STA 2002, Summer 2024
Probability and Statistics II

Chi-square Goodness-of-fit Test

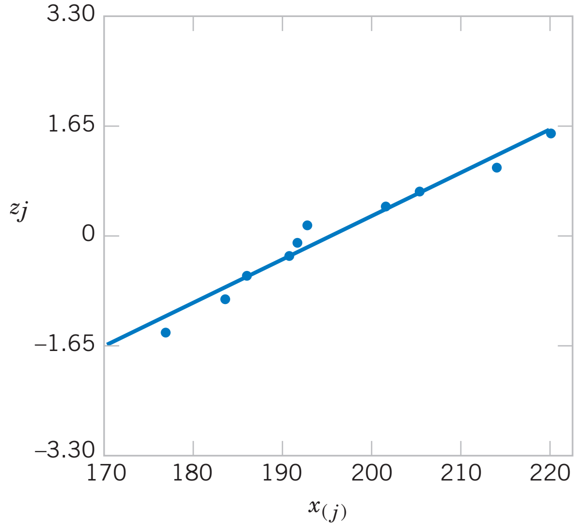
Goodness-of-fit test

- The hypothesis-testing procedures that we have discussed in previous sections are mostly designed for problems in which the population or probability distribution is **known** and the hypotheses involve the **parameters** of the distribution.

- We **do not know the underlying distribution** of the population, and we wish to **test the hypothesis that a particular distribution will be satisfactory as a population model**.
  - For example, test the hypothesis that the population is normal.

Reading: Chapter 9.1 in our textbook.

Overview



- Probability plot (Q-Q plot): plot and show
- Now: formal way to test the hypothesis
- Given *n* samples
- Discretize the sample space into *k* intervals
- Let O_i be the observed frequency in the *i*-th interval
- From hypothesized probability distribution, we compute the expected frequency in the *i*-th interval, denoted as E_i
- The chi-square test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Recap: Multinomial Distribution

Multinomial Distribution

- Let an experiment have *k* mutually exclusive and exhaustive outcomes, $A_1, A_2, \dots, A_k$
- $p_i = P(A_i), i = 1, \dots, k$; $p_1 + p_2 + \cdots p_k = 1$
- Repeat the experiment *n* independent times, let Y_i represent the number of times the experiment results in $A_i, i = 1, \dots, k$.
- $Y_1 + Y_2 + \cdots + Y_k = n$; they follow the Multinomial Distribution

$$(Y_1, \dots, Y_k) \sim \text{Multinomial}(n, p_1, \dots, p_k)$$

PMF:

$$f(y_1, \dots, y_k) = \frac{n!}{y_1!\, y_2! \cdots y_k!} p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$$

(Recall that $y_k = n - y_1 - y_2 - \cdots - y_{k-1}$.)

Goodness-of-fit test

Theorem: If $(Y_1, \dots, Y_k)$ follow the Multinomial Distribution

$$(Y_1, \dots, Y_k) \sim \text{Multinomial}(n, p_1, \dots, p_k),$$

then:

$$\chi^2 = \sum_{i=1}^{k} \frac{(Y_i - np_i)^2}{np_i} \xrightarrow{d} \chi^2(k-1), \qquad as\ n \to \infty$$

<u>k-1</u>: only k-1 of the proportions can vary.

$$y_k = n - y_1 - y_2 - \cdots - y_{k-1}$$

We call χ^2 the chi-square test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

- **O_i: the observed frequency**
- **E_i: the expected frequency**

Example

Two categories: $k = 2$

- Flip a coin *n* times, the probability of observing a HEAD is p_1, so the probability of observing a TAIL is $p_2 = 1 - p_1,\ 0 < p_1 < 1$.

- Let Y_1 denotes the total number of Heads, and $Y_2 = n - Y_1$ be the total number of Tails.

$$Y_1 \sim Bin(n, p_1) \qquad Y_2 = n - Y_1 \sim Bin(n, p_2)$$

$$\Rightarrow \quad Z = \frac{Y_1 - np_1}{\sqrt{np_1(1 - p_1)}} \sim N(0,1) \qquad \text{Central Limit Theorem (CLT)}$$

$$\Rightarrow \quad Z^2 = \frac{(Y_1 - np_1)^2}{np_1(1 - p_1)} \sim \chi^2(1) = \chi^2(k - 1)$$

Example

$$Y_1 \sim Bin(n, p_1) \qquad Y_2 = n - Y_1 \sim Bin(n, p_2) \qquad p_2 = 1 - p_1$$

$$Z^2 = \frac{(Y_1 - np_1)^2}{np_1(1-p_1)} \overset{approx}{\sim} \chi^2(1)$$

$$= \frac{(Y_1 - np_1)^2}{np_1(1-p_1)}(1 - p_1 + p_1)$$

$$= \frac{(Y_1 - np_1)^2}{np_1} + \frac{(Y_1 - np_1)^2}{n(1-p_1)}$$

$$(Y_1 - np_1)^2 = (n - Y_1 - n[1-p_1])^2 = (Y_2 - np_2)^2$$

$$= \frac{(Y_1 - np_1)^2}{np_1} + \frac{(Y_2 - np_2)^2}{np_2}$$

observed numbers

expected numbers

$$= \sum_{i=1}^{2} \frac{(\boxed{Y_i} - \boxed{np_i})^2}{np_i} \overset{approx}{\sim} \chi^2(1).$$

Goodness-of-fit test

Data

$$Y_1, \dots, Y_n \sim \text{Multinomial}(n, p_1, \dots, p_k)$$

Null Hypothesis

$$H_0 : p_i = p_{i0}, \quad i = 1, 2, \dots, k,$$

Alternative Hypothesis

$$H_1 : H_0 \text{ is not true.}$$

(Chi-square) Test Statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(y_i - np_{i0})^2}{np_{i0}}$$

Distribution under H_0

$$\chi^2 \sim \chi^2(k-1)$$

Critical (Rejection) region

$$\chi^2 > \chi^2_\alpha(k-1)$$

at significance level α

p-value $P(\chi^2(k-1) > \chi^2)$

Example

Example 9.1-1 If persons are asked to record a string of random digits, such as

$$3 \quad 7 \quad 2 \quad 4 \quad 1 \quad 9 \quad 7 \quad 2 \quad 1 \quad 5 \quad 0 \quad 8 \ldots,$$

we usually find that they are reluctant to record the same or even the two closest numbers in adjacent positions. And yet, in true random-digit generation, the probability of the next digit being the same as the preceding one is $p_{10} = 1/10$, the probability of the next being only one away from the preceding (assuming that 0 is one away from 9) is $p_{20} = 2/10$, and the probability of all other possibilities is $p_{30} = 7/10$. We shall test one person's concept of a random sequence by asking her to record a string of 51 digits that seems to represent a random-digit generation. Thus, we shall test

$$H_0\colon\ p_1 = p_{10} = \frac{1}{10},\ \ p_2 = p_{20} = \frac{2}{10},\ \ p_3 = p_{30} = \frac{7}{10}.$$

The critical region for an $\alpha = 0.05$ significance level is $\chi^2 \geq \chi^2_{0.05}(2) = 5.991$. The sequence of digits was as follows:

5	8	3	1	9	4	6	7	9	2	6	3	0
8	7	5	1	3	6	2	1	9	5	4	8	0
3	7	1	4	6	0	4	3	8	2	7	3	9
8	5	6	1	8	7	0	3	5	2	5	2	

51 digits, 50 pairs. e.g, (5,8), (8,3) …

Example

We went through this listing and observed how many times the next digit was the same as or was one away from the preceding one:

	Frequency	Expected Number
Same	0	$50(1/10) = 5$
One away	8	$50(2/10) = 10$
Other	42	$50(7/10) = 35$
Total	50	50

The computed chi-square statistic is

$$\frac{(0-5)^2}{5} + \frac{(8-10)^2}{10} + \frac{(42-35)^2}{35} = 6.8 > 5.991 = \chi^2_{0.05}(2).$$

Thus, we would say that this string of 51 digits does not seem to be random. ■

Example

- Let X denote the number of heads that occur when four coins are tossed at random. Under the assumption that the four coins are independent and the probability of heads on each coin is 1/2, $X \sim Bin\left(4, \frac{1}{2}\right)$.
- 100 repetitions of this experiment resulted in 0,1,2,3, and 4 heads being observed on 7,18,40,31, and 4 trials.
- Do these results support the assumption?

	observed	expected
$A_1 = \{0\}$	7	6.26
$A_2 = \{1\}$	18	25
$A_3 = \{2\}$	40	37.5
$A_4 = \{3\}$	31	25
$A_5 = \{4\}$	4	6.25

$$H_0\colon\ p_i = p_{i0}, \qquad i = 1, 2, \ldots, 5,$$

$$p_{10} = p_{50} = \binom{4}{0}\left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625,$$

$$p_{20} = p_{40} = \binom{4}{1}\left(\frac{1}{2}\right)^4 = \frac{4}{16} = 0.25,$$

$$p_{30} = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{6}{16} = 0.375.$$

Example

- 100 repetitions of this experiment resulted in 0,1,2,3, and 4 heads being observed on 7,18,40,31, and 4 trials.

$$H_0:\ p_i = p_{i0}, \qquad i = 1,2,\ldots,5,$$

$$p_{10} = p_{50} = \binom{4}{0}\left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625,$$

$$p_{20} = p_{40} = \binom{4}{1}\left(\frac{1}{2}\right)^4 = \frac{4}{16} = 0.25,$$

$$p_{30} = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{6}{16} = 0.375.$$

	observed	expected
$A_1 = \{0\}$	7	6.26
$A_2 = \{1\}$	18	25
$A_3 = \{2\}$	40	37.5
$A_4 = \{3\}$	31	25
$A_5 = \{4\}$	4	6.25

- Chi-square test statistic:

$$\frac{(7-6.25)^2}{6.25} + \frac{(18-25)^2}{25} + \frac{(40-37.5)^2}{37.5} + \frac{(31-25)^2}{25} + \frac{(4-6.25)^2}{6.25} = 4.47 < \chi^2_{0.05}(4) = 9.488$$

- Fail to reject the null hypothesis.

Goodness-of-fit test (with unspecified parameter)

Example

Data

$$X_1, \dots, X_n \sim \mathrm{Bin}(4, p)$$

***p* unknown**

Null Hypothesis

$$H_0 : p_i = p_{i0}, \quad i = 1, 2, \dots, k,$$

$$p_{i0} = P(A_i) = \frac{4!}{(i-1)!\,(5-i)!} p^{i-1}(1-p)^{5-i}, i = 1,2,\dots,5$$

p_{0i} also unknown (depend on *p*)

	observed	expected
$A_1 = \{0\}$	7	$100(1-p)^4$
$A_2 = \{1\}$	18	$100 * 4p(1-p)^3$
$A_3 = \{2\}$	40	$100 * 6p^2(1-p)^2$
$A_4 = \{3\}$	31	$100 * 4p^3(1-p)^1$
$A_5 = \{4\}$	4	$100p^4$

$\Downarrow$

Estimate *p* from data and plug-in

Goodness-of-fit test
(with unspecified parameter)

Estimate p

	observed	expected
$A_1 = \{0\}$	7	$100(1-p)^4$
$A_2 = \{1\}$	18	$100 * 4p(1-p)^3$
$A_3 = \{2\}$	40	$100 * 6p^2(1-p)^2$
$A_4 = \{3\}$	31	$100 * 4p^3(1-p)^1$
$A_5 = \{4\}$	4	$100p^4$

(depend on p)

$$\chi^2(p) = \frac{(y_1 - 100(1-p)^4)^2}{100(1-p)^4} + \frac{(y_2 - 100 * 4p(1-p)^3\ )^2}{100 * 4p(1-p)^3} + \frac{(y_3 - 100 * 6p^2(1-p)^2\ )^2}{100 * 6p^2(1-p)^2} + \frac{(y_4 - 100 * 4p^3(1-p)^1\ )^2}{100 * 4p^3(1-p)^1} + \frac{(y_5 - 100p^4\ )^2}{100p^4}$$

Estimate p as the $\tilde{p}$ that minimizes $\chi^2(p)$

Called the minimum chi-square estimator

Goodness-of-fit test (with unspecified parameter)

Estimate *p* using the minimum chi-square estimator and plug in to calculate the test statistic

	observed	expected
$A_1 = \{0\}$	7	$100(1-\tilde{p})^4$
$A_2 = \{1\}$	18	$100 * 4\tilde{p}(1-\tilde{p})^3$
$A_3 = \{2\}$	40	$100 * 6\tilde{p}^2(1-\tilde{p})^2$
$A_4 = \{3\}$	31	$100 * 4\tilde{p}^3(1-\tilde{p})^1$
$A_5 = \{4\}$	4	$100\tilde{p}^4$

$$\chi^2(\tilde{p}) = \frac{(y_1 - 100(1-\tilde{p})^4)^2}{100(1-\tilde{p})^4} + \frac{(y_2 - 100 * 4\tilde{p}(1-\tilde{p})^3)^2}{100 * 4\tilde{p}(1-\tilde{p})^3} + \frac{(y_3 - 100 * 6\tilde{p}^2(1-\tilde{p})^2)^2}{100 * 6\tilde{p}^2(1-\tilde{p})^2} + \frac{(y_4 - 100 * 4\tilde{p}^3(1-\tilde{p})^1)^2}{100 * 4\tilde{p}^3(1-\tilde{p})^1} + \frac{(y_5 - 100\tilde{p}^4)^2}{100\tilde{p}^4}$$

$$\boldsymbol{\chi^2(\tilde{p}) \sim \chi^2(5-1-1)}$$

the degrees of freedom is further reduced by one for each parameter estimated by the minimum chi-square technique.

Goodness-of-fit test (with unspecified parameter)

Data

$Y_1, \dots, Y_n \sim \text{Multinomial}(n, p_1, \dots, p_k)$

***k* categories**

Null Hypothesis

$H_0 : p_i = p_{i0}, \quad i = 1, 2, \dots, k,$

p_{0i} unknown (depend on parameters)

***d* unknown parameters**

Estimate *d* parameters using minimum chi-square estimator

(Chi-square) Test Statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(y_i - n\tilde{p}_{0i})^2}{n\tilde{p}_{0i}}$$

$\tilde{p}_{0i}$ is calculated using the minimum chi-square estimator

Distribution under H_0

$$\chi^2 \sim \chi^2(k - 1 - d)$$

Critical (Rejection) region

$$\chi^2 > \chi^2_\alpha(k - 1 - d)$$

at significance level α

p-value $P(\chi^2_\alpha(k - 1 - d) > \chi^2)$

Goodness-of-fit test (with unspecified parameter)

Data

$$Y_1, \dots, Y_n \sim \text{Multinomial}(n, p_1, \dots, p_k)$$

***k* categories**

Null Hypothesis

$$H_0 : p_i = p_{i0}, \quad i = 1, 2, \dots, k,$$

p_{0i} unknown (depend on parameters)

***d* unknown parameters**

One last question: How to find the minimum chi-square estimator?

For example, the minimum chi-square estimator can be difficult to find:

$$\chi^2(p) = \frac{(y_1 - 100(1-p)^4)^2}{100(1-p)^4} + \frac{(y_2 - 100 * 4p(1-p)^3\ )^2}{100 * 4p(1-p)^3} + \frac{(y_3 - 100 * 6p^2(1-p)^2\ )^2}{100 * 6p^2(1-p)^2} + \frac{(y_4 - 100 * 4p^3(1-p)^1\ )^2}{100 * 4p^3(1-p)^1} + \frac{(y_5 - 100p^4\ )^2}{100p^4}$$

difficult to find the minimizer

In practice, we can use some reasonable methods of estimating the parameters. (Maximum likelihood estimate is satisfactory.)

Example

Estimate *p*

	observed	expected
$A_1 = \{0\}$	7	$100(1-p)^4$
$A_2 = \{1\}$	18	$100 * 4p(1-p)^3$
$A_3 = \{2\}$	40	$100 * 6p^2(1-p)^2$
$A_4 = \{3\}$	31	$100 * 4p^3(1-p)^1$
$A_5 = \{4\}$	4	$100p^4$

(depend on *p*)

$$\chi^2(p) = \frac{(y_1 - 100(1-p)^4)^2}{100(1-p)^4} + \frac{(y_2 - 100 * 4p(1-p)^3\ )^2}{100 * 4p(1-p)^3} + \frac{(y_3 - 100 * 6p^2(1-p)^2\ )^2}{100 * 6p^2(1-p)^2} + \frac{(y_4 - 100 * 4p^3(1-p)^1\ )^2}{100 * 4p^3(1-p)^1} + \frac{(y_5 - 100p^4\ )^2}{100p^4}$$

minimum chi-square estimator

$\tilde{p}$ =0.5189

$\chi^2(\tilde{p})$ =**3.90**

$\chi^2_{0.05}(3)$ =7.81

Fail to reject the null hypothesis.

Or we can use a reasonable estimator: the MLE is $X_i \sim Bin(k, p)$

$$\hat{p} = \frac{\sum X_i}{kn} = \frac{18 + 2 * 40 + 3 * 31 + 4 * 4}{400} = 0.5175$$

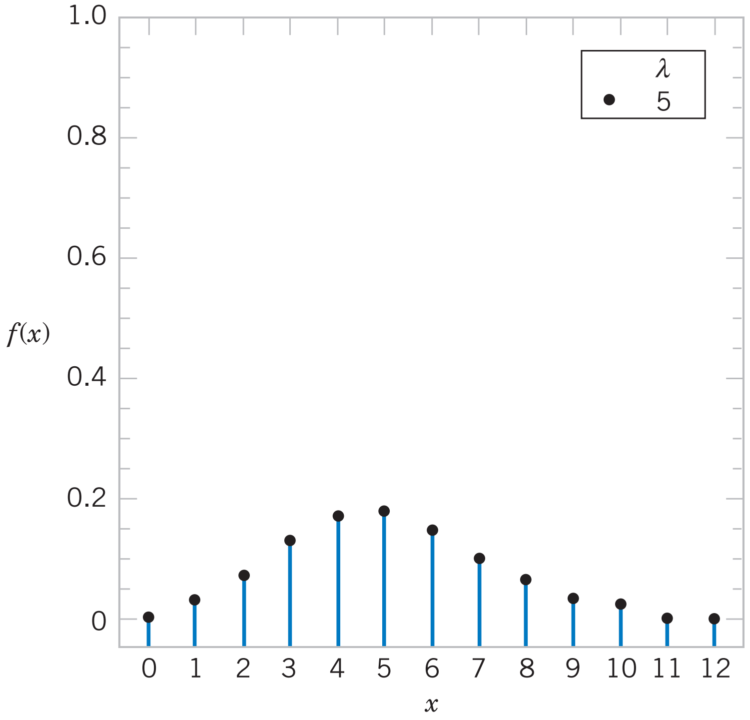
$$\chi^2(0.5175) = \mathbf{3.91} < \chi^2_{0.05}(3) = 7.81$$

Testing Poisson

Poisson Distribution

The number of defects in printed circuit boards is hypothesized to follow a Poisson distribution. A random sample of $n = 60$ printed boards has been collected, and the following number of defects observed.

Number of Defects	Observed Frequency
0	32
1	15
2	9
3	4



Testing Poisson

The mean of the assumed Poisson distribution in this example is unknown and must be estimated from the sample data. The estimate of the mean number of defects per board is the sample average, that is, $(32 \cdot 0 + 15 \cdot 1 + 9 \cdot 2 + 4 \cdot 3)/60 = 0.75$. From the Poisson distribution with parameter 0.75, we may compute p_i, the theoretical, hypothesized probability associated with the ith class interval. Since each class interval corresponds to a particular number of defects, we may find the p_i as follows:

$$p_1 = P(X = 0) = \frac{e^{-0.75}(0.75)^0}{0!} = 0.472$$

$$p_2 = P(X = 1) = \frac{e^{-0.75}(0.75)^1}{1!} = 0.354$$

$$p_3 = P(X = 2) = \frac{e^{-0.75}(0.75)^2}{2!} = 0.133$$

$$p_4 = P(X \geq 3) = 1 - (p_1 + p_2 + p_3) = 0.041$$

The expected frequencies are computed by multiplying the sample size $n = 60$ times the probabilities p_i. That is, $E_i = np_i$. The expected frequencies follow:

Number of Defects	Probability	Expected Frequency
0	0.472	28.32
1	0.354	21.24
2	0.133	7.98
3 (or more)	0.041	2.46

Since the expected frequency in the last cell is less than 3, we combine the last two cells:

Number of Defects	Observed Frequency	Expected Frequency
0	32	28.32
1	15	21.24
2 (or more)	13	10.44

Testing Poisson

Null hypothesis: H_0: The form of the distribution of defects is Poisson.

Alternative hypothesis: H_1: The form of the distribution of defects is not Poisson.

Test statistic: The test statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \frac{(32 - 28.32)^2}{28.32} + \frac{(15 - 21.24)^2}{21.24} + \frac{(13 - 10.44)^2}{10.44} = 2.94$$

Number of Defects	Observed Frequency	Expected Frequency
0	32	28.32
1	15	21.24
2 (or more)	13	10.44

Rejection region: Since one parameter in the Poisson distribution has been estimated, the chi-square statistic will have 3-1-1=1 degree of freedom. We use the significance level 0.05, then we will reject H_0 If $\chi^2 > \chi^2_{0.05}(1)$.

Conclusion: Since $\chi^2 < \chi^2_{0.05}(1) = 3.84$, fail to reject H_0

p-value: $\mathrm{P}(\chi^2(1) > 2.94) = 0.086$

Example 9.1-3

Testing Poisson

Let X denote the number of alpha particles emitted by barium-133 in one tenth of a second. The following 50 observations of X were taken with a Geiger counter in a fixed position:

7	4	3	6	4	4	5	3	5	3
5	5	3	2	5	4	3	3	7	6
6	4	3	11	9	6	7	4	5	4
7	3	2	8	6	7	4	1	9	8
4	8	9	3	9	7	7	9	3	10

The experimenter is interested in determining whether X has a Poisson distribution. To test H_0: X is Poisson, we first estimate the mean of X—say, λ—with the sample mean, $\bar{x} = 5.4$, of these 50 observations. We then partition the set of outcomes for this experiment into the sets $A_1 = \{0, 1, 2, 3\}$, $A_2 = \{4\}$, $A_3 = \{5\}$, $A_4 = \{6\}$, $A_5 = \{7\}$, and $A_6 = \{8, 9, 10, \ldots\}$. (Note that we combined $\{0, 1, 2, 3\}$ into one set A_1 and $\{8, 9, 10, \ldots\}$ into another A_6 so that the expected number of outcomes for each set would be at least five when H_0 is true.) In Table 9.1-1, the data are grouped and the estimated probabilities specified by the hypothesis that X has a Poisson distribution with an estimated $\widehat{\lambda} = \bar{x} = 5.4$ are given.

Table 9.1-1 Grouped Geiger counter data

	Outcome					
	A_1	A_2	A_3	A_4	A_5	A_6
Frequency	13	9	6	5	7	10
Probability	0.213	0.160	0.173	0.156	0.120	0.178
Expected ($50p_i$)	10.65	8.00	8.65	7.80	6.00	8.90

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{[13 - 50(0.213)]^2}{50(0.213)} + \cdots + \frac{[10 - 50(0.178)]^2}{50(0.178)}$$

$$= 2.763 < 9.488 = \chi^2_{0.05}(4),$$

degree of freedom: 6-1-1=4

H_0 is not rejected at the 5% significance level. That is, with only these data, we are quite willing to accept the model that X has a Poisson distribution.

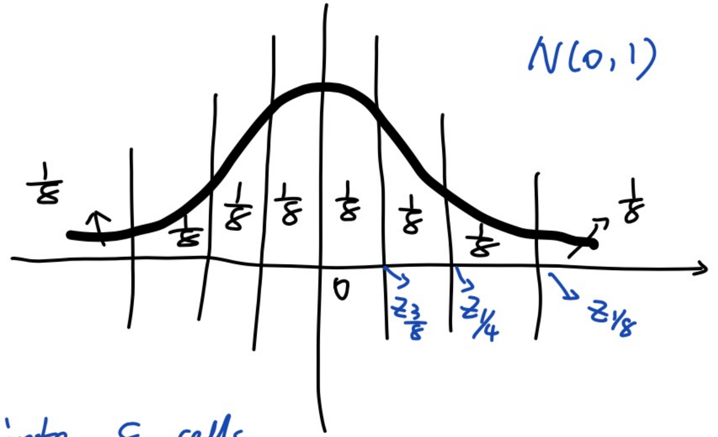
Example: Testing normal

> A manufacturing engineer is testing a power supply used in a notebook computer and, using $\alpha = 0.05$, wishes to determine whether output voltage is adequately described by a normal distribution. Sample estimates of the mean and standard deviation of $\bar{x} = 5.04$ V and $s = 0.08$ V are obtained from a random sample of $n = 100$ units.

Estimate *d*=2 parameters

- A common practice in constructing the time intervals used in the chi-square test is to choose the intervals so that the expected frequencies p_i are equal for all cells.
- Suppose we want to choose $k = 8$ cells, then the 8 intervals will be chosen such that p_i=1/8

e.g. for standard normal



Divide R into 8 cells

Example: Testing normal

if $X \sim N(\mu, \sigma^2)$

		probability p_{0i}
A_1:	$(-\infty, \ -z_{1/8}\sigma+\mu)$	0.125
A_2:	$[-z_{1/8}\sigma+\mu, \ -z_{1/4}\sigma+\mu)$	0.125
A_3:	$[-z_{1/4}\sigma+\mu, \ -z_{3/8}\sigma+\mu)$	0.125
A_4.	$[-z_{3/8}\sigma+\mu, \ \mu)$	0.125
A_5.	$[\mu, \ z_{3/8}\sigma+\mu)$	0.125
A_6	$[z_{3/8}\sigma+\mu, \ z_{1/4}\sigma+\mu)$	0.125
A_7.	$[z_{1/4}\sigma+\mu, \ z_{1/8}\sigma+\mu)$	0.125
A_8	$[z_{1/8}\sigma+\mu, \ +\infty)$	0.125

$\hat{\mu} = 5.04$. $\quad \hat{\sigma} = 0.08$

$z_{1/8} = 1.15 \quad z_{1/4} = 0.67 \quad z_{3/8} = 0.32$

Class Interval	Observed Frequency o_i	Expected Frequency E_i
$x < 4.948$	12	12.5
$4.948 \leq x < 4.986$	14	12.5
$4.986 \leq x < 5.014$	12	12.5
$5.014 \leq x < 5.040$	13	12.5
$5.040 \leq x < 5.066$	12	12.5
$5.066 \leq x < 5.094$	11	12.5
$5.094 \leq x < 5.132$	12	12.5
$5.132 \leq x$	14	12.5
Totals	100	100

$\Rightarrow A_1$: $(-\infty, \ -1.15\times 0.08 + 5.04) = (-\infty, 4.948)$

A_2: $[4.948, \ -0.67\times 0.08+5.04) = [4.948, 4.986)$

$\vdots$

Example: Testing normal

Null hypothesis: H_0: The form of the distribution is normal.

Alternative hypothesis: H_1: The form of the distribution is nonnormal.

Test statistic: The test statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \frac{(12 - 12.5)^2}{12.5} + \frac{(14 - 12.5)^2}{12.5} + \cdots + \frac{(14 - 12.5)^2}{12.5}$$
$$= 0.64$$

Class Interval	Observed Frequency o_i	Expected Frequency E_i
$x < 4.948$	12	12.5
$4.948 \leq x < 4.986$	14	12.5
$4.986 \leq x < 5.014$	12	12.5
$5.014 \leq x < 5.040$	13	12.5
$5.040 \leq x < 5.066$	12	12.5
$5.066 \leq x < 5.094$	11	12.5
$5.094 \leq x < 5.132$	12	12.5
$5.132 \leq x$	14	12.5
Totals	100	100

Reject H_0 if: Since two parameters in the normal distribution have been estimated, the chi-square statistic above will have $k - d - 1 = 8 - 2 - 1 = 5$ degrees of freedom. We will use a fixed significance level test with $\alpha = 0.05$. Therefore, we will reject H_0 If $\chi^2 > \chi^2_{0.05}(5)$.

Conclusion: Since $\chi^2 < \chi^2_{0.05}(5) = 11.07$, fail to reject H_0

p-value: $P(\chi^2(5) > 0.64) = 0.9861$

Summary

- Discretize the sample space into ***k*** intervals
- Let $\boldsymbol{O_i}$ **be the observed frequency** in the *i*-th interval
- From hypothesized probability distribution, we compute the **expected frequency in the *i*-th interval, denoted as** $\boldsymbol{E_i}$
- The chi-square test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

- Under the null, $\boldsymbol{\chi^2 \sim \chi^2(k-1-d)}$
- ***d*** **is the number of parameters that have been estimated**