APM 4663/5663—Fall 2024
Assignment #4
First due date: Sunday, Dec 1, 2024

100% for graduate students: 36 pts. 100% for undergraduate students: 30 pts.

1. (6 pts.) Show that every simple, triangle-free, planar graph is 4-colorable. *(Hint: show first that such a graph has a vertex of degree at most* 3.*)*

2. (12 pts.) Consider the formal power series $f(x) = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ and $g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. Compute *by hand* the first five coefficients (i.e., up to the coefficient of x^4) of the following formal power series (solution must be hand-written and not typed, indicate partial results):
   (a) $h(x) = e^{f(x)}$,
   (b) $k(x) = \log(1 + g(x))$,
   (c) $m(x) = (h \circ k)(x)$.

3. (6 pts.) Let n, k be positive integers. Use generating functions to find the number of integral solutions to the inequality $x_1 + x_2 + \cdots + x_k \leq n$ such that $x_i \geq 2i$ for all $i = 1, 2, \ldots, k$.

4. (8 pts.) Let the sequence $\{a_n\}$ be defined by the recurrence relation $a_{n+2} = 3a_{n+1} - 2a_n + 3$ for $n \geq 0$ with initial conditions $a_0 = a_1 = 1$, and let $\{a_n\}_{n=0}^{\infty} \overset{\text{ops}}{\longleftrightarrow} A(x)$. Find $A(x)$, the partial fraction decomposition of $A(x)$, and derive a closed formula for a_n.

5. (6 pts.) Find the generating function with respect to length of the set of $\{0, 1\}$-strings not containing the substring 0110.

6. (6 pts.) Show that the number of partitions of n in which parts can occur at most seven times but not exactly once or six times is equal to the number of partitions of n in which all parts give remainder 2, 3, or 4 when divided by 6.

7. (6 pts.) For every positive integer m, find the number of planted plane trees on $2m+1$ vertices in which the number of children (downward branches) at every vertex is even.

Students choosing the Computer Science Option should do problems 1–4, and

1. (18 pts.) Write a computer program that computes the first $m + 1$ coefficients of the k^{th} power, the reciprocal, and the inverse of a given power series $f(x) = \sum_{i=0}^{n} a_i x^i$ (note that for a given $f(x)$ some of these will not exist). Specifications of the program:

   Input: The input is given in a file in the following format: the first line contains just the values of n, k, and m in this order (these are nonnegative integers), then the next $n+1$ lines contain the first $n+1$ coefficients (i.e. $a_0, a_1, \ldots, a_n$, which are **real** numbers) of the power series $f(x)$. Sample files will be found in Moodle.

   Output: The program should print into a file whether the reciprocal or the inverse exists, then print the first $m + 1$ coefficients of each of the power series $f^k(x)$, $\frac{1}{f(x)}$, and $f^{-1}(x)$, whichever exists (the coefficients are real numbers, so use at least 8-digit precision).

   E-mail me the source code of the program and the resulting outputs for the given sample inputs as simple text files. In addition, e-mail me documentation of the program (this may partially be in the source code), explaining the data structures, main variables, and the **algorithm you used**.