Regular Assignment 1 Clean

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Matthew P. Horvath JR.

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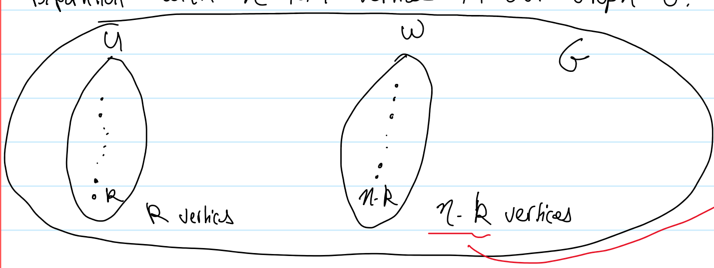
Claim: Prove the number of edges in every simple bipartite graph on $n \geq 2$ vertices is at most $\lfloor n^2/4 \rfloor$

Pf: Assume we have k vertices on 1 side of bipartition with n total vertices in our graph G.



Co-Pilot Tip/hint

The maximum # of edges is when $v \in |U|$ connects to every $v \in |W|$. This would be equal to $k \cdot (n-k)$. Now, maximize this expression

$$\frac{d}{dk}\left(k\cdot(n-k)\right) \to kn - k^2 \to \frac{d}{dk} \to n - 2k$$

Statisticshowto.com to help with how to maximize a fn

Now, we set this equal to 0, and find the critical points

$$n - 2k = 0 \to n = 2k \to k = n/2$$

Sub this back into our original expression

$$k\cdot(n-k) \to n/2 \cdot (\underbrace{n - n/2}_{\to\, n/2}) \to n/2 \cdot n/2 \to \boxed{n^2/4}$$

Now, why do we round down?

→ edge # must be a whole number, so we must round

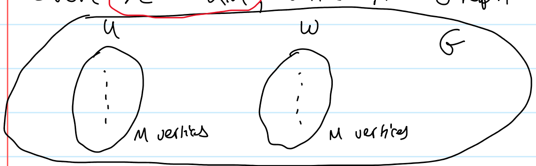
→ we round down because it is "at most."

thus, $n^2/4 \to \lfloor n^2/4 \rfloor$ □

Is this Sharp?
Well, lets show an even and odd n that test this upper bound.

Co-Pilot Hint

even $n = 2m$, with the Graph G:

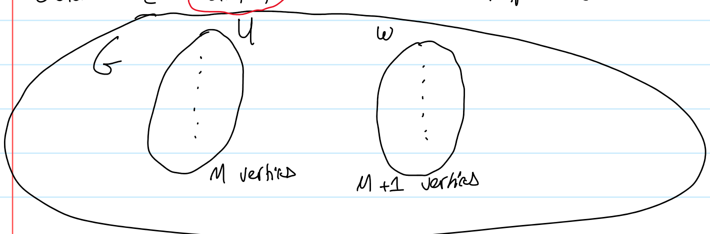


The # of edges must be $m \cdot m = \underline{m^2}$
Using the Constraint of $\lfloor n^2/4 \rfloor$, and with $n = 2m$,
we get $(2m)^2/4 = 4m^2/4 = m^2/1 = \underline{m^2}$, which is good for our G graph above ✓

Now, check odd

Co-Pilot Hint

odd: $n = 2m+1$ vertices in Graph G



# of edges would be $m \cdot m+1 \rightarrow \underline{m^2 + m}$.
Using the Constrain $\lfloor n^2/4 \rfloor$, with $n = 2m+1$, we get

$$\left\lfloor \frac{(2m+1)^2}{2} \right\rfloor \rightarrow \left\lfloor \frac{(4m^2+4m+2)}{2} \right\rfloor \rightarrow \left\lfloor m^2 + m + 1/4 \right\rfloor$$

$(2m+1)^2 \rightarrow 4m^2+4m+2$

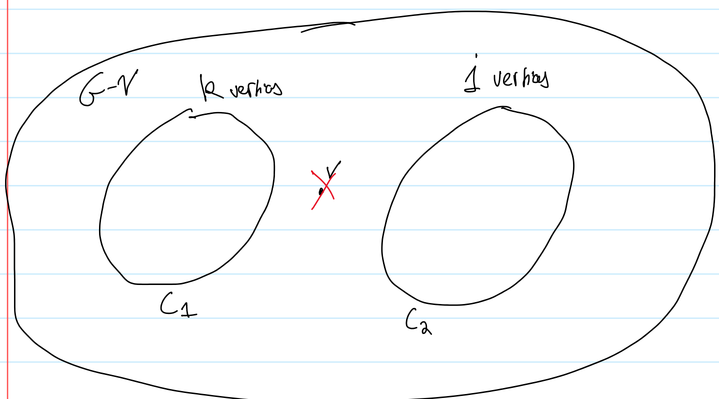
$\rightarrow$ Now, this round down means we can drop the fraction $1/4$, leaving us with $\underline{m^2+m}$, which matches our above graph G ✓

2 Claim: Prove that if G is a simple graph on at least $n \geq 3$ vertices s.t. $\deg_G(v) \geq \lceil n/2 \rceil$ for every vertex $v \in V(G)$, then the Graph $G-v$ is connected for every vertex v of G.

Pf: Using Contradiction

Assume: G is simple with $n \geq 3$ vertices, and $\deg_G(v) \geq \lceil \frac{n}{2} \rceil$ for every $v \in V(G)$

W.T.S. $\exists v \in V(G)$ s.t. $G-v$ is <u>Not</u> Connected.



our $G-v$ graph Must have 2 Components, C_1, C_2 to be not Connected. We also know we have -1 vertices, so lets say $n-1$ total vertices.

$n-1 = k+j$ vertices, and $\deg(v) \geq \lceil n/2 \rceil$ in G

WLOG: Assume $k \geq j. \rightarrow k \geq \lceil n/2 \rceil$ } GPT Really helped to remind me about WLOG Principle!

$$n-1 = k+j \rightarrow j = (n-1) - k \rightarrow j \leq (n-1) - \lceil n/2 \rceil$$

$\hookrightarrow$ If $k \geq \lceil n/2 \rceil$, then k is at least half or more of $n-1$ } GPT Hint

But, $j \leq (n-1) - k$ Means that j is strictly half or nearly half of $n-1$, so,

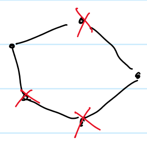
$k+j = n-1$ cannot work if k is sufficiently large $\lightning$

Is it Sharp?

Need to Test lower bound, likely with n for odd + even with n s.t. $\deg_G(v) \geq \lceil n/2 \rceil - 1$.

odd $n = 5$ vertices, our deg is $\lceil n/2 \rceil = \frac{5}{2}\uparrow = 3 - 1 = \underline{2}$

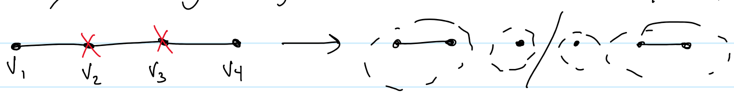
5 vertices and deg 2 $\rightarrow$ C_5 graph



, we can delet any vertex and we are still Conected. X

Even $n = 4 \rightarrow$ deg constraint is $\lceil \frac{4}{2} \rceil - 1 \rightarrow 2 - 1 = \underline{1}$

4 vertices, and deg ≥ 1, we can look at a P_4 Graph



If we cut out either of the v_2, v_3 vertices, we no longer have a Conected graph, so the -1 does not work. So, $\lceil n/2 \rceil$ is Sharp. ✓

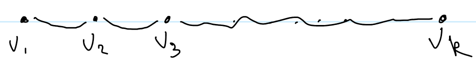
(3) Claim: let $\delta \geq 2$ be any Integer. Prove that every Simple Graph G, Satisfying $\delta_{min}(G) \geq \delta$ has a Cycle of length at least $\delta + 1$.

Assume the Path $P_1 = v_1, v_2, v_3, \cdots v_k$ to be the longest Path in our Simple Graph G.

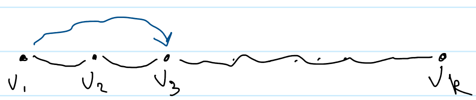
GPT asked me "what can you assume about the neighbors?" which helped!

$P_1 = v_1 \; v_2 \; v_3 \; v_4 \; \ldots \; v_{k-1} \; v_k$

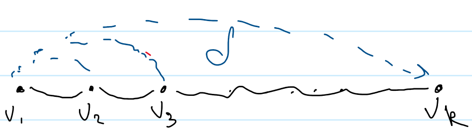
All of the endpoints (v_1, v_k) neighbors must already reside on the given path, otherwise, we could extend this path from the endpoint to its neighbor. This new Path would be called P_2, but it would have a length longer than P_1, so this can't be. So, any neighbors of V_1 must be on the Path P_1.



From $\delta_{min}(G) \geq \delta$, and $\delta \geq 2$, so V_1 must have at least 2 neighbors, so, there must be at least A cycle of length 2.



But, we know V_1 must have δ neighbors from constraints, and it is simple G (No Parallel edges) so, V_1 must have an edge At least δ vertices away.



Thus, At worst, If the longest Path away was δ vertices, it would be a Cycle of $\delta + 1$ length

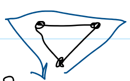
$\square$

Is it Sharp?

Show this lower bound of length, at least $\delta+1$ for cycle length.

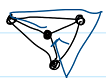
Well, lets start with some specific cases for δ.

$\delta = 2$, Min degree $= 2$



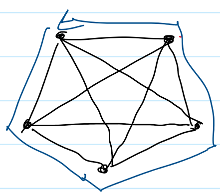
$\rightarrow$ A Cycle length $= \underline{3} = \delta + 1$

$\delta = 3$



$\rightarrow$ A Cycle length is $\underline{4} = \delta + 1$

$\delta = 4$



$\rightarrow$ A cycle of length $5 = \delta + 1$

GPT helped!

Now, in General, and Complete Graph K_δ will have a Cycle of at least length $\delta + 1$ ✓

(4) Claim: Prove that in every connected graph, every two paths of maximum length have a common vertex.

Pf: Contradiction

Assume $\Rightarrow$ Connected graph, so there is a path between any 2 vertices.

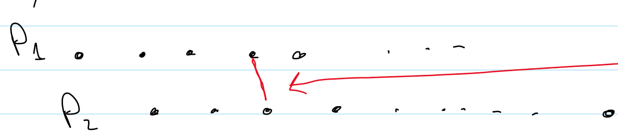
Assume $\Rightarrow$ we have connected graph, and two paths of maximum length

Q: Contradiction $\Rightarrow$ They do not share a common vertex

$\hookrightarrow$ Pf: Two paths, largest, P_1, P_2

Its a connected graph, so every vertex in P_1 must have a path to every vertex in P_2, and vice versa.

So, P_1 and P_2 must be connected via some edge (From Assumptions)



However, now, you have a path that is the length of P_1 and P_2 $\boxed{+1}$, thus P_1 and P_2 must not have been the maximum length paths ↯