↳ $n-2 \geq n_1 \geq \lceil \frac{n+1}{2} \rceil + 1$

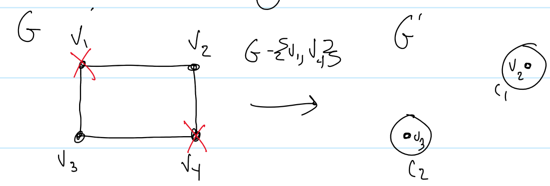
↳ $n-2 \geq \lceil \frac{n+1}{2} \rceil + 1 \rightarrow n \geq \lceil \frac{n+1}{2} \rceil + 3$

→ This gives us a lower limit on the n vertex count. ↯

$\square$

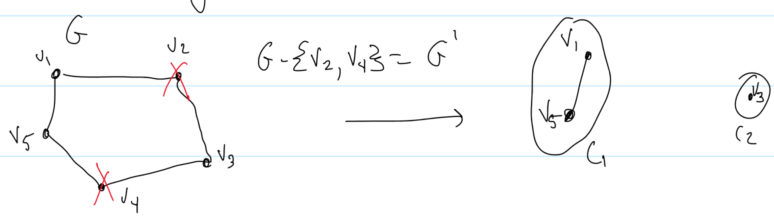
Sharpness: Prove this Result is sharp for $n \geq 4$ by showing lower bound $\lceil \frac{n+1}{2} \rceil$ on degree is replaced by $\lceil \frac{n+1}{2} \rceil - 1$ in Assumption:

$n = 4 \rightarrow \deg = \lceil \frac{4+1}{2} \rceil - 1 \rightarrow 3 - 1 = 2$



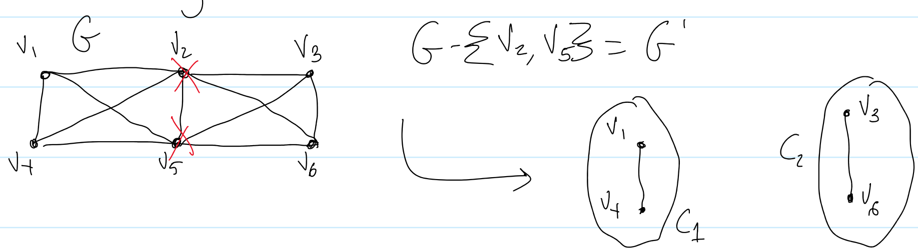
disconnected ✓

$n = 5 \rightarrow \deg = \lceil \frac{5+1}{2} \rceil - 1 = 3 - 1 = 2$



disconnected ✓

$n = 6 \rightarrow \deg = \lceil \frac{7}{2} \rceil = 4 - 1 = 3$



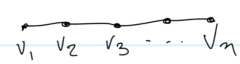
Same for $n = 7, 8, \ldots$

disconnected ✓

2) Let $\delta \geq 2$, Prove that every Simple Graph G Satisfying $\delta_{min}(G) \geq \delta$ and Containing No triangles Contains a Cycle of length $\geq 2\delta$

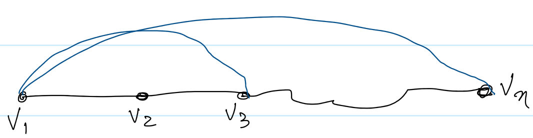
Pf: Lets Define Path P in G that is a longest Path

$P =$



V_1 Must have $\deg \geq \delta$, So it Must have other vertices incident to it. But, since this is longest Path, it Must be incident to vertices along Path Somewhere, otherwise this would not be the longest Path.

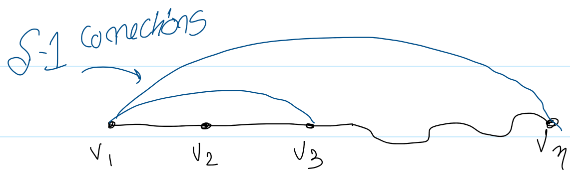
$P =$



Also, due to triangle Constraint, It Cannot be incident to a vertex along Path that would create a triangle. This also creates our Cycle, $\underline{C}$, which would Contain The Path P, + whatever edge Conects V_1 to some $v \in P$ to satisfy degree Constraint.

Now, to show length of Cycle, GPT Help →

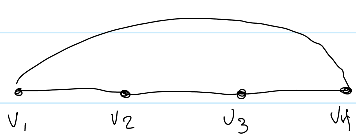
Since each vertex, due to Min degree Constraint, allows Multiple Connections from v_1. But, you must skip every other vertex to avoid Triangles → so, with these $\delta-1$ additional edges, each skipping a vertex, we should get a length that is δ away, twice over, for a total Cycle length of $\geq 2\delta$. $\square$



Sharpness: Show we C/N guarantee the existence of a cycle of length $\geq 2\delta+1$.

Well, look at $\delta=2$:

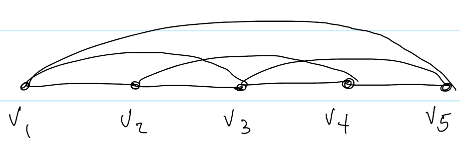
$2\delta+1 = \underline{5}$



→ Max Cycle is 4, Not 5 ✓

$\delta=3$:

$2\delta+1 = 7$



Max Cycle is 5, Not 7 ✓

3) A) Show that for every simple connected graph G, if G has a Eulerian Tour, then its line graph $L(G)$ also has a Eulerian Tour.

→ Pf: W.T.S. The $L(G)$ is connected, and every vertex has an even degree.

$L(G)$ is connected: →

Since G is connected, there exists a path between any 2 vertices $v, u \in V(G)$. So, every edge $e \in E(G)$ can be reached from any other edge too. GPT Hint → Consider e_1, e_2, If they share a common vertex, v, then in the $L(G)$, the vertices corresponding to e_1, e_2 are adjacent. And, since G is connected, you can find a path that can visit any edge, meaning every edge in G can be connected to every edge in $L(G)$, via shared vertices, thus, $L(G)$ is connected.

Okay, so $L(G)$ is connected, now we need to show each $v \in V(L(G))$ has even degree, which would show it has a Eulerian Tour.

lets look at edge e in G, with endpoints u, v. The degree of the Corresponding vertex for e in L(G) is determined by the # of edges incident to u and v. GPT Hint → lets say u has degree d_u, and v has d_v, then we can show:

$$\deg_{L(G)}(e) = (d_u - 1) + (d_v + 1) = d_u + d_v - 2$$

What does that tell us about it being even, odd?

Okay, well every vertex must have even degree in G, by Assumption (Eulerian Tour), so we know that d_v and d_u must also be even in L(G) →

$d_v + d_u$ → still even

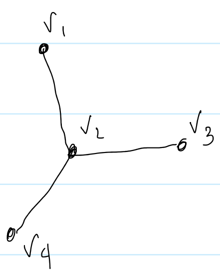
-2 → still even

So in L(G), each vertex must have an even degree, thus, it has a Eulerian Tour

□

B) Find a Simple Connected Graph G with $|E(G)| \geq 3$ that has no Eulerian Tour, But, L(G) does have one.

→ w.t.s. G should have $\exists v \in V(G)$ with odd degree



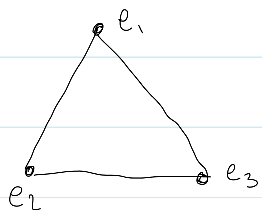
$\deg_G(v_2) = 3 = $ odd

→ No Eulerian Tour in G

$L(G) \rightarrow$ $e_1 = v_1 v_2$

$e_2 = v_2 v_3$

$e_3 = v_4 v_2$



Cycle $\rightarrow$ $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_1$

all even deg

✓