T1

Задача №1

$\xi \sim R(0, \theta) \quad \theta > 0$ вер. модель

$\vec{X}_n$ - выборка объема n

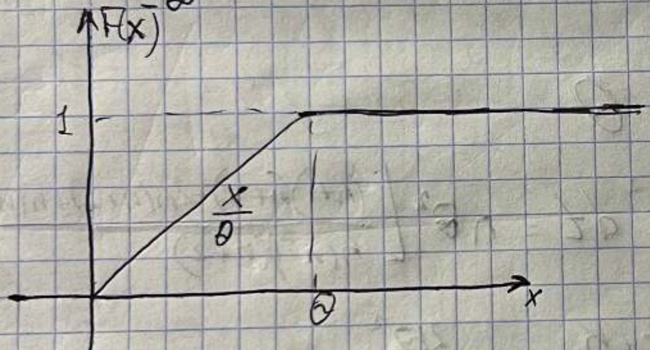
$\tilde{\theta}_1 = 2\bar{x} = 2\frac{1}{n}\sum_{i=1}^{n} x_i$

$\tilde{\theta}_2 = \min X_i$

$\tilde{\theta}_3 = \max X_i$

$\tilde{\theta}_4 = x_1 + \frac{1}{(n-1)} \cdot \sum_{i=2}^{n} x_i$

$M[\xi] = \int_{-\infty}^{+\infty} x \cdot p(x)\, dx = \int_0^{\theta} \frac{x}{\theta} dx = \frac{\theta}{2}$



$p(x) = \frac{1}{\theta} \{(0, \theta)\}$

$M[\xi^2] = \frac{\theta^2}{3}$

$D[\xi] = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$

$\tilde{\theta}_1 = 2 \cdot \bar{x}$

$\forall \theta > 0 \quad M[\tilde{\theta}_1] = \theta$

- $M\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{n}\sum_{i=1}^{n} M[X_i] = 2M[\xi] = \theta \Rightarrow$ несмещ (+)
- $D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum x_i\right] = \frac{4}{n^2}\sum D[x_i] = \frac{4}{n} \cdot D[\xi] =$

$= \frac{\theta^2}{3n} \to 0$ при $n \to \infty$ $\Rightarrow$ по дост. усл. сост. эта оценка.

$\tilde{\Theta}_2 = \min(X_i)$

$$M[\tilde{\Theta}_2] = \int_{-\infty}^{+\infty} y\,\varphi(y)\,dy$$

$$\Phi(y) = 1 - (1 - F(y))^n$$

$$\varphi(y) = \Phi'(y) = n(1 - \underbrace{F(y)}_{\frac{y}{\theta}})^{n-1} \cdot \underbrace{p(y)}_{\frac{1}{\theta}\{(0,\theta)\}}$$

$$M[\tilde{\Theta}_2] = \int_0^{\theta} n\left(1 - \frac{y}{\theta}\right)^{n-1} \cdot y \cdot \frac{1}{\theta}\,dy =$$

$$= -\int_1^0 n \cdot t^{n-1}(1-t) \cdot \theta\,dt = \int_0^1 n \cdot \theta \cdot t^{n-1}\,dt - \int_0^1 n \cdot \theta \cdot t^n\,dt =$$

$$= \theta\left[1 - \frac{n}{n+1}\right] = \frac{\theta}{n+1} \Rightarrow \text{смещ.}$$

$\tilde{\Theta}_2' = (n+1)x_{min}$ – несмещ.

$$M[\tilde{\Theta}_2'] = \theta$$

$$M[\tilde{\Theta}_2^2] = -\int_1^0 n t^{n-1}\,\theta^2 (1-t)^2\,dt = n\theta^2\left[\frac{(n+1)(n+2) - 2n(n+2) + n(n+1)}{n(n+1)(n+2)}\right] =$$

$$= \theta^2 \frac{\cancel{n^2} + \cancel{3n} + 2 - \cancel{2n^2} - \cancel{4n} + \cancel{n^2} + \cancel{n}}{(n+1)(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$$

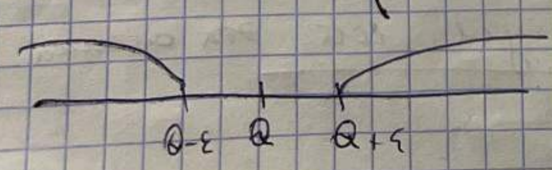
$$D[\tilde{\Theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2\left[\frac{2(n+1) - (n+2)}{(n+1)^2(n+2)}\right] =$$

$$= \theta^2\left[\frac{2n + \cancel{2} - n - \cancel{2}}{(n+1)^2(n+2)}\right] = \theta^2\left[\frac{n}{(n+1)^2(n+2)}\right] \xrightarrow[n\to\infty]{} 0\ ?$$

$$D[\tilde{\Theta}_2'] = \frac{\cancel{(n+1)^2} \cdot n \cdot \theta^2}{\cancel{(n+1)^2}(n+2)} = \frac{n \cdot \theta^2}{n+2} \not\to 0$$ дост. усл. не работает

$\tilde{\Theta}_2'$ – по определению исслед. на сост.

$$\forall \theta > 0\ \ \forall \varepsilon > 0\ \ P\left(|\tilde{\Theta}_2' - \theta| \geq \varepsilon\right) \xrightarrow[n\to\infty]{} 0$$



$$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta + \varepsilon) =$$

$$= P((n+1)\cdot x_{min} \geq \theta + \varepsilon) = P\left(x_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\theta+\varepsilon}{n+1}\right) =$$

$$= 1 - \left(1 - \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right) = \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \xrightarrow[n\to\infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0$$

не явл. состоят.

$\tilde{\theta}_2$ $\quad P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

$$P(\tilde{\theta}_2 < \theta - \varepsilon) + \underbrace{P(\underset{x_{min}}{\underset{\shortparallel}{\tilde{\theta}_2}} > \theta + \varepsilon)}_{\substack{\shortparallel \\ 0}}$$

т.к. $x \in [0, \theta]$, а $\varepsilon > 0$

$$P(x_{min} < \theta - \varepsilon) = \underset{\varepsilon < \theta}{P(\theta - \varepsilon)} = 1 - \left(1 - \frac{\theta - \varepsilon}{\theta}\right)^n =$$

$$= 1 - \left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1$$ => и эта оценка явл. не состоятель

- $\tilde{\theta}_3 = x_{max}$

$$M[\tilde{\theta}_3] = \int_{-\infty}^{\infty} z\,\psi(z)\,dz = \int_0^{\theta} n\cdot\frac{z^n}{\theta^n}\,dz = \frac{n}{n+1}\theta$$ => ~~несм~~ смещ

$$\tilde{\theta}_3' = \frac{n+1}{n}\cdot x_{max} \to$$ несм

$$\Psi(z) = (F(z))^n$$

$$\psi(z) = \Psi'(z) = n\cdot(F(z))^{n-1}\cdot \underset{\frac{1}{\theta}\{(0,\theta)\}}{\overset{\shortparallel}{p(z)}} =$$

$$= n\cdot\left(\frac{z}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}$$

$$\underset{D[\tilde{\theta}_3]}{\overset{\shortparallel}{\cancel{M[\tilde{\theta}_3]}}} = \frac{n}{n+2}\cdot\theta^2 - \frac{n^2}{(n+1)^2}\cdot\theta^2 = \frac{n\cdot\theta^2}{(n+2)\cdot(n+1)^2}$$

$$\overset{\shortparallel}{M[\tilde{\theta}_3^2]}$$

$$D[\tilde{\theta}_3'] = \frac{(n+1)^2}{n^2}\cdot D[\tilde{\theta}_3] = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0$$ => сост.

$\tilde{\Theta}_3$ – по определ. , $\forall \theta>0$, $\forall \varepsilon>0$

$$P(|\tilde{\Theta}_3 - \theta| \ge \varepsilon) = P(X_{max} \le \theta-\varepsilon) + \underbrace{P(X_{max} \ge \theta+\varepsilon)}_{0} =$$

$= (F(\theta-\varepsilon))^n \Rightarrow$ состает.

$0<\varepsilon<\theta$: $\left(\frac{\theta-\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0$

$\varepsilon \ge \theta$: $(0)^n \xrightarrow[n\to\infty]{} 0$

док-ть $\tilde{\Theta}_3'$ несм. по опр.

$\tilde{\Theta}_3'$ – по опр. сост.

$\forall \theta>0, \forall \varepsilon>0$

$$P\left(\left|\tilde{\Theta}_3' - \theta\right| \ge \varepsilon\right) = \underset{①}{P\left(\frac{n+1}{n}\cdot x_{max} \le \theta - \varepsilon\right)} + \underset{②}{P\left(\frac{n+1}{n}\cdot x_{max} \ge \theta +\right.}$$

② $P\left(x_{max} \ge \frac{n}{n+1}\cdot\theta+\varepsilon\right) = P\left(\frac{n+1}{n}\cdot x_{max} \ge \theta+\varepsilon\right)$

т.к. $X\in[0,\theta]$, $\theta>0$, $\varepsilon>0$ и $\frac{n+1}{n}>1 \Rightarrow P\left(\frac{n+1}{n}\cdot x_{max}\ge\theta+\varepsilon\right) =$

① $\left(F\left(\frac{(\theta-\varepsilon)\cdot n}{n+1}\right)\right)^n = \left(\frac{(\theta-\varepsilon)\cdot n}{\theta\cdot(n+1)}\right)^n = \left(\frac{n}{n+1}\right)^n\cdot\left(\frac{\theta-\varepsilon}{\theta}\right)^n =$

$= \left(1-\frac{1}{n+1}\right)^{n+1}\cdot\left(1-\frac{\varepsilon}{\theta}\right)^n\cdot\left(1-\frac{1}{n+1}\right)^{-1} = B$

I при $0<\varepsilon<\theta$: $B\to e^{-1}\cdot 0\cdot 1 = 0$

II при $\varepsilon\ge\theta$: $B\to 0$