Lezione 11 - AMI

Titolo nota 10/11/2022

√ $\lim_{x\to 0} \frac{\text{sen}(3x)}{x}$ √ $\lim_{x\to 0} \frac{\text{sen}(\text{sen}(x))}{x}$

√ $\lim_{x\to 0} \frac{\text{sen}(x^2)}{x^2}$ √ $\lim_{x\to 0} (\cos(x))^{\frac{1}{x^2}}$

√ $\lim_{x\to 0} (1+\sin(x))^{\frac{1}{x}}$ $\lim_{x\to 0^{\pm}} e^{\frac{1}{x}}$

√ $\lim_{x\to 0} \frac{\ln(1+\sin(x))}{x\cos(x)}$ $\lim_{x\to +\infty} x\sin\left(\frac{1}{x}\right)$

$\lim_{x\to 0} x\log(x)$ $\lim_{\to 0} x^x$

$$\lim_{x\to 0} \frac{\sin(3x)}{3\cdot x}\cdot 3 = \lim_{y\to 0} \frac{\sin(y)}{y}\cdot 3 = \boxed{3}$$

$y = 3x$

$y \to 0$

$$\lim_{\xi\to 0} \frac{\sin(\xi)}{\xi} = 1$$

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{y\to 0^+} \frac{\sin(y)}{y} = \boxed{1}$$

$y = x^2$

$y \to 0^+$

1 limite not

$$\lim_{x\to 0} \frac{\sin(\sin(x))}{x}\cdot\frac{\sin(x)}{\sin(x)} = 1\cdot 1 = \boxed{1}$$

1 per limite not. con $\xi = \sin(x)$

$$\lim_{x\to 0} \frac{\ln(1+\sin(x))}{x\cdot\cos(x)}\cdot\frac{\sin(x)}{\sin(x)} = \boxed{1}$$

$\frac{\ln(1+\sin(x))}{\sin(x)} \to 1$ per ②

$\cos(x) \to 1$ per continuità di $\cos(x)$

$\frac{\sin(x)}{x} \to 1$ per ① $t=\sin(x)$

① $\lim_{t\to 0} \frac{\ln(1+t)}{t} = 1$

② $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$

$$\lim_{x\to 0} \frac{\ln(1+\cos(x))}{\cos(x)} = \lim_{t\to 1} \frac{\ln(1+t)}{t} = \ln(2)$$

$\cos(x)=t$

$t\to 1$

⚠ ATTENZIONE CHE VALGANO LE CONDIZIONI DI LIMITE CORRETTE

$$\lim_{x\to 0^+} (1+\sin(x))^{\frac{1}{x}} = \lim_{t\to +\infty} \left(1+\sin\left(\frac{1}{t}\right)\right)^t$$

$t=\frac{1}{x} \rightsquigarrow x=\frac{1}{t}$

$t\to +\infty$

① $\lim_{x\to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$

② $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

$\sin(x)=t \to x=\arcsin(t)$

$t\to 0$

$$\lim_{t\to 0} (1+t)^{\frac{1}{\arcsin(t)}\cdot\frac{t}{t}} = \lim_{t\to 0} (1+t)^{\frac{1}{t}\cdot\frac{t}{\arcsin(t)}} = \lim_{t\to 0} \left[(1+t)^{\frac{1}{t}}\right]^{\frac{t}{\arcsin(t)}}$$

$(1+t)^{\frac{1}{t}} \to e$ per ②

$\frac{t}{\arcsin(t)} \to 1$ L.N.

$$= \boxed{e^1}$$

II° modo

$$\lim_{x\to 0} (1+\sin(x))^{\frac{1}{x}} = \lim_{x\to 0} \left(\left(1+\sin(x)\right)^{\frac{1}{\sin(x)}\cdot\sin(x)}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0} \left[(1+\sin(x))^{\frac{1}{\sin(x)}}\right]^{\frac{\sin(x)}{x}} = e^1 = e$$

$(1+\sin(x))^{\frac{1}{\sin(x)}} \to e$ per ① con $t=\sin(x)$

$\frac{\sin(x)}{x} \to 1$ per L.N.

$$\lim_{x\to 0} (\cos(x))^{\frac{1}{x^2}} = 1^{+\infty} \quad F.I.$$

$$= \lim_{x\to 0} \left(1 + \underbrace{\boxed{\cos(x)-1}}_{\downarrow\, 0}\right)^{\frac{1}{\boxed{\cos(x)-1}} \cdot \frac{\boxed{\cos(x)-1}}{x^2}}$$

$$\boxed{\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}} = \lim_{x\to 0} \underbrace{\left[\left(1+ m\right)^{\frac{1}{m}}\right]}_{\downarrow\, e}^{\underbrace{\frac{\cos(x)-1}{x^2}}_{\downarrow\, -\frac{1}{2}}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

perché $m \to 0$

$$\lim_{x\to x_0} (f(x))^{g(x)} \underset{\text{se } f(x)\to 1}{=} \lim_{x\to x_0} e^{g(x)\cdot(f(x)-1)} = e^{\lim_{x\to x_0} g(x)\cdot(f(x)-1)}$$

$$\lim_{x\to +\infty} x \sin\left(\frac{1}{x}\right) = +\infty \cdot 0 \quad F.I.$$

$$\underset{\substack{t=\frac{1}{x}\\ t\to 0^+\\ x=\frac{1}{t}}}{=} \lim_{t\to 0^+} \frac{1}{t}\cdot \sin(t) = \lim_{t\to 0^+} \frac{\sin(t)}{t} = 1$$

$$\lim_{x\to 0^+} x\cdot \ln(x) = 0\cdot(-\infty) \quad F.I. \qquad \ln(1+\underbrace{(x-1)}_{\downarrow\,-1})$$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \underset{\substack{t=\frac{1}{x} \quad x=\frac{1}{t}\\ t\to +\infty}}{=} \lim_{t\to +\infty} \frac{\ln\left(\frac{1}{t}\right)}{t} = \lim_{t\to +\infty} \frac{\ln(1)-\ln(t)}{t} =$$

$$= \lim_{t\to +\infty} -\frac{\ln(t)}{t} = 0$$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)} = e^0 = \boxed{1}$$

VOGLIO SCRIVERE $x^x = e^{\text{qualcosa}}$

$$x \cdot \ln(x) = \ln(x^x) = \text{qualcosa}$$

$$\boxed{f(x)^{g(x)} \underset{(f(x)>0)}{=} e^{\ln(f(x))\cdot g(x)}}$$

DA TENERE A MENTE

- Metodo del confronto a 2 e a 3
- Dimostrazioni limiti notevoli trigonometrici.

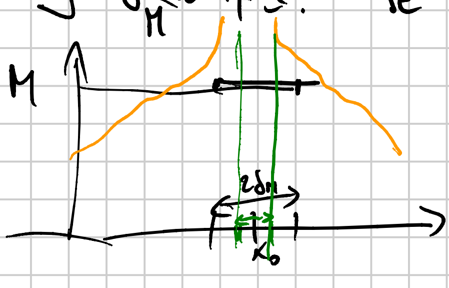
Teo. Siano $f(x)$, $g(x)$ due funzioni definite in un intorno di x_0. Supponiamo che

(1) $\exists\, \delta$ t.c. $f(x) \geq g(x)$ per $x \in (x_0-\delta, x_0+\delta) \setminus \{x_0\}$

(2) $\lim_{x\to x_0} g(x) = +\infty$

Allora $\lim_{x\to x_0} f(x) = +\infty$

Dim. (2) $\Rightarrow$ $\forall\, M>0$ $\exists\, \delta_M < \delta$ t.c. se $|x-x_0|<\delta_M$ allora $\boxed{g(x) \geq M}$



ma poiché $|x-x_0|<\delta_M<\delta$, per (1) ho

$$\boxed{f(x)\geq g(x)}$$

$\leadsto$ $f(x)\geq g(x)\geq M \rightarrow f(x)\geq M$

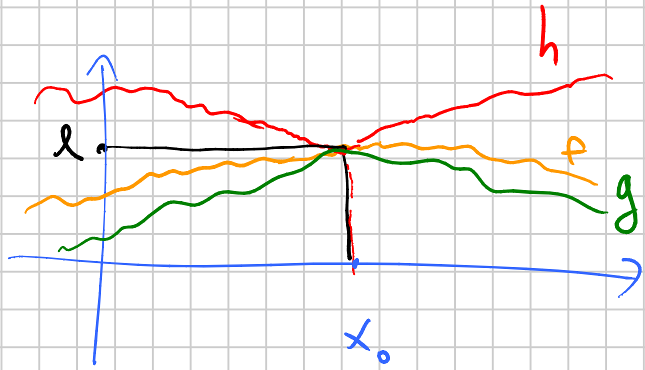
$\square$.

Teo. Siano f, g, h funzioni definite in un intorno di x_0. Supponiamo che

① $\exists\, \delta>0$ t.c. $g(x)\leq f(x)\leq h(x)$ $\forall\, x\in(x_0-\delta, x_0+\delta)\setminus\{x_0\}$

② $\lim_{x\to x_0} g(x)=\lim_{x\to x_0} h(x)=\ell$

$\lim_{x\to x_0} f(x)=\ell$.



Dim. $\forall\, \varepsilon>0$, per ② abbiamo

$\exists\, \delta_\varepsilon'$ tale che $\ell-\varepsilon\leq g(x)\leq \ell+\varepsilon$ se $|x-x_0|<\delta_\varepsilon'$

$\exists\, \delta_\varepsilon''$ " " $\ell-\varepsilon\leq h(x)\leq \ell+\varepsilon$ se $|x-x_0|<\delta_\varepsilon''$

Ora scelgo $\delta_\varepsilon=\min\{\delta_\varepsilon', \delta_\varepsilon'', \delta\}$. Se $|x-x_0|<\delta_\varepsilon$

$$\underset{\delta\leq\delta_\varepsilon'}{\ell-\varepsilon\leq} g(x)\underbrace{\leq f(x)\leq}_{\delta_\varepsilon\leq\delta} h(x)\underset{\delta_\varepsilon\leq\delta_\varepsilon''}{\leq \ell+\varepsilon}$$

$\hookrightarrow \quad l-\varepsilon \leq f(x) \leq l+\varepsilon$ $\square$.

$$\lim_{x \to +\infty} x+\sin(x) = +\infty + \text{ind},$$

ma $\underbrace{x+\sin(x)}^{f(x)} \geq \overbrace{x-1}^{g(x)} \quad \forall x \in \mathbb{R}$

$$\lim_{x \to +\infty} \overbrace{x-1}^{g(x)} = +\infty - 1 = +\infty$$

$$\rightsquigarrow \lim_{x \to +\infty} x+\sin(x) = +\infty.$$

$$\lim_{x \to +\infty} \frac{\sin(x)}{x} = \frac{\text{ind.}}{+\infty}$$

$(x>0)$ uso $-1 \leq \sin(x) \leq 1$

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$$

$$g(x) \leq f(x) \leq h(x)$$

1

2 $\lim_{x \to +\infty} -\frac{1}{x} = 0 = \lim_{x \to +\infty} \frac{1}{x}$

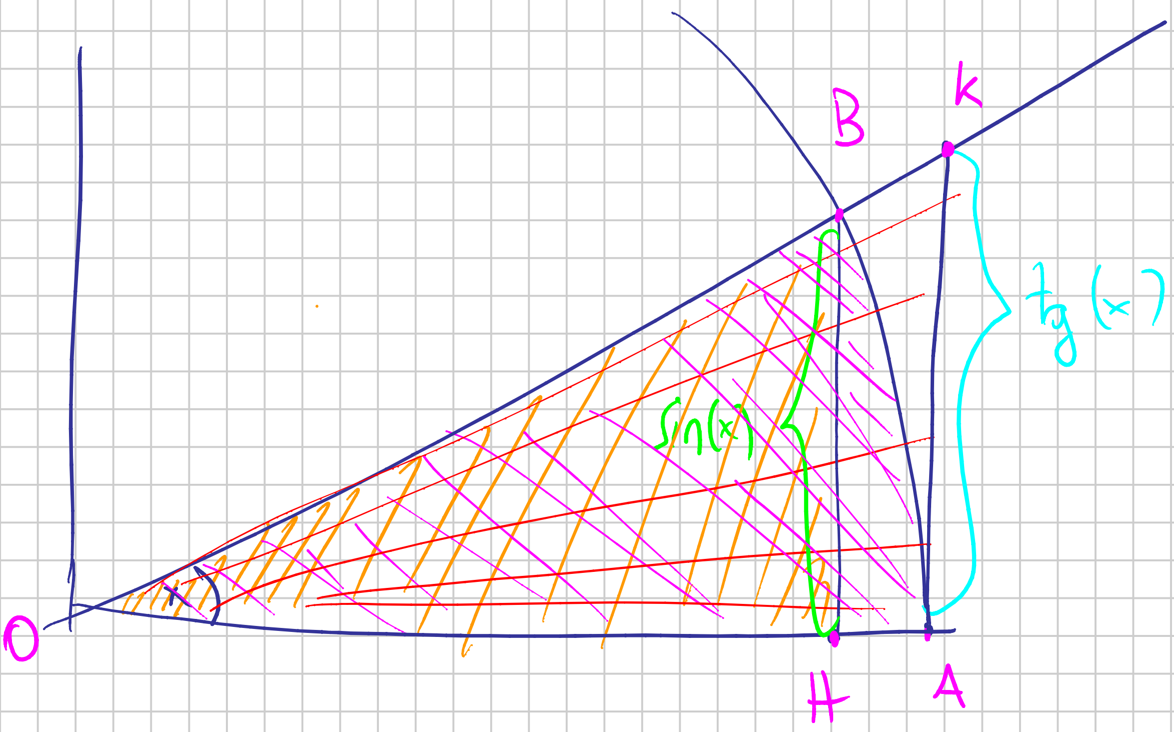
Teo. del confronto $\longrightarrow \lim_{x \to +\infty} \frac{\sin(x)}{x} = 0$

Oss. In generale se $|g(x)| \leq M$ in un intorno di $x \to x_0$ e $\lim_{x \to x_0} f(x) = 0$ allora

$$\lim_{x \to x_0} g(x) \cdot f(x) = 0$$

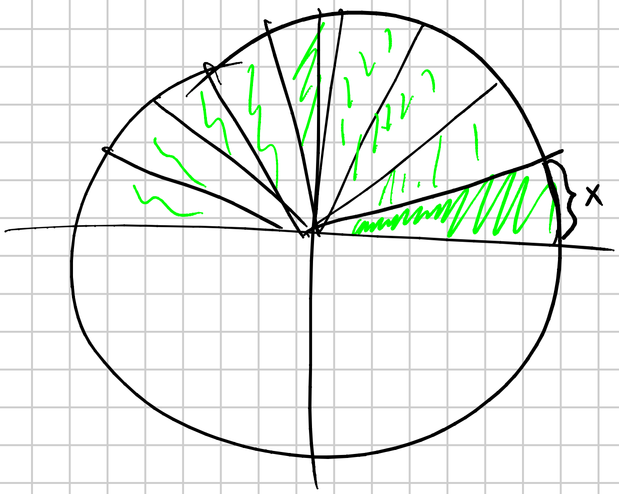
infinitesimo × LIMITATO = infinitesimo.

0



$$A_{\overset{\triangle}{OHB}} \le A_{\overset{\frown}{OAB}} \le A_{\overset{\triangle}{OAK}}$$

$$\frac{\operatorname{sen}(x)\cdot\cos(x)}{2} \le \frac{x}{2} \le \frac{tg(x)}{2}$$



$\rightarrow A \cdot n = \frac{\pi}{2}$

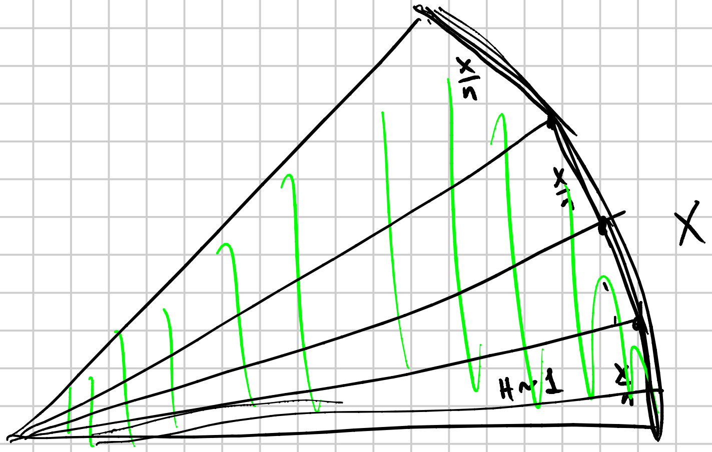
$A = \frac{\pi}{2\cdot n}$

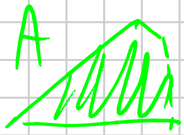
$x = \frac{m\pi}{n}$ $\quad A(x) = m\, A\left(\frac{\pi}{n}\right) = \frac{m\pi}{2n}$

$= \frac{x}{2}$

Se $A(x) = \frac{x}{2}$ per ogni $x = \frac{m}{n}\cdot\pi$

Ma allora $A(x) = \frac{x}{2}$ $\forall\, x \in [0, \pi]$





$$\simeq \frac{\cancel{n}\cdot\frac{x}{\cancel{n}}\cdot 1}{2} = \frac{x}{2}$$

$$\boxed{x > 0}$$

$$\frac{\operatorname{sen}(x)\cdot\cos(x)}{x} \leq \frac{\cancel{x}}{x} \leq \frac{\operatorname{sen}(x)}{\cos(x)\, x}$$

$$\frac{\operatorname{sen}(x)}{x}\cdot\cos(x) \leq 1 \leq \frac{\operatorname{sen}(x)}{x\cdot\cos(x)}$$

$$\underbrace{\cos(x)}_{\substack{\downarrow x\to 0 \\ 1}} \leq \underbrace{\frac{\operatorname{sen}(x)}{x}}_{\substack{\downarrow \\ \text{Teo. confronto} \\ \downarrow \\ 1}} \leq \underbrace{\frac{1}{\cos(x)}}_{\substack{\downarrow x\to 0 \\ 1}}$$

Ho dimostrato che $\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$

$$\lim_{x\to 0^-} \frac{\sin(x)}{x} = \lim_{\substack{y\to 0^+ \\ x=-y}} \frac{\sin(-y)}{-y} = \lim_{y\to 0^+} \frac{\cancel{+}\sin(y)}{\cancel{+}y} = 1$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \cdot \frac{1 + \cos(x)}{1 + \cos(x)} = \lim_{x \to 0} \frac{1^2 - \cos(x)^2}{x^2(1 + \cos(x))} \overset{\sin(x)^2 + \cos(x)^2 = 1}{=}$$

$$= \lim_{x \to 0} \frac{\sin(x)^2}{x^2(1 + \cos(x))} = \lim_{x \to 0} \Big[\underbrace{\frac{\sin(x)}{x}}_{\to 1}\Big]^2 \cdot \underbrace{\frac{1}{1 + \cos(x)}}_{1+\cos(x) \to 1} = \frac{1}{2}$$

D.