№6

$$p(x) = \begin{cases} \frac{\theta - 1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$$

арпмп

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \left(\theta - 1\right)^n \cdot \prod_{i=1}^{n} x_i^{\theta} \{x_{min} \geq 1\}$$

$$\cancel{\ln L = n \cdot \ln \frac{\theta - 1}{x^\theta} \to max}$$

$$\cancel{(\ln L)'_\theta = n \cdot \frac{x^\theta}{\theta - 1} \cdot (\theta - 1) \cdot (\theta) \frac{1}{x^{\theta - 1}} = \frac{n \cdot x}{1} \theta = 0}$$

$$(\ln L)'_\theta = \left(n \cdot \ln(\theta - 1) - \theta \left(\sum_{i=1}^{n} \ln x_i\right)\right)'_\theta$$

$$(\ln L)'_\theta = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow$$

$$\Rightarrow \theta = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-n}{(\theta - 1)^2} < 0 \Rightarrow \text{лок. max.}$$

$$\tilde{\theta} = 1 + \frac{1}{\overline{\ln x_i}}$$

б) $$\frac{\partial p}{\partial \theta} = \frac{1 - \theta \ln x + \ln x}{x^\theta}$$

$$\int_1^{+\infty} \frac{\partial p}{\partial \theta} dx = \int_1^{+\infty} \left[(1 - \theta) \cdot x^{-\theta} \ln x + x^{-\theta}\right] dx = \frac{1}{\theta - 1} - \int_1^{+\infty} x^{-\theta} dx = 0$$

$$\frac{\partial^2 p}{\partial \theta^2} = \ln x \left(\frac{\theta \cdot \ln x - \ln x - 2}{x^\theta}\right)$$

$$\int_1^{+\infty} \frac{\partial^2 p}{\partial x^2} dx = \int_1^{+\infty} \left((\theta - 1) \cdot x^{-\theta} \ln^2 x - 2x^{-\theta} \ln x\right) dx = 0.$$

$$\int_1^{x} \frac{\theta - 1}{x^\theta} dx \Rightarrow x = 2^{\frac{1}{\theta - 1}}$$

$g(\hat\theta) = \sqrt[\hat\theta-1]{2}$, $\nabla g(\theta) = 2^{\frac{1}{\theta-1}} \cdot \ln(2) \cdot (-1) \cdot \frac{1}{(\theta-1)^2}$

Проверим на непр $I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right]$

$\ln p = \ln(\theta-1) - \theta \ln x$

$\left(\frac{\partial \ln p}{\partial \theta}\right)^2 = \left(\frac{1}{\theta-1} - \ln x\right)^2$

$I(\theta) = \int\limits_1^{+\infty} \left(\frac{1}{(\theta-1)^2} + (-2)\frac{\ln x}{\theta-1} + \ln^2 x\right) \frac{\theta-1}{x^\theta}\, dx =$

$= \int\limits_1^{+\infty} \left(\frac{x^{-\theta}}{\theta-1} - 2\frac{\ln x}{x^\theta} + (\theta-1)\left(\frac{\ln^2 x}{x^\theta}\right)\right) dx =$

$= \frac{1}{(\theta-1)^2} - 2\left(\ln \frac{x^{1-\theta}}{1-\theta}\Big|_1^{+\infty} - \int\limits_1^{+\infty} \frac{x^{-\theta}}{1-\theta}\, dx\right) +$ $\downarrow 0$

$+ \int\limits_1^{+\infty} \frac{(\theta-1)\ln^2 x}{x^\theta}\, dx = \frac{-1}{(1-\theta)^2} - \int\limits_1^{+\infty} \ln^2 x \, d(x^{1-\theta}) =$

$= -1 \cdot \frac{1}{(1-\theta)^2} - \left(\ln^2 x \cdot x^{1-\theta}\Big|_1^{+\infty} - 2\int\limits_1^{+\infty} \frac{\ln x}{x} x^{1-\theta}\, dx\right) =$ $\downarrow 0$

$= \frac{-1}{(1-\theta)^2} + 2\left(\frac{x^{1-\theta} \cdot \ln x}{1-\theta}\Big|_1^{+\infty} - \int\limits_1^{+\infty} \frac{x^{-\theta}}{1-\theta}\, dx\right) = \frac{-1}{(1-\theta)^2}$ $\downarrow 0$

$+ 2\frac{1}{(1-\theta)^2} = \frac{1}{(1-\theta)^2} \Rightarrow$ непр на $(1, +\infty)$

$$\sqrt{n} \cdot \frac{(g(\hat\theta) - g(\theta)) \cdot (\hat\theta - 1)}{2^{\frac{1}{\hat\theta-1}} \cdot \ln 2} \to N(0,1)$$

$$g(\tilde{\theta}) - \frac{1{,}96 \cdot 2^{\frac{1}{\hat{\theta}-1}} \cdot \ln 2}{\sqrt{n}\,(\hat{\theta}-1)} < \tilde{x} < g(\hat{\theta}) + \frac{1{,}35 \cdot 2^{\frac{1}{\hat{\theta}-1}}}{\sqrt{n} \cdot (\hat{\theta}-1)}$$

c) рез для параметра θ, то $\nabla g = 1$

$$\frac{\sqrt{n} \cdot (\hat{\theta} - \theta)}{\hat{\theta} - 1} \rightsquigarrow N(0,1)$$

$$t_1 < \sqrt{n} \cdot \overline{\ln x} \cdot \left(1 + \frac{1}{\overline{\ln x}} - \theta\right) < t_2$$

$$\Downarrow$$

$$1 - \frac{t_2 - \sqrt{n}}{\sqrt{n}\,\overline{\ln x}} < \theta < 1 - \frac{t_1 - \sqrt{n}}{\sqrt{n}\,\overline{\ln x}}$$