Exercise solutions for Reinforcement Learning: An Introduction [2nd Edition]

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Chapter 1

Exercise 1.1: Self-Play
Suppose, instead of playing against a random opponent, the reinforcement learning algorithm described above played against itself, with both sides learning. What do you think would happen in this case? Would it learn a different policy for selecting moves?

Solution:

- Early games would be random and unstructured.
- Over time, the agent would improve by reinforcing winning moves and avoiding losing ones.
- Eventually, the agent would converge toward optimal play, leading to most games ending in a draw.
- The RL agent would effectively learn to play tic-tac-toe perfectly, unable to lose but also unable to win against a similarly skilled opponent (itself).

Exercise 1.2 Symmetries
Many tic-tac-toe positions appear different but are really the same because of symmetries. How might we amend the learning process described above to take advantage of this? In what ways would this change improve the learning process? Now think again. Suppose the opponent did not take advantage of

symmetries. In that case, should we? Is it true, then, that symmetrically equivalent positions should necessarily have the same value?

Solution:

- Rather than treating each possible board configuration as unique, the RL agent should group board positions that are equivalent under symmetry. This would reduce the number of unique states the agent needs to learn/investigate.

- No, we shouldn't exploit symmetries if the opponent doesn't exploit them either. If the opponent has a policy that behaves differently at states that are symmetric to each other, then the agent couldn't expoit this if it treated symmetric states the same. In this case symmetrically equivalent positions should not have the same value.

Exercise 1.3 Greedy Play
Suppose the reinforcement learning player was greedy, that is, it always played the move that brought it to the position that it rated the best. Might it learn to play better, or worse, than a nongreedy player? What problems might occur?

Solution:

- Case 1, The opponent is deterministic (always plays the same move at a certain state):
  In the beginnig, all of the non-winning states have a 50% rating, so the greedy player would choose a random move. If this move leads to winning, its estimate increases, so in the following rounds the agent will choose it again and again. If it leads to losing, then its estimate decreases, so the agent will choose another move in the following rounds. This way the agent will find a set of steps that will always lead to winning (if they exist).

- Case 2, The opponent is non-deterministic:
  Suppose at first try the agent finds a set of steps which always produce an estimate that is $> 50\%$. The greedy agent will always choose these steps (the others were initialized at 50%), but there might be a set of steps that achieve better winrate. In this case there is no guarantee

that the agent finds the best policy.
(This is a special case, the winrate can be arbitrarily small)

Exercise 1.4 Learning from Exploration
Suppose learning updates occurred after all moves, including exploratory moves. If the step-size parameter is appropriately reduced over time (but not the tendency to explore), then the state values would converge to a different set of probabilities. What (conceptually) are the two sets of probabilities computed when we do, and when we do not, learn from exploratory moves? Assuming that we do continue to make exploratory moves, which set of probabilities might be better to learn? Which would result in more wins?

Solution:

- If we don't update after exploratory moves: the value is the probability that a greedy agent wins from that state.
- If we update after exploratory moves: the value is the probability that an agent that is prone to explore wins from that state.
- If we continue to explore, then we should include the exploratory moves into the updates.

Exercise 1.5 Other Improvements
Can you think of other ways to improve the reinforcement learning player? Can you think of any better way to solve the tic-tac-toe problem as posed?
Solution:

- Incorporate domain knowledge (Heuristics and Rules): e.g. always take center.
- Use of Value Function Approximation: use a neural network or other function approximator to estimate the value function.

Chapter 2

Exercise 2.1
In ε-greedy action selection, for the case of two actions and $\varepsilon = 0.5$, what is the probability that the greedy action is selected?

Solution:
P(greedy action is selected) $= P$(greedy action is selected | exploratory step)$\cdot$ P(exploratory step) $+$ P(greedy action is selected | non-exploratory step) $\cdot$ P(non-exploratory step) $= 0.5 \cdot 0.5 + 1 \cdot 0.5 = 0.75$.

Exercise 2.2: Bandit example
Consider a k-armed bandit problem with $k = 4$ actions, denoted 1, 2, 3, and 4. Consider applying to this problem a bandit algorithm using ε-greedy action selection, sample-average action-value estimates, and initial estimates of $Q_1(a) = 0$, for all a. Suppose the initial sequence of actions and rewards is $A_1 = 1$, $R_1 = 1$, $A_2 = 2$, $R_2 = 1$, $A_3 = 2$, $R_3 = 2$, $A_4 = 2$, $R_4 = 2$, $A_5 = 3$, $R_5 = 0$. On some of these time steps the ε case may have occurred, causing an action to be selected at random. On which time steps did this definitely occur? On which time steps could this possibly have occurred?

Solution:

$t =$	0	1	2	3	4	5
$Q_t(1)$	0	**1**	1	1	1	1
$Q_t(2)$	0	0	**1**	**1.5**	**1.67**	1.67
$Q_t(3)$	0	0	0	0	0	**0**
$Q_t(4)$	0	0	0	0	0	0

- ε case definitely occured at $t \in \{2, 5\}$
- It may have occured at any other time, too, just by chance the agent may have choosen the greedy step

Exercise 2.3
In the comparison shown in Figure 2.2, which method will perform best in the long run in terms of cumulative reward and probability of selecting the best action? How much better will it be? Express your answer quantitatively.

Solution:
As $t \to \infty$ we have $Q_t(a) \to q_*(a)$ for all a. The agent with $\varepsilon = 0.1$ will choose the correct action only 90% of the time, while the agent with $\varepsilon = 0.01$ will choose it 99% of the time.

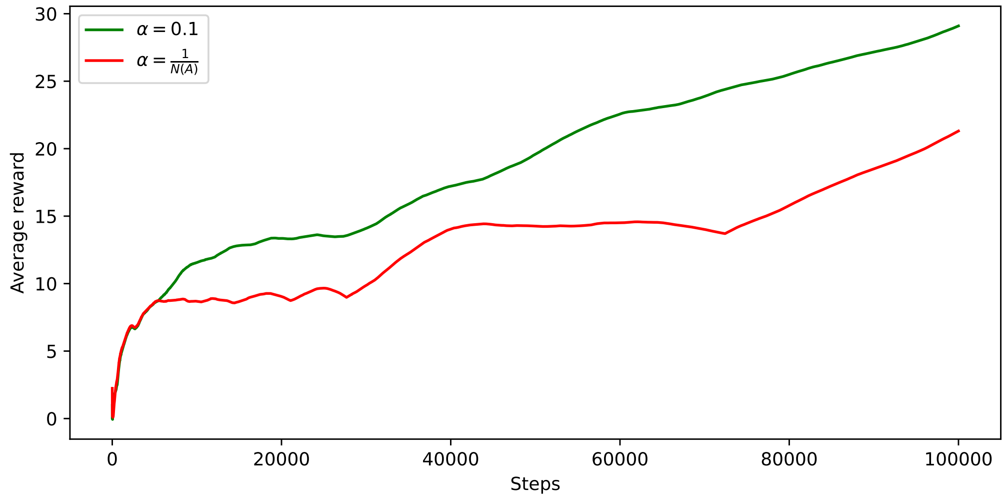
Exercise 2.4
If the step-size parameters, a_n, are not constant, then the estimate Q_n is a weighted average of previously received rewards with a weighting different from that given by (2.6). What is the weighting on each prior reward for the general case, analogous to (2.6), in terms of the sequence of step-size parameters?

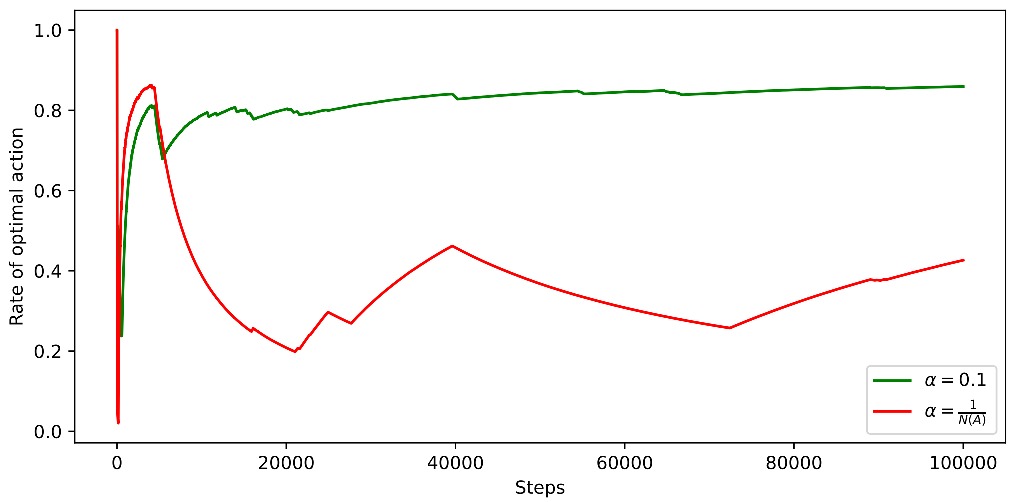
Solution:

$$
\begin{aligned}
Q_{n+1} &= Q_n + \alpha_n \left[R_n - Q_n\right] \\
&= \alpha_n R_n + (1-\alpha_n) Q_n \\
&= \alpha_n R_n + (1-\alpha_n)\left[\alpha_{n-1} R_{n-1} + (1-\alpha_{n-1}) Q_{n-1}\right] \\
&= \alpha_n R_n + (1-\alpha_n)\alpha_{n-1} R_{n-1} + (1-\alpha_n)(1-\alpha_{n-1}) Q_{n-1} \\
&= \left(\prod_{i=1}^{n}(1-\alpha_i)\right) Q_1 + \sum_{i=1}^{n} \alpha_i R_i \prod_{k=i+1}^{n}(1-\alpha_k)
\end{aligned}
\tag{1}
$$

Exercise 2.5 (programming)
Design and conduct an experiment to demonstrate the difficulties that sample-average methods have for nonstationary problems. Use a modified version of the 10-armed testbed in which all the $q_*(a)$ start out equal and then take independent random walks (say by adding a normally distributed increment with mean 0 and standard deviation 0.01 to all the $q_*(a)$ on each step). Prepare plots like Figure 2.2 for an action-value method using sample averages, incrementally computed, and another action-value method using a constant step-size parameter, $\alpha = 0.1$. Use $\varepsilon = 0.1$ and longer runs, say of 10,000 steps.

Solution:
See the notebook.





Exercise 2.6: Mysterious Spikes

The results shown in Figure 2.3 should be quite reliable because they are averages over 2000 individual, randomly chosen 10-armed bandit tasks. Why, then, are there oscillations and spikes in the early part of the curve for the optimistic method? In other words, what might make this method perform particularly better or worse, on average, on particular early steps?

Solution:

The agent might choose the optimal action correctly, then through chance it could receive high rewards initially, so it will choose it many times, thus increasing the % Optimal rate. After a while, again by chance, its estimate of the reward could decrease, so it would choose another suboptimal action, and the % Optimal action would decrease.

Exercise 2.7: Unbiased Constant-Step-Size Trick
In most of this chapter we have used sample averages to estimate action values because sample averages do not produce the initial bias that constant step sizes do (see the analysis leading to (2.6)). However, sample averages are not a completely satisfactory solution because they may perform poorly on nonstationary problems. Is it possible to avoid the bias of constant step sizes while retaining their advantages on nonstationary problems? One way is to use a step size of

$$\beta_t \doteq \alpha/\bar{o}_t, \tag{2}$$

where $\alpha > 0$ is a conventional constant step size and $\bar{o}_t$ is a trace of one that starts at 0:

$$\bar{o}_{t+1} = \bar{o}_t + \alpha(1 - \bar{o}_t) \tag{3}$$

for $t \geq 1$ and with $\bar{o}_1 \doteq \alpha$.

Carry out an analysis like that in (2.6) to show that Q_n is an exponential recency-weighted average without initial bias.

Solution:

$\bar{o}_{t+1} = \bar{o}_t + \alpha(1 - \bar{o}_t) = \bar{o}_t(1 - \alpha) + \alpha$

$$\begin{aligned} \bar{o}_1 &= \alpha \\ \bar{o}_2 &= \alpha + \alpha(1-\alpha) = 2\alpha - \alpha^2 \\ \bar{o}_t &= \sum_{i=1}^{t} \alpha(1-\alpha)^{t-i} \\ &= 1 - (1-\alpha)^t \end{aligned} \tag{4}$$

$$\begin{aligned} Q_{n+1} &= Q_n + \beta_n \left[R_n - Q_n\right] \\ &= Q_n + \frac{\alpha}{1-(1-\alpha)^n} \left[R_n - Q_n\right] \end{aligned} \tag{5}$$

The β_n is decreasing, so it is recency-weighted. The contribution of the initial estimate Q_1 diminishes to zero as t increases, the system effectively forgets the initial bias after a sufficient number of updates.

Exercise 2.8: UCB Spikes
In Figure 2.4 the UCB algorithm shows a distinct spike in performance on the 11th step. Why is this? Note that for your answer to be fully satisfactory it must explain both why the reward increases on the 11th step and why it decreases on the subsequent steps. Hint: If $c = 1$, then the spike is less prominent.

Solution:
At the timesteps $t \leq 10$ there is always an action a for which $N_t(a) = 0$, so the agent will always choose that action. After the 10th step the agent will choose greedily so the average reward increases. After some timesteps the uncertainty term of some other action overtakes the previously chosen action, and the agent starts to explore again, so the average reward decreases.

Exercise 2.9
Show that in the case of two actions, the soft-max distribution is the same as that given by the logistic, or sigmoid, function often used in statistics and artificial neural networks.

Solution:

$$\begin{aligned} P(A_t = 1) &= \frac{e^{H_t(1)}}{\sum_{b=1}^{2} e^{H_t(b)}} = \frac{e^{H_t(1)}}{e^{H_t(1)} + e^{H_t(0)}} = \frac{1}{1 + e^{-(H_t(1) - H_t(0))}} \\ &= \sigma(H_t(1) - H_t(0)) \end{aligned} \tag{6}$$

Exercise 2.10
Suppose you face a 2-armed bandit task whose true action values change randomly from time step to time step. Specifically, suppose that, for any time step, the true values of actions 1 and 2 are respectively 10 and 20 with probability 0.5 (case A), and 90 and 80 with probability 0.5 (case B). If you are not able to tell which case you face at any step, what is the best expected reward you can achieve and how should you behave to achieve it? Now suppose that on each step you are told whether you are facing case A or case B

(although you still don't know the true action values). This is an associative search task. What is the best expected reward you can achieve in this task, and how should you behave to achieve it?

Solution
If we can't differentiate the 2 cases:
Suppose the agent chooses action 1 with probability p. Then the expected reward is:

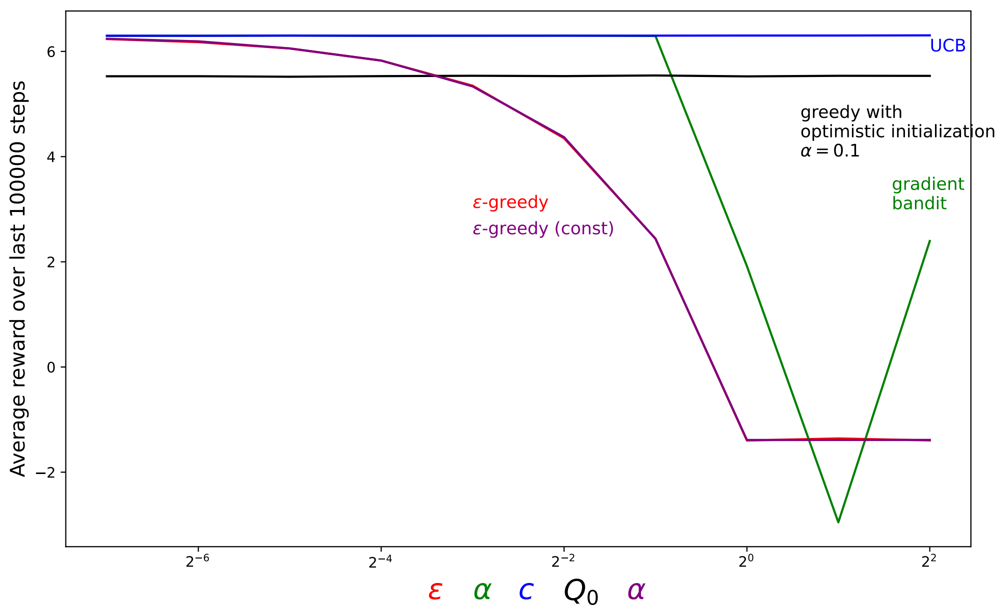
$$\begin{aligned}\mathbb{E}[R] &= P(A) \cdot (pR_A(1) + (1-p)R_A(2)) + P(B) \cdot (pR_B(1) + (1-p)R_B(2)) \\ &= 0.5(10p + 20(1-p)) + 0.5(90p + 80(1-p)) \\ &= 50\end{aligned} \tag{7}$$

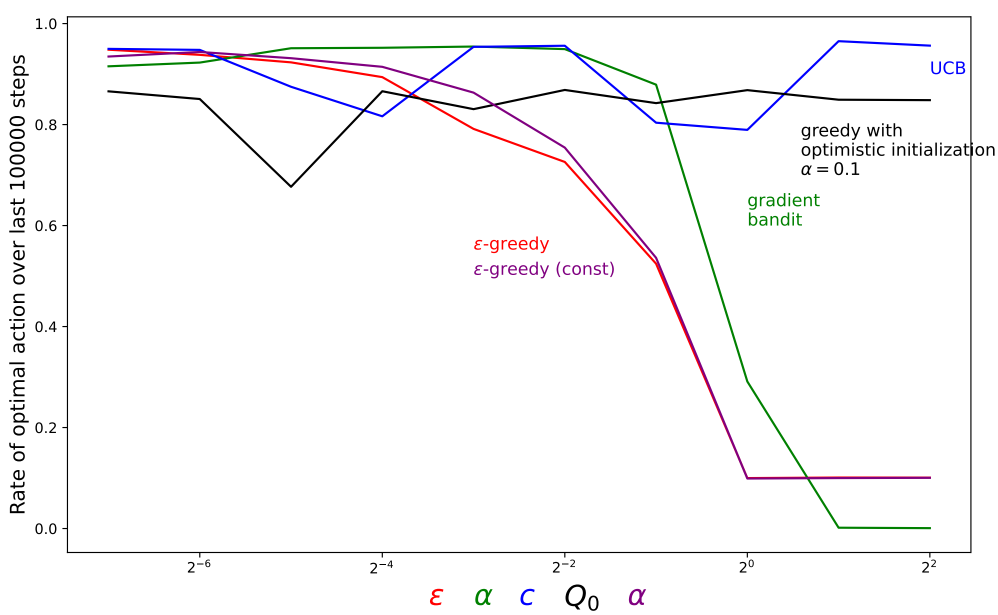
If we *can* differentiate the 2 cases:
We want to maximize the reward, so in case A the agent should choose action 2 and in case B it should choose action 1.

$$\begin{aligned}\mathbb{E}[R] &= P(A) \cdot R_A(2) + P(B) \cdot R_B(1) \\ &= 0.5 \cdot 20 + 0.5 \cdot 90 \\ &= 55\end{aligned} \tag{8}$$

Exercise 2.11 (programming)
Make a figure analogous to Figure 2.6 for the nonstationary case outlined in Exercise 2.5. Include the constant-step-size ε-greedy algorithm with $\alpha = 0.1$. Use runs of 200,000 steps and, as a performance measure for each algorithm and parameter setting, use the average reward over the last 100,000 steps.

Solution:
See the notebook.


Average reward over last 100000 steps
UCB
greedy with
optimistic initialization
α = 0.1
gradient
bandit
ε-greedy
ε-greedy (const)
ε α c Q0 α


Rate of optimal action over last 100000 steps
UCB
greedy with
optimistic initialization
α = 0.1
gradient
bandit
ε-greedy
ε-greedy (const)
ε α c Q0 α