APM 4663/5663—Fall 2024—Assignment #1

First due date: Sunday, September 29, 2024

100% for graduate students: 44 pts. 100% for undergraduate students: 40 pts.

1. (6 pts.) Prove that the number of edges in every simple bipartite graph on $n \geq 2$ vertices is at most $\lfloor n^2/4 \rfloor$.

   (2 pts.) Show that this upper bound on the number of edges is sharp by giving an example of a graph for each n that achieves the bound.

2. (6 pts.) Prove that if G is a simple graph on $n \geq 3$ vertices such that $\deg_G(v) \geq \lceil \frac{n}{2} \rceil$ for every vertex $v \in V(G)$, then the graph $G - v$ is connected for every vertex v of G.

   (2 pts.) Show that this lower bound on the degrees is necessary by giving an example of a graph for each n such that $\deg_G(v) \geq \lceil \frac{n}{2} \rceil - 1$ for every vertex $v \in V(G)$, and G has a vertex whose deletion results in a disconnected graph.

3. (6 pts.) Let $\delta \geq 2$ be any integer. Prove that every simple graph G satisfying $\delta_{\min}(G) \geq \delta$ has a cycle of length at least $\delta + 1$.

   (2 pts.) Show that this lower bound on the length of the cycle we can find in G is sharp by giving an example of a simple graph for each δ such that $\delta_{\min}(G) \geq \delta$ and the longest cycle in G has length $\delta + 1$.

4. (6 pts.) Prove that in every connected graph every two paths of maximum length must have a common vertex.

5. Let n be any positive integer. We say that a graph G can be *decomposed into triangles* if and only if there is a set of subgraphs of G such that each subgraph is isomorphic to K_3, and each edge of G appears in exactly one of the subgraphs.

   (a) (2 pts.) Find an explicit decomposition into triangles of K_7 and of K_9. *(Hint: number the vertices, and provide only the triples of the decompositions.)*

   (b) (6 pts.) Prove that if the complete graph K_n can be decomposed into triangles, then $n - 1$ or $n - 3$ is divisible by 6.

6. (6 pts.) Show that every loopless graph G has a bipartite spanning subgraph H with the property that $\deg_H(v) \geq \frac{\deg_G(v)}{2}$ for every vertex $v \in V(G)$.

Students choosing the Computer Science Option should do problems 1–4 and the following problem:

1. (14 pts.) Write a computer program to find an Eulerian tour or trail in a graph.

   Input: The input graph is given in a file in the following format: the first line contains just the number of vertices (which are assumed to be labeled with the numbers 1 up to the number of vertices), and then the edges are listed by specifying the two endpoints (each line contains info about one edge, loops and multiple edges are possible). Sample files can be found in Moodle.

   Output: The program should print into a file whether the graph has an Eulerian trail or not (use complete English sentences). If it has, print the vertices in an order that gives an Eulerian trail as well, and specify whether it is closed or not. If the graph has no Eulerian trail, give reasons why (e.g., graph has edges in two different components; more than two vertices have an odd degree).

   E-mail me the source code of the program as a simple text file and the resulting outputs for the given sample inputs. In addition, e-mail me documentation of the program (this may partially be in the source code), explaining the data structures, the main variables, and how your program finds the Eulerian tour/trail.