Internal Assessment

Mathematics – Higher Level

Investigation of trains' braking process and motion from the mathematical perspective

Introduction

Travelling by train is an everyday part of my life as my home-town is approximately 80 kilometres from the school. Since an early age I have had a lot of conversations with my grandfather regarding the trains – how they have changed over time, how much they can carry on, what distance they can cross etc. Mechanics is an essential part of physics and at high school level it includes analysis of motion of a point mass. We examined on physics lessons that the braking distance of a car increases with square of the velocity because its kinetic energy increases by that rate but the braking distance does not depend on the mass of the car. I wondered how the gathered formula would work for trains, considering their braking system is likely to be more complicated and their mass is significantly higher. Knowledge of calculus broadened my opportunities in physics and I decided to research the braking process of trains from mathematical perspective.

The aim of this internal assessment is to investigate the motion of train after it has started braking. I will attempt to find the formula for the velocity of and distance crossed by a train over time once it has started braking, try to find out the braking distance of the train and determine its dependence on the initial velocity of the train at which it starts braking.

Properties of the train and specifications of the theoretical model

The braking forces that the train experiences depend on multiple parameters of the train and surroundings such as the dryness of the rails, shabbiness and material of the wheels, state of the pressure in the brakes etc. Specific details were found on the internet, so the calculations could become more concrete and comparable with the real world.

Quantity	Value of the quantity in basic units
Mass of the train m[kg][1]	$4.5*10^{6}$
Rolling resistance coefficient b[m][2]	$5*10^{-4}$
Coefficient of friction μ[-][3]	0.70
Radius of the wheel R[m]	0.2
Velocity at which the train starts braking v[m/s]	50
Area resistivity of the wheel[4] [Ωm^2]	10^{-6}
Magnetic field strength[5] B[T]	1.1

[1] Brzdná dráha
[2] Trenie
[3] Friction and Friction Coefficients
[4] Elert, G
[5] H, L.

Area of the magnetic patch A[m²]	10^{-4}

Table 1. Properties selected for further calculations.

The properties of the train are obtained from more sources as I could not find one source that would include all necessary information. Some of the properties, such as the radius of the train's wheel is estimated.

Theory

The train drag can be classified in three groups: the friction between the wheels and rails, the friction caused by the pantograph system and the aerodynamic drag[6]. The train's braking process is a complicated issue as the train has numerous different brakes and braking systems which ensure a safe discontinuance of the train. This study will be limited to investigation of the rolling frictional force, force of frictional braking and magnetic braking.

Rolling friction between the wheels and rails

The rolling friction experiences every object that performs rotational movement. Without this friction the object would be only sliding. A train experiences this frictional force during the whole time of its motion and it is not primarily used for the braking, thus this force is undesired. The rolling friction is satisfied by the equation: $F_R = F_N \frac{b}{R}$ where F_N is the normal force, b is the rolling resistance coefficient and R is radius of the train's wheel[7].

The normal force is a reaction of the gravitational force and assuming the train is not moving on an inclined plane, these two forces have equal size and opposite direction. The rolling friction is slowing down the train's motion as some part of the train's kinetic energy is transformed into thermal energy. It points in the opposite direction of the motion as shown in the figure below.

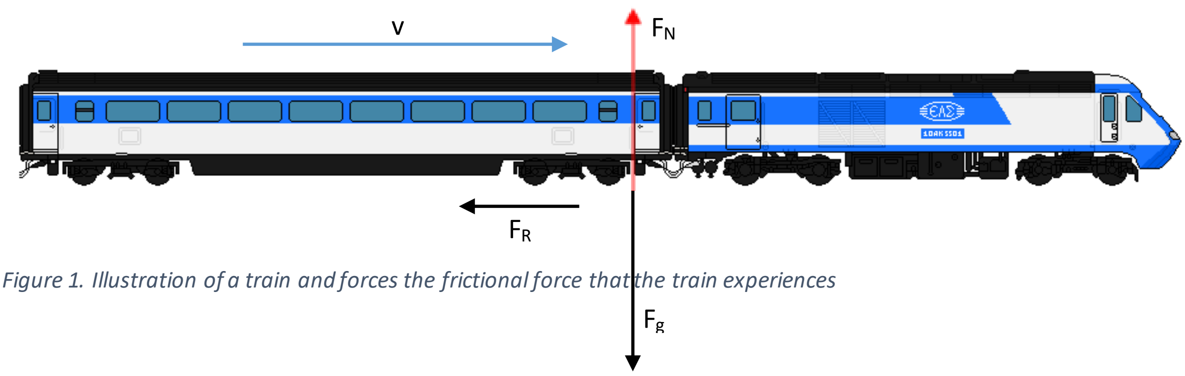

Figure 1. Illustration of a train and forces the frictional force that the train experiences

[6] Tian,H
[7] Rolling resistance

Acceleration is the rate of change of velocity and is given by the net force acting on an object. Considering the rolling friction would be the only force acting on the train once the locomotive's tractive effort stops, it would give the train an acceleration: $a = \frac{F_R}{m}$ where m is the mass of the train. Using the second Newton's law, the equation can be rewritten as:

$$m\frac{dv}{dt} = -mg\frac{b}{R} \tag{1}$$

Which is a first-order linear differential equation. The minus sign is used because the force is acting against the motion of the train and thus it decelerates. The tractive force of the locomotive kept the train moving at velocity v_0. When the locomotive stops this function, the train starts slowing down. The time before the train started braking is irrelevant in this investigation. Therefore, formula *1* will be examined within time interval starting at zero when the train started braking at initial velocity v_0 and time t at which the train will have a certain velocity v.

$$\int_{v_0}^{v}\frac{dv}{dt}dt = -g\frac{b}{R}\int_0^t dt \tag{2}$$

The obtained relation which expresses velocity of the train as a function of time is:

$$v(t) = v_0 - g\frac{b}{R}t \tag{3}$$

In a velocity versus time diagram (Figure 1), gradient of the function would be acceleration of the train and the displacement crossed by the train in a given time interval would be represented by the area below the curve in this interval.

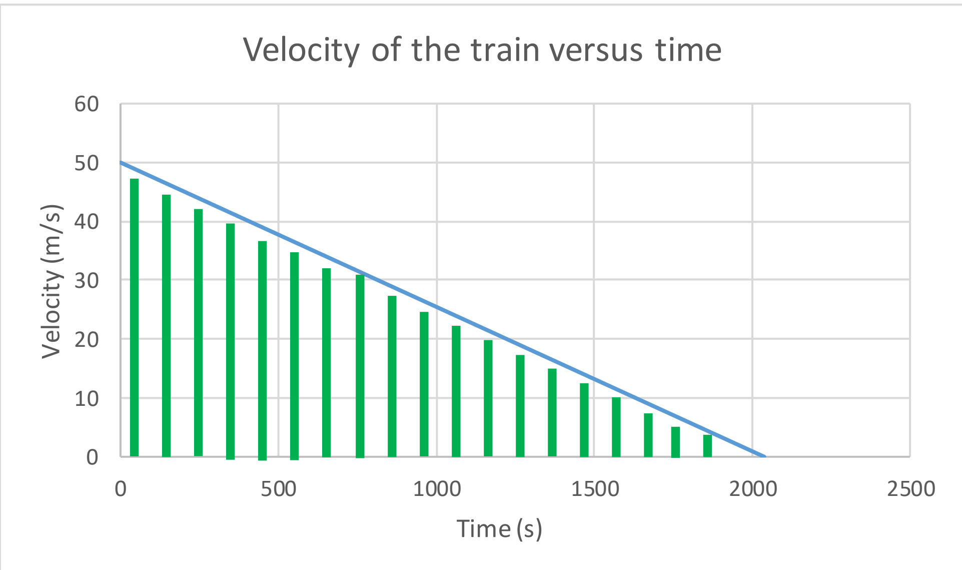

Figure 2. Velocity of the train over time when no braking system is applied. The area under the curve represents the displacement travelled by the train over time.

Hence, the displacement travelled by the train is the integral of the velocity (equation *3*) of the train with respect to time. In other words:

$$s(t) = \int v\,dt \tag{2}$$

Substituting equation (3) for the velocity, it is obtained:

$$s(t) = \int (v_0 - g\frac{b}{R}t)\,dt \tag{3}$$

Equation *5* will be again integrated with respect to time in interval from 0 to t as the process before the braking started is not aim of this investigation. This integral can be split into two parts:

$$s(t) = \int_0^t v_0\,dt - \int_0^t g\frac{b}{R}t\,dt = v_0 t - \frac{gb}{2R}\,t^2 \tag{6}$$

In which case the integrant constant s_0 can be omitted as the distance crossed by the train before it started braking is not involved in the braking process which is examined. The braking distance of the train can be either determined from the Figure 1 as the area under the curve until the train's velocity is negative, or can be calculated in two ways.

Method 1 - using the formula for the distance

The braking distance of the train is the same as the maximum of formula *6*. Since the equation *6* is a polynomial function of second order, there are more ways how to calculate maximum of the function: by using equation for calculation of the y-coordinate of vertex of parabola, calculating the discriminant etc. Nevertheless, a function reaches its local maximums and minimums when its first derivation equals zero, that is, when the gradient of the function is equal zero. The instantaneous velocity at a given point of time is the first derivative of distance with respect to time at that time. Hence, equation *3* can be just set to zero and calculated as follows:

$$0 = v_0 - g\frac{b}{R}t \tag{4}$$

Which provides calculation for the time at which the train reaches zero velocity as:

$$t = \frac{Rv_0}{gb} \tag{5}$$

This time can be now used in the equation for the distance:

$$d = v_0\frac{Rv_0}{gb} - \frac{gb}{2R}(\frac{Rv_0}{gb})^2 \tag{6}$$

And the resulting equation for the braking distance is:

$$d = v_0^2 \frac{R}{2gb} \tag{7}$$

Energetic perspective

When the train starts braking it has $inetic\ energy = \frac{1}{2}mv^2$. When the train stops, its kinetic energy is zero and the energy has been transformed into thermal energy. The thermal energy is the work performed by the rolling friction over the braking distance. This work can be calculated as:

$$W = \int \overline{F_R} \, . \, d\bar{s} \tag{8}$$

The vector of force is parallel with the displacement vector. Their dot product is therefore equal to the product of these vectors' size. Hence, this integration simply becomes

$$W = \int F \, ds = mg \frac{b}{R} s \tag{9}$$

Kinetic energy must equal the work performed by the rolling friction:

$$\frac{1}{2}mv^2 = mg \frac{b}{R} s \tag{10}$$

From which the braking distance can be calculated as:

$$s(t) = v_0^2 \frac{R}{2gb} \tag{11}$$

For the given parameters of the train, the braking distance would be approximately $s = 50968.40m$. Using excel, the result for the braking distance from the graph was $s_{graph} = 50950m$ which is a precise enough correlation. The result obtained with excel is less precise because it is not analytically determined but numerically calculated. Nonetheless, both results are unrealistic as the braking distance of trains varies from a hundred meter up to maximum a few kilometres. Moreover, this result is independent on the mass of the train which raises suspicion as the inertia and momentum of a heavier object is higher than of a lighter object which should imply a bigger braking distance. The determined braking distance of the train is proportional to the square of the initial velocity at which the train starts braking and this relation is portrayed in figure below. It corresponds with the rise in kinetic energy and the fact that the rate of loss of energy is independent on the velocity.

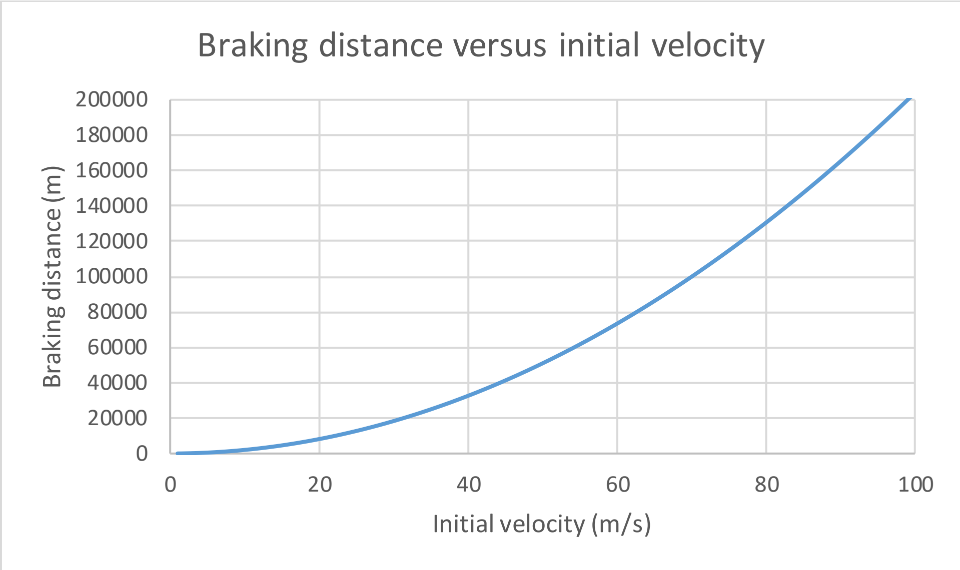

Figure 3 Braking distance of the train versus initial velocity at which the train starts braking when no braking is applied

Frictional brake force

The frictional brake force is caused by more factors. The most important parts are the adhesion traction and the friction in the axle bearings. The adhesion traction is caused by the railway air brake which pulls out levers that rub with the rails and thus cause the static friction. This brake force is intended and the train experiences this force only during the braking process. Static friction is not to be misunderstood with the rolling friction and it is given by the equation: $F = F_N\mu$ where F_N is the normal force acting on the train and μ is the coefficient of friction.

Taking into consideration the frictional brake force combined with the rolling friction, the net force that the train experiences would be their sum because they both point in the same direction. Hence, the deceleration of the train would be given by the equation:

$$m\frac{dv}{dt} = -mg\mu - mg\frac{b}{R} \qquad (12)$$

Which is again a linear differential equation of the first order. Under the same circumstances as in the previous case, this equation can be further calculated as:

$$\int \frac{dv}{dt}dt = \int -g(\mu + \frac{b}{R})\ dt \qquad (16)$$

The train starts braking with velocity v_0 at zero time and its instantaneous at time t is v:

$$\int_{v_0}^{v} dv = -g\left(\mu + \frac{b}{R}\right)\int_0^t dt \qquad (17)$$

And hence the train's velocity at a certain time will be given by the equation:

$$v(t) = v_0 - g\left(\mu + \frac{b}{R}\right)t \tag{13}$$

In order to determine the distance travelled by the train over a certain period of time, formula *18* will be integrated with respect to time:

$$s(t) = \int_0^t [v_0 - g(\mu + \frac{b}{R})t]\, dt \tag{14}$$

Omitting the initial distance the train had crossed before it started braking, the final formula for the distance as a function of time is:

$$s(t) = v_0 t - \frac{g\left(\mu + \frac{b}{R}\right)}{2} t^2 \tag{15}$$

The braking distance will be found as the maximum of equation *20*. As already mentioned, the maximum of a parabolic function can be found by using formula for the vertex. The X-value of the vertex of parabola is:

$$X_V = -\frac{b}{2a} \tag{21}$$

Which in this case is:

$$T_V = \frac{v_0}{g\left(\mu + \frac{b}{R}\right)} \tag{16}$$

Where T_V is the time at which the train reaches zero velocity. Substituting formula *22* in formula *20*, it is obtained:

$$d = v_0 \frac{v_0}{g\left(\mu + \frac{b}{R}\right)} - \frac{g\left(\mu + \frac{b}{R}\right)}{2} \left(\frac{v_0}{g\left(\mu + \frac{b}{R}\right)}\right)^2 \tag{17}$$

And the braking distance of the train can be calculated as:

$$d = \frac{v_0^2}{2g\left(\mu + \frac{b}{R}\right)} \tag{18}$$

For given parameters the braking distance of the train now would be approximately $d = 181.38m$ which is a much more realistic result.

Nevertheless, the obtained equations are still independent on the mass of the train. Kinetic energy of an object is proportional to the mass of the object. The rolling friction force and the frictional brake force are, however, also proportional to the mass of the train. Hence, although a heavier train would have a higher kinetic energy, the rate of loss of the kinetic energy would increase as well and therefore its braking distance would be independent on the mass of the train. This, does not entirely correspond with the reality, as it is known that a heavier object will have a longer braking distance than a lighter object if other properties are the same. This is because there is an upper limit for the frictional forces above which an increase in mass does not increase their size, only the kinetic energy of the train. For that reason, it can be observed in the real world that the braking distance of heavier vehicles is longer than of the lighter ones as long as other parameters remain equal for both of them.

Magnetic braking

Both forces that have been mentioned are independent on the train's velocity. While doing the young physicist tournament, I encountered phenomena known as magnetic braking. It grabbed my attention and I decided to research this braking further for this internal assessment. Many train companies nowadays start to use magnetic brakes this type of braking. Magnetic braking is usually accomplished by conducting wheels and a permanent magnet placed above the surface of the wheels. It is based on the principle of the electromagnetic induction and eddy currents. A conductor placed in a magnetic field experiences a flux linkage flowing through it. If the conductor and the magnetic field are in a relative motion, there is a change in the flux linkage in the conductor over time. This relative motion is simply attained by the rotation of the wheels that establishes motion of the train. Michael Faraday found that in such a case, there is an induced electromotive force in the conductor that is equal to the negative rate of change of the magnetic flux. If the loop does not have an infinite resistance, the induced electromotive force will cause a current flowing through the conductor. Considering the wheels are conductive so they do not have an infinite electrical resistance, there will be indeed a current flowing in them. By Lenz's law "the induced electromotive force will be in such a direction as to oppose the change in the magnetic flux that created the current." Hence, the resultant magnetic force acting on the wheels and thus on the train will point in the opposite direction of the motion.

The magnetic force acting on the train can be calculated as[8]:

$$F(v) = \frac{vAB^2}{\sigma} \tag{19}$$

where B is strength of the magnetic field, A is area of the magnetic patch, σ is area resistivity of the wheel and v is velocity of the train. Area of the magnetic patch is estimated to be $A = 10^{-4} m^2$.

The calculation of velocity and distance as a function of time becomes rather complicated when all of the mentioned forces are combined. Hence, only effect of the magnetic braking force will be examined. The train's motion can be described as:

$$m\frac{dv}{dt} = -\frac{vAB^2}{\sigma} \tag{26}$$

[8] F. Method of calculating eddy currents of a conductor

Equation *26* is a first-order ordinary differential equation. Calculation of this differential equation proceeds as:

$$\frac{1}{v}\frac{dv}{dt} = -\frac{AB^2}{\sigma m} \tag{20}$$

In order to determine equation for the velocity, equation *27* will be integrated with respect to time:

$$\int \frac{1}{v}\frac{dv}{dt}dt = -\frac{AB^2}{\sigma m}\int dt \tag{21}$$

The train started with initial velocity v_0 at zero time, so a certain velocity v will be reached in time t. Hence:

$$\int_{v_0}^{v} \frac{1}{v}dv = -\frac{AB^2}{\sigma m}\int_0^t dt \tag{22}$$

Which results in:

$$\ln\frac{v}{v_0} = -\frac{AB^2}{\sigma m}t \tag{30}$$

In order to obtain the relation for velocity v as a function of time, this simple logarithmic equation can be simplified as:

$$\mathrm{v(t)} = v_0 e^{-\frac{AB^2}{\sigma m}t} \tag{31}$$

The relation of velocity as a function of time when the magnetic braking is used is plotted in the next figure:

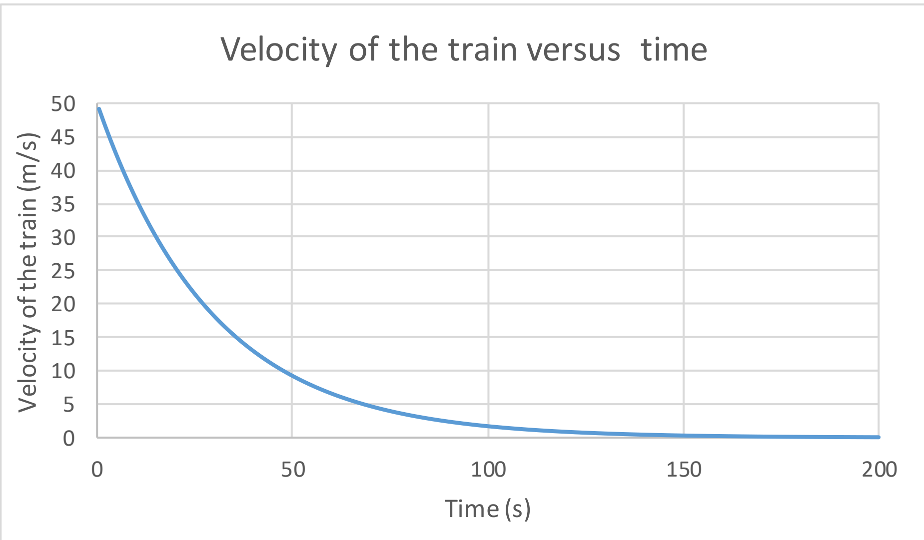

Figure 4. Velocity of the train versus time when magnetic braking is used. The area under the curve is the distance travelled by the train while it slows down.

This result is very interesting. While the previous braking forces were independent on the train's velocity, this braking force is proportional to the velocity of the train. The train's velocity exponentially decreases with time, nevertheless, it would never entirely slow down. The reason is logical – the braking force is proportional to the velocity. When the velocity of the train is high, the rate of loss of the velocity is also big and the train efficiently slows down. As the train's velocity decreases, the braking force that the train experiences also decreases and thus the force would not entirely stop the train.

It might be deduced that this braking force is indeed efficient for high-speed trains. If there is an unexpected situation, such as a car on the railway crossing, this force makes the train slow down in a short period of time and when combined with the frictional brake force, the train would quickly stop.

In order to obtain a formula for the distance crossed by the train as a function of time after the braking process has started, equation *31* will be integrated with reference to time:

$$\mathrm{s(t)} = \int v_0 e^{-\frac{AB^2}{\sigma m}t}\mathrm{dt} \tag{32}$$

This integral can be calculated using the substitution:

$$u = -\frac{AB^2}{\sigma m}t \tag{33}$$

And using this substitution in equation 32:

$$\mathrm{s(t)} = v_0 \int e^u \mathrm{dt} \tag{34}$$

Now, it is necessary that dt is expressed in terms of du:

$$dt = -\frac{\sigma m}{AB^2}du \tag{35}$$

The obtained expression *35* for dt in terms of du can now be used in equation *34*:

$$\mathrm{s(t)} = -\frac{\sigma m}{AB^2}v_0 \int e^u \mathrm{du} \tag{36}$$

Which became a simple integral. Once again, the time before the braking process is not relevant and therefore expression *36* will be integrated within boundaries 0 and t. Also, the integrant constant that represents displacement travelled by the train before the braking process starts will be not considered. Thus, it can be obtained:

$$\mathrm{s(t)} = -\frac{\sigma m}{AB^2}v_0 e^{-\frac{AB^2}{\sigma m}t} \tag{37}$$

Equation *37* expresses displacement of the train as a function of time. In order to obtain a relation for the distance crossed by the train over a certain time period, it is necessary to calculate the difference in the train's displacement over this time. Thus, braking distance would be the difference between the displacement of the train at zero time and infinite time. This can be expressed as:

$$\mathrm{d} = s(infinite) - s(0) \tag{38}$$

In the infinite time, the displacement would already be zero. The displacement then becomes:

$$d = -s(0) = \frac{\sigma m}{AB^2} v_0 e^{-\frac{AB^2}{\sigma m}t} \quad (39)$$

For given parameters the result became $d = 1455.71m$. The most fascinating thing about relation *39* is that the braking distance is directly proportional to the initial velocity and does not increase with its square. Also, it depends on the mass of the train which is realistic as heavier freight trains have longer braking distances. Although the braking distance for given parameters is rather big, this is caused by the evasion of the frictional brake force and rolling friction.

Further study – combination of the forces and aerodynamic drag force

It would be interesting to examine effect of these forces combined and investigate aerodynamic drag force, which is proportional to the square of the velocity. I examined a model in which all braking forces were applied, but the process of braking became rather complicated and did not fit into the range of this internal assessment. Nevertheless, I manually calculated that the velocity of the train as a function of time would be:

$$v(t) = v_0 e^{-\frac{AB^2 t}{\sigma m}} + \frac{mg(b\sigma + \mu\sigma R)[e^{-\frac{AB^2 t}{\sigma m}} - 1]}{AB^2 R} \quad (40)$$

And displacement of the train would behave according to the equation:

$$s(t) = -\frac{\sigma m}{AB^2}\{e^{-\frac{AB^2}{\sigma m}t}[v_0 + \frac{mg\sigma(b + \mu R)}{AB^2 R}] + \frac{g(b + \mu R)}{R}t\} \quad (41)$$

It is necessary to note here that equation *41* would not reach zero in infinite time because it contains a term proportional to time. Calculation of the braking distance would become more complicated.

Conclusion

Forces applied	Braking distance [m]
Rolling friction	50968.40
Rolling friction and frictional brake force	181.38
Magnetic brake force	1455.71

Table 2. Table summarizing the braking distance of different theoretical models.

The primary aim of this internal assessment was to investigate the train's motion using the mathematical apparatus. Braking distances for adequate models were calculated and the results are summarized in table above. Every result has certain limitations and the combined model would be still imprecise as the frictional forces in real life depend on the mass of the object and the aerodynamic drag was not involved.

Nevertheless, the result is still amazing as it has been obtained that using the magnetic braking the braking distance would be proportional to the velocity at which the train starts braking. This is interesting because even the fastest trains could be very efficiently slowed down and braked.

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