№3

$$p(x) = \begin{cases} e^{-x/\theta} \cdot \frac{1}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \theta > 0$$

$$M[\xi] = \int_{-\infty}^{+\infty} x \cdot e^{-x/\theta} \frac{1}{\theta} dx = \int_{0}^{+\infty} x \cdot e^{-x/\theta} \cdot \frac{1}{\theta} dx = \begin{cases} u = x, & dv = \frac{e^{-\frac{x}{\theta}}}{\theta} dx \\ du = 1 dx & v = -e^{-\frac{x}{\theta}} \end{cases}$$

$$\Big\} = -x \cdot e^{-x/\theta} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-\frac{x}{\theta}} dx = \theta$$

$$M[\tilde{\theta}_1] = M\left[\frac{\sum_{i=1}^{n} x_i}{n}\right] = \frac{n}{n} \cdot M[\xi] = \theta \Rightarrow \text{несмещен.} \ \oplus$$

$$F(x) = \int_0^x e^{-\frac{t}{\theta}} \cdot \frac{1}{\theta} dt = \begin{cases} 1 - e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$n = 3$, $k = 2$:

$$\mathscr{X}(x) = 3 \cdot e^{-x/\theta} \cdot \frac{1}{\theta} \cdot C_2^1 \cdot \left(1 - 1 + e^{-\frac{x}{\theta}}\right)^1 \cdot \left(1 - e^{-\frac{x}{\theta}}\right)^1$$

$$\mathscr{X}(x) = 3 \cdot e^{-x/\theta} \cdot \frac{1}{\theta} \cdot 2 \cdot e^{-\frac{x}{\theta}} \left(1 - e^{-x/\theta}\right)$$

$$\mathscr{X}(x) = \frac{6}{\theta} \cdot e^{-\frac{2x}{\theta}} \left(1 - e^{-\frac{x}{\theta}}\right)$$

$$M[\tilde{\theta}_2] = \int_0^{+\infty} x \cdot e^{-\frac{2x}{\theta}} \cdot \frac{6}{\theta} \left(1 - e^{-\frac{x}{\theta}}\right) dx = \frac{5}{6} \theta - \text{смещ.}$$

$$\tilde{\theta}_2^{(1)} = \frac{6}{5} \tilde{\theta}_2 \Rightarrow \text{несмещ.} \ \oplus$$

б) $D[\tilde{\theta}_1] = D\left[\frac{1}{n}\sum x_i\right] = \frac{1}{n^2} n \cdot D[\xi] \underset{\text{при } n=3}{=} \frac{1}{3}\cdot\theta^2 = \frac{\theta^2}{3}$

г) $M[\tilde{\theta}_2] = \frac{5}{6}\cdot\theta$

$$M[\tilde{\theta}_2^2] = M[X_{(2)}^2] = \frac{6}{\theta}\int_0^{+\infty}\left(\overset{①}{x^2\cdot e^{-\frac{2x}{\theta}}} - \overset{②}{x^2\cdot e^{-\frac{3x}{\theta}}}\right)dx =$$

$$= \frac{6}{\theta}\Big(\underbrace{-\frac{\theta}{2}\cdot x^2 e^{-\frac{2x}{\theta}}\Big|_0^{+\infty} + \frac{\theta}{3}x^2\cdot e^{-\frac{3x}{\theta}}\Big|_0^{+\infty}}_{0} + 2\cdot\frac{\theta}{2}\int_0^{+\infty} x\cdot e^{-\frac{2x}{\theta}}dx -$$

$$- 2\cdot\frac{\theta}{3}\int_0^{+\infty} x\cdot e^{-\frac{3x}{\theta}}dx\Big)^{*} = \frac{6}{\cancel{\theta}}\left(\frac{\theta^{\cancel{3}\,2}}{4} - \frac{2\cdot\theta^{\cancel{3}\,2}}{27}\right) = \frac{19}{18}\cdot\theta^2$$

$*$ (2 раза по частям сначала ①, потом ②) ① $u = x^2$, $dv = e^{-\frac{2x}{\theta}}dx$, $du = 2x$, $v = -\frac{\theta}{2}\cdot e^{-\frac{2x}{\theta}}$ ② $u = x^2$, $dv = e^{-\frac{3x}{\theta}}dx$, $du = 2x$, $v = -\frac{\theta}{3}\cdot e^{-\frac{3x}{\theta}}$

$\Downarrow$

$$D[\tilde{\theta}_2] = M[\tilde{\theta}_2^2] - M^2[\tilde{\theta}_2] = \frac{13}{36}\theta^2$$

$$D[\tilde{\theta}_2'] = \frac{\cancel{36}}{25}\cdot\frac{13}{\cancel{36}}\theta^2 = \frac{13\cdot\theta^2}{25}$$

<u>Итог:</u> $D[\tilde{\theta}_1] < D[\tilde{\theta}_2'] \Rightarrow \tilde{\theta}_1$ - эффективнее.

в) $I(\theta) = M\left[\left(\frac{\partial \ln p(x,\theta)}{\partial\theta}\right)^2\right] = M\left[\left(\frac{\partial \ln\left(e^{-x/\theta}/\theta\right)}{\partial\theta}\right)^2\right] =$

$= M\left[\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2\right] = \frac{1}{\theta^4}\cdot M[\xi] - \frac{2}{\theta^3}\cdot M[\xi] + \frac{1}{\theta^2} = \frac{1}{\theta^2} > 0$ ③ $\oplus$ для того, чтобы вероятн. мод. была опт.

$\oplus$ $\in C[(0,+\infty)]$

① $p(x,\theta) = p(x,\theta)$ - непр. дифф. на $(0,+\infty)$ $\oplus$

② $\frac{\partial}{\partial\theta}\int_A p(x,\theta)\,dx \overset{?}{=} \int_A \frac{\partial}{\partial\theta}p(x,\theta)\,dx$

$$① \quad I_1 = \frac{\partial}{\partial\theta}\int_A p(x,\theta)\,dx = \frac{\partial}{\partial\theta}\int_0^{+\infty} e^{-x/\theta}\cdot\frac{1}{\theta}\,dx =$$

$$= \frac{\partial}{\partial\theta}\left(\underset{0-(-1)}{-e^{-\frac{x}{\theta}}\Big|_0^{+\infty}}\right) = 0$$

(II) $\int\limits_0^{+\infty} \frac{\partial}{\partial\theta}\left(e^{-x/\theta}\cdot\frac{1}{\theta}\right)dx = \int\limits_0^{+\infty}\left(\frac{x\cdot e^{-\frac{x}{\theta}}}{\theta^3} - \frac{e^{-x/\theta}}{\theta^2}\right)dx =$

$= \frac{1}{\theta^3}\underbrace{\int\limits_0^{+\infty} x\cdot e^{-x/\theta}}_{\theta^2} - \frac{1}{\theta^2}\underbrace{\int\limits_0^{+\infty} e^{-x/\theta}dx}_{\theta} = \frac{1}{\theta} - \frac{1}{\theta} = 0$

$\Downarrow$

(2) $+$

$\Downarrow$

по опр. регулярн. вер. модели

$\boxed{(1)+}\boxed{(2)+}\boxed{(3)+} \Rightarrow$ вер. модель регулярна

- $D[\tilde{\theta}_1] = \frac{\theta^2}{3}$ – огр. на $\forall$ компакте из $(0, +\infty)$
$\left([a,b] \subset (0,+\infty)\right) \Rightarrow \forall\, \theta \in [a,b] \hookrightarrow \frac{\theta^2}{3} \le \frac{b^2}{3}$

$\Rightarrow$ по дост. усл. регулярной оценки $\Rightarrow$ оценка $\tilde{\theta}_1$ – регуляр.

- $D[\tilde{\theta}_2'] = \frac{13}{25}\theta^2$ – огр. на $\forall$ компакте из $(0, +\infty)$ (аналог. $D[\tilde{\theta}_1]$)

$\Rightarrow$ по дост. усл. регул. оценки: $\tilde{\theta}_3'$ – регулярна

- все условия нер-ва Крамера-Рао соблюдены:

$\forall\, \theta \in \Theta \quad D[\tilde{\theta}] \ge \frac{1}{n\cdot I(\theta)} = \frac{\theta^2}{3}$

$D[\tilde{\theta}_1] = \frac{\theta^2}{3} = \frac{\theta^2}{3}$ по дост. условию эффек.

оценка $\tilde{\theta}_1$ будет ~~на~~ эфф.

А т.к. $\exists!$ эфф. оценка, то $\tilde{\theta}_3'$ – не будет эффект.