Exercise solutions for Reinforcement Learning: An Introduction [2nd Edition]

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Chapter 1

Exercise 1.1: Self-Play
Suppose, instead of playing against a random opponent, the reinforcement learning algorithm described above played against itself, with both sides learning. What do you think would happen in this case? Would it learn a different policy for selecting moves?

Solution:

- Early games would be random and unstructured.
- Over time, the agent would improve by reinforcing winning moves and avoiding losing ones.
- Eventually, the agent would converge toward optimal play, leading to most games ending in a draw.
- The RL agent would effectively learn to play tic-tac-toe perfectly, unable to lose but also unable to win against a similarly skilled opponent (itself).

Exercise 1.2 Symmetries
Many tic-tac-toe positions appear different but are really the same because of symmetries. How might we amend the learning process described above to take advantage of this? In what ways would this change improve the learning process? Now think again. Suppose the opponent did not take advantage of

symmetries. In that case, should we? Is it true, then, that symmetrically equivalent positions should necessarily have the same value?

Solution:

- Rather than treating each possible board configuration as unique, the RL agent should group board positions that are equivalent under symmetry. This would reduce the number of unique states the agent needs to learn/investigate.

- No, we shouldn't exploit symmetries if the opponent doesn't exploit them either. If the opponent has a policy that behaves differently at states that are symmetric to each other, then the agent couldn't expoit this if it treated symmetric states the same. In this case symmetrically equivalent positions should not have the same value.

Exercise 1.3 Greedy Play

Suppose the reinforcement learning player was greedy, that is, it always played the move that brought it to the position that it rated the best. Might it learn to play better, or worse, than a nongreedy player? What problems might occur?

Solution:

- Case 1, The opponent is deterministic (always plays the same move at a certain state):
  In the beginnig, all of the non-winning states have a 50% rating, so the greedy player would choose a random move. If this move leads to winning, its estimate increases, so in the following rounds the agent will choose it again and again. If it leads to losing, then its estimate decreases, so the agent will choose another move in the following rounds. This way the agent will find a set of steps that will always lead to winning (if they exist).

- Case 2, The opponent is non-deterministic:
  Suppose at first try the agent finds a set of steps which always produce an estimate that is $> 50\%$. The greedy agent will always choose these steps (the others were initialized at 50%), but there might be a set of steps that achieve better winrate. In this case there is no guarantee

that the agent finds the best policy.
(This is a special case, the winrate can be arbitrarily small)

Exercise 1.4 Learning from Exploration
Suppose learning updates occurred after all moves, including exploratory moves. If the step-size parameter is appropriately reduced over time (but not the tendency to explore), then the state values would converge to a different set of probabilities. What (conceptually) are the two sets of probabilities computed when we do, and when we do not, learn from exploratory moves? Assuming that we do continue to make exploratory moves, which set of probabilities might be better to learn? Which would result in more wins?

Solution:

- If we don't update after exploratory moves: the value is the probability that a greedy agent wins from that state.
- If we update after exploratory moves: the value is the probability that an agent that is prone to explore wins from that state.
- If we continue to explore, then we should include the exploratory moves into the updates.

Exercise 1.5 Other Improvements
Can you think of other ways to improve the reinforcement learning player? Can you think of any better way to solve the tic-tac-toe problem as posed?
Solution:

- Incorporate domain knowledge (Heuristics and Rules): e.g. always take center.
- Use of Value Function Approximation: use a neural network or other function approximator to estimate the value function.

Chapter 2

Exercise 2.1
In ε-greedy action selection, for the case of two actions and $\varepsilon = 0.5$, what is the probability that the greedy action is selected?

Solution:
$P(\text{greedy action is selected}) = P(\text{greedy action is selected} \mid \text{exploratory step}) \cdot P(\text{exploratory step}) + P(\text{greedy action is selected} \mid \text{non-exploratory step}) \cdot P(\text{non-exploratory step}) = 0.5 \cdot 0.5 + 1 \cdot 0.5 = 0.75$.

Exercise 2.2: Bandit example
Consider a k-armed bandit problem with $k = 4$ actions, denoted 1, 2, 3, and 4. Consider applying to this problem a bandit algorithm using ε-greedy action selection, sample-average action-value estimates, and initial estimates of $Q_1(a) = 0$, for all a. Suppose the initial sequence of actions and rewards is $A_1 = 1$, $R_1 = 1$, $A_2 = 2$, $R_2 = 1$, $A_3 = 2$, $R_3 = 2$, $A_4 = 2$, $R_4 = 2$, $A_5 = 3$, $R_5 = 0$. On some of these time steps the ε case may have occurred, causing an action to be selected at random. On which time steps did this definitely occur? On which time steps could this possibly have occurred?

Solution:

$t =$	0	1	2	3	4	5
$Q_t(1)$	0	**1**	1	1	1	1
$Q_t(2)$	0	0	**1**	**1.5**	**1.67**	1.67
$Q_t(3)$	0	0	0	0	0	**0**
$Q_t(4)$	0	0	0	0	0	0

- ε case definitely occured at $t \in \{2, 5\}$
- It may have occured at any other time, too, just by chance the agent may have choosen the greedy step

Exercise 2.3
In the comparison shown in Figure 2.2, which method will perform best in the long run in terms of cumulative reward and probability of selecting the best action? How much better will it be? Express your answer quantitatively.

Solution:
As $t \to \infty$ we have $Q_t(a) \to q_*(a)$ for all a. The agent with $\varepsilon = 0.1$ will choose the correct action only 90% of the time, while the agent with $\varepsilon = 0.01$ will choose it 99% of the time.

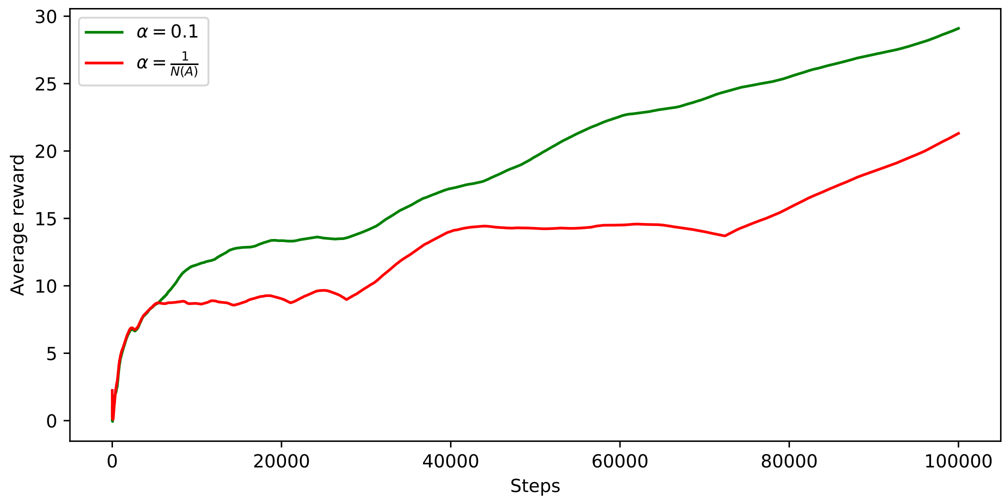
Exercise 2.4
If the step-size parameters, a_n, are not constant, then the estimate Q_n is a weighted average of previously received rewards with a weighting different from that given by (2.6). What is the weighting on each prior reward for the general case, analogous to (2.6), in terms of the sequence of step-size parameters?

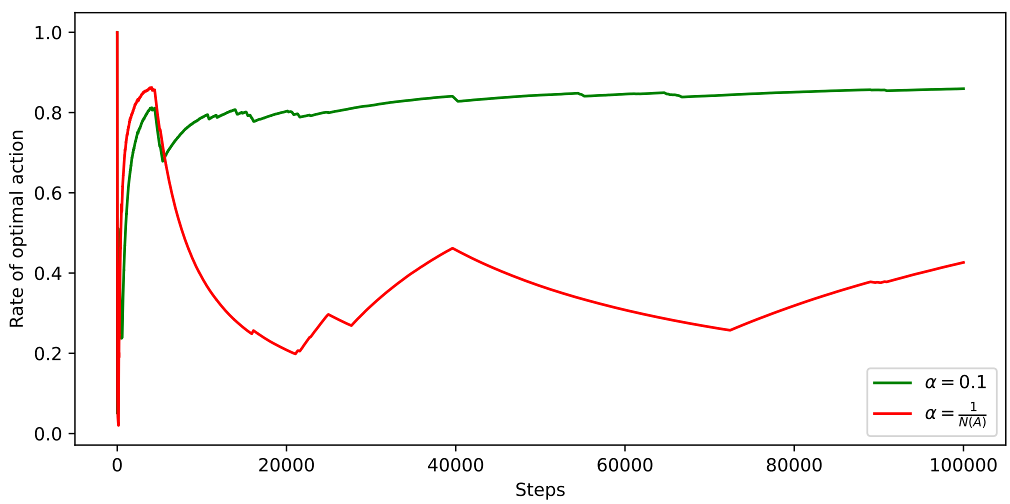
Solution:

$$
\begin{aligned}
Q_{n+1} &= Q_n + \alpha_n \left[R_n - Q_n\right] \\
&= \alpha_n R_n + (1 - \alpha_n) Q_n \\
&= \alpha_n R_n + (1 - \alpha_n) \left[\alpha_{n-1} R_{n-1} + (1 - \alpha_{n-1}) Q_{n-1}\right] \\
&= \alpha_n R_n + (1 - \alpha_n)\alpha_{n-1} R_{n-1} + (1 - \alpha_n)(1 - \alpha_{n-1}) Q_{n-1} \\
&= \left(\prod_{i=1}^{n} (1 - \alpha_i)\right) Q_1 + \sum_{i=1}^{n} \alpha_i R_i \prod_{k=i+1}^{n} (1 - \alpha_k)
\end{aligned}
\tag{1}
$$

Exercise 2.5 (programming)
Design and conduct an experiment to demonstrate the difficulties that sample-average methods have for nonstationary problems. Use a modified version of the 10-armed testbed in which all the $q_*(a)$ start out equal and then take independent random walks (say by adding a normally distributed increment with mean 0 and standard deviation 0.01 to all the $q_*(a)$ on each step). Prepare plots like Figure 2.2 for an action-value method using sample averages, incrementally computed, and another action-value method using a constant step-size parameter, $\alpha = 0.1$. Use $\varepsilon = 0.1$ and longer runs, say of 10,000 steps.

Solution:
See the notebook.





Exercise 2.6: Mysterious Spikes
The results shown in Figure 2.3 should be quite reliable because they are averages over 2000 individual, randomly chosen 10-armed bandit tasks. Why, then, are there oscillations and spikes in the early part of the curve for the optimistic method? In other words, what might make this method perform particularly better or worse, on average, on particular early steps?

Solution:
The agent might choose the optimal action correctly, then through chance it could receive high rewards initially, so it will choose it many times, thus

increasing the % Optimal rate. After a while, again by chance, its estimate of the reward could decrease, so it would choose another suboptimal action, and the % Optimal action would decrease.

Exercise 2.7: Unbiased Constant-Step-Size Trick
In most of this chapter we have used sample averages to estimate action values because sample averages do not produce the initial bias that constant step sizes do (see the analysis leading to (2.6)). However, sample averages are not a completely satisfactory solution because they may perform poorly on nonstationary problems. Is it possible to avoid the bias of constant step sizes while retaining their advantages on nonstationary problems? One way is to use a step size of

$$\beta_t \doteq \alpha / \bar{o}_t, \tag{2}$$

where $\alpha > 0$ is a conventional constant step size and $\bar{o}_t$ is a trace of one that starts at 0:

$$\bar{o}_{t+1} = \bar{o}_t + \alpha(1 - \bar{o}_t) \tag{3}$$

for $t \geq 1$ and with $\bar{o}_1 \doteq \alpha$.

Carry out an analysis like that in (2.6) to show that Q_n is an exponential recency-weighted average without initial bias.

Solution:

$\bar{o}_{t+1} = \bar{o}_t + \alpha(1 - \bar{o}_t) = \bar{o}_t(1 - \alpha) + \alpha$

$$\begin{aligned} \bar{o}_1 &= \alpha \\ \bar{o}_2 &= \alpha + \alpha(1 - \alpha) = 2\alpha - \alpha^2 \\ \bar{o}_t &= \sum_{i=1}^{t} \alpha(1 - \alpha)^{t-i} \\ &= 1 - (1 - \alpha)^t \end{aligned} \tag{4}$$

$$\begin{aligned} Q_{n+1} &= Q_n + \beta_n \left[R_n - Q_n\right] \\ &= Q_n + \frac{\alpha}{1 - (1 - \alpha)^n} \left[R_n - Q_n\right] \end{aligned} \tag{5}$$

The β_n is decreasing, so it is recency-weighted. The contribution of the initial estimate Q_1 diminishes to zero as t increases, the system effectively forgets the initial bias after a sufficient number of updates.

Exercise 2.8: UCB Spikes
In Figure 2.4 the UCB algorithm shows a distinct spike in performance on the 11th step. Why is this? Note that for your answer to be fully satisfactory it must explain both why the reward increases on the 11th step and why it decreases on the subsequent steps. Hint: If $c = 1$, then the spike is less prominent.

Solution:
At the timesteps $t \leq 10$ there is always an action a for which $N_t(a) = 0$, so the agent will always choose that action. After the 10th step the agent will choose greedily so the average reward increases. After some timesteps the uncertainty term of some other action overtakes the previously chosen action, and the agent starts to explore again, so the average reward decreases.

Exercise 2.9
Show that in the case of two actions, the soft-max distribution is the same as that given by the logistic, or sigmoid, function often used in statistics and artificial neural networks.

Solution:

$$\begin{aligned} P(A_t = 1) &= \frac{e^{H_t(1)}}{\sum_{b=1}^{2} e^{H_t(b)}} = \frac{e^{H_t(1)}}{e^{H_t(1)} + e^{H_t(0)}} = \frac{1}{1 + e^{-(H_t(1) - H_t(0))}} \\ &= \sigma(H_t(1) - H_t(0)) \end{aligned} \tag{6}$$

Exercise 2.10
Suppose you face a 2-armed bandit task whose true action values change randomly from time step to time step. Specifically, suppose that, for any time step, the true values of actions 1 and 2 are respectively 10 and 20 with probability 0.5 (case A), and 90 and 80 with probability 0.5 (case B). If you are not able to tell which case you face at any step, what is the best expected reward you can achieve and how should you behave to achieve it? Now suppose that on each step you are told whether you are facing case A or case B

(although you still don't know the true action values). This is an associative search task. What is the best expected reward you can achieve in this task, and how should you behave to achieve it?

Solution
If we can't differentiate the 2 cases:
Suppose the agent chooses action 1 with probability p. Then the expected reward is:

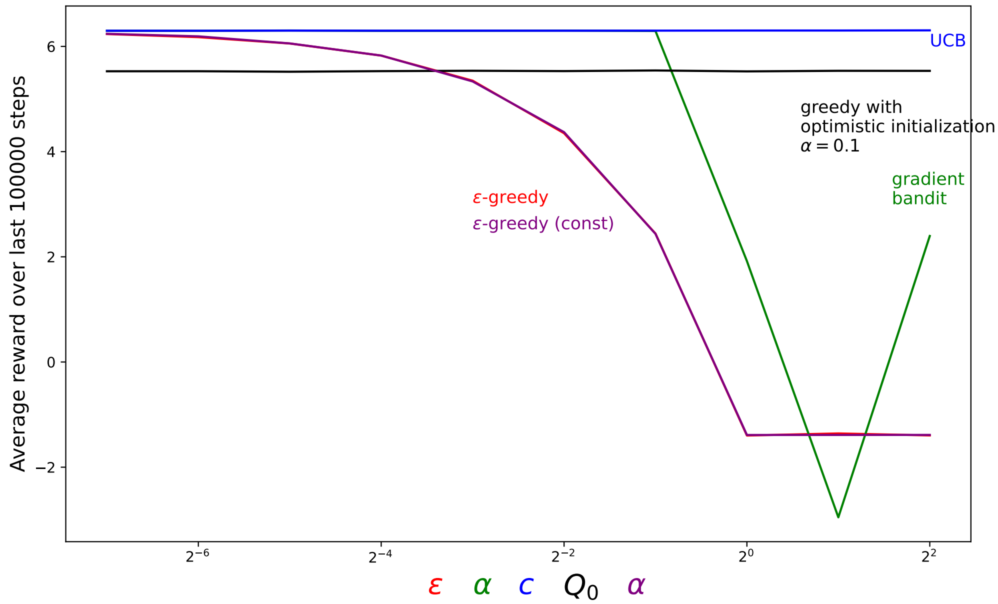
$$\begin{aligned}\mathbb{E}[R] &= P(A) \cdot (pR_A(1) + (1-p)R_A(2)) + P(B) \cdot (pR_B(1) + (1-p)R_B(2)) \\ &= 0.5(10p + 20(1-p)) + 0.5(90p + 80(1-p)) \\ &= 50\end{aligned} \tag{7}$$

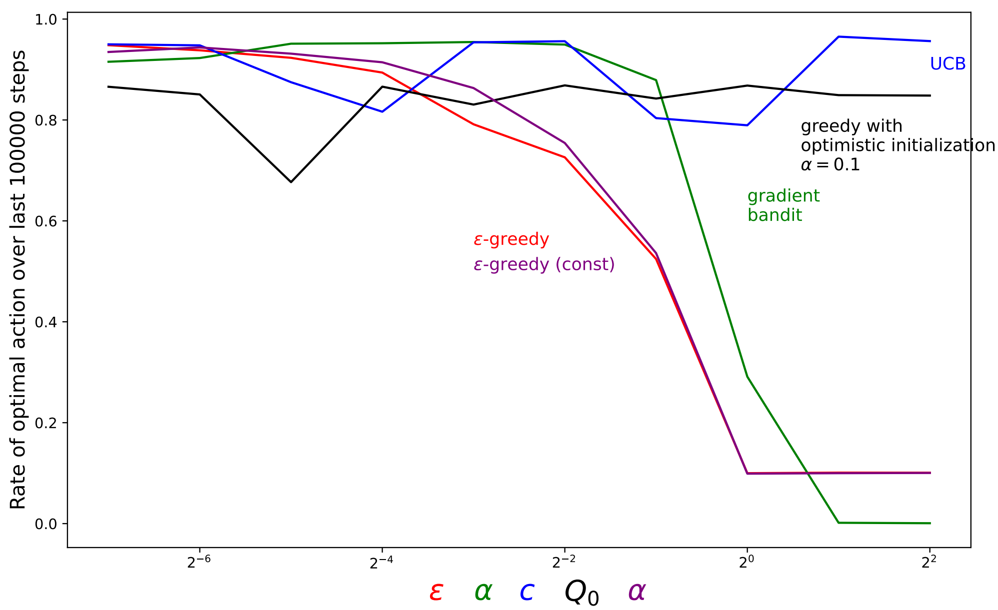
If we *can* differentiate the 2 cases:
We want to maximize the reward, so in case A the agent should choose action 2 and in case B it should choose action 1.

$$\begin{aligned}\mathbb{E}[R] &= P(A) \cdot R_A(2) + P(B) \cdot R_B(1) \\ &= 0.5 \cdot 20 + 0.5 \cdot 90 \\ &= 55\end{aligned} \tag{8}$$

Exercise 2.11 (programming)
Make a figure analogous to Figure 2.6 for the nonstationary case outlined in Exercise 2.5. Include the constant-step-size ε-greedy algorithm with $\alpha = 0.1$. Use runs of 200,000 steps and, as a performance measure for each algorithm and parameter setting, use the average reward over the last 100,000 steps.

Solution:
See the notebook.


Average reward over last 100000 steps
UCB
greedy with
optimistic initialization
$\alpha = 0.1$
gradient
bandit
ε-greedy
ε-greedy (const)
ε α c Q_0 α


Rate of optimal action over last 100000 steps
UCB
greedy with
optimistic initialization
$\alpha = 0.1$
gradient
bandit
ε-greedy
ε-greedy (const)
ε α c Q_0 α

Chapter 3

Exercise 3.1

Devise three example tasks of your own that fit into the MDP framework, identifying for each its states, actions, and rewards. Make the three examples as different from each other as possible. The framework is abstract and flexible and can be applied in many different ways. Stretch its limits in some way in at least one of your examples.

Solution:

Example 1: Chess Player Program
Task Description: A chess-playing program aims to play chess in a way that conceals its identity as a non-human player.

- **States (S):**
  - Current board position (the arrangement of pieces).
  - Time remaining for each player.
  - Opponent's previous move history.
- **Actions (A):**
  - Make a legal move
  - Pause (to simulate human behavior, potentially to analyze).
  - Offer a draw or resignation.
- **Rewards (R):**
  - +1 for winning the game.
  - -1 for drawing or losing.
  - -0.01 for making moves that are too optimal, which might reveal the program's identity (e.g., moving in a way that a human wouldn't).
  - +0.01 for making "human-like" moves (less optimal but more deceptive).

Example 2: Person Taking an IQ Test
Task Description: A person takes an IQ test with the goal of maximizing their score.

- **States (S):**
  - Current question number.
  - The question itself.
  - Time left for the test.
- **Actions (A):**
  - Answer the question (select one of multiple-choice options).
  - Think.
  - Skip the question.
- **Rewards (R):**
  - +1 for a correct answer.
  - 0 for a skipped question.
  - -1 for an incorrect answer.

Example 3: Elevator Control System
Task Description: An elevator system optimizes its operation to minimize wait times and maximize efficiency in a multi-story building.

- **States (S):**
  - Current floor of the elevator.
  - Status of the elevator (moving up, moving down, idle).
  - Status of the door (open, closed)
  - Floors where people wish to go up.
  - Floors where people wish to go down.
  - Current load (number of passengers in the elevator based on weight).
  - Buttons pressed (which floors the passengers in the elevator wish to go to).

- **Actions (A):**
  - Move up.
  - Move down.
  - Open doors.
  - Close doors.
  - Wait on the current floor.
- **Rewards (R):**
  - -1 for each second the passengers spend waiting.
  - -1 for unnecessary movements (e.g., moving to a floor with no waiting passengers).

Exercise 3.2

Is the MDP framework adequate to usefully represent all goal-directed learning tasks? Can you think of any clear exceptions?

Solution:

There are situations where the MDP framework is not adequate.

- Continous action spaces
- Non-Markovian environments
- High-Dimensional state spaces

Example: self-driving for cars have continous action spaces (steering the wheel, rate of acceleration) and high-dimensional state spaces (images of the environment).

Exercise 3.3

Consider the problem of driving. You could define the actions in terms of the accelerator, steering wheel, and brake, that is, where your body meets the machine. Or you could define them farther out—say, where the rubber meets the road, considering your actions to be tire torques. Or you could define them farther in—say, where your brain meets your body, the actions being muscle twitches to control your limbs. Or you could go to a really high level and say that your actions are your choices of where to drive. What is the right level, the right place to draw the line between agent and environment? On what basis is one location of the line to be preferred over another? Is there any fundamental reason for preferring one location over another, or is it a free choice?

Solution:

The right place to draw the line between agent and environment is highly dependant on the problem we are modeling/trying to solve. If we want to create a route finding algorithm, then the highest level of choice would be ideal, but if we want to look at the problem in a neuroscientific way, then the brain and muscle option would be fine.
There is no fundamental reason for preferring one location over another, it is only good for using the advantages of abstraction.

Exercise 3.4

Give a table analogous to that in Example 3.3, but for $p(s', r|s, a)$. It should have columns for s, a, s', r, and $p(s', r|s, a)$, and a row for every 4-tuple for which $p(s', r|s, a) > 0$.

Solution:

s	a	s'	r	$p(s', r\|s, a)$
high	search	high	r_{search}	α
high	search	low	r_{search}	$1 - \alpha$
low	search	high	-3	$1 - \beta$
low	search	low	r_{search}	β
high	wait	high	r_{wait}	1
low	wait	low	r_{wait}	1
low	recharge	high	0	1

Exercise 3.5

The equations in Section 3.1 are for the continuing case and need to be modified (very slightly) to apply to episodic tasks. Show that you know the modifications needed by giving the modified version of (3.3).

Solution:

The original equation:

$$\sum_{s' \in S} \sum_{r \in R} P(s', r \mid s, a) = 1, \quad \forall s \in S, \forall a \in A(s)$$

This would exclude the terminal states $S^+ \setminus S$, so we wouldn't count the cases where a state goes from non-terminal to a terminal state.
The modified equation:

$$\sum_{s' \in S^+} \sum_{r \in R} P(s', r \mid s, a) = 1, \quad \forall s \in S^+, \forall a \in A(s)$$

Exercise 3.6

Suppose you treated pole-balancing as an episodic task but also used discounting, with all rewards zero except for 1 upon failure. What then would the return be at each time? How does this return differ from that in the discounted, continuing formulation of this task?

Solution:

$$G_t = \sum_{k=0}^{T-t} \gamma^k r_{t+k+1} = 0 + 0 + 0 + \cdots + -\gamma^{T-t} = -\gamma^{T-t}$$

The reward in the discounted, continuing formulation:

$$G_t = \sum_{k \in K} -\gamma^{k-t}$$

Where K is the number of time steps before failure (as well as to the times of later failures).

Exercise 3.7

Imagine that you are designing a robot to run a maze. You decide to give it a reward of +1 for escaping from the maze and a reward of zero at all other times. The task seems to break down naturally into episodes—the successive runs through the maze—so you decide to treat it as an episodic task, where the goal is to maximize expected total reward (3.7). After running the learning agent for a while, you find that it is showing no improvement in escaping from the maze. What is going wrong? Have you effectively communicated to the agent what you want it to achieve?

Solution:

The agent receives +1 reward regardless of the timesteps it takes it to leave the maze. Because of this, it can't distinguish a good policy from a bad one.

Exercise 3.8

Suppose $\gamma = 0.5$ and the following sequence of rewards is received $R_1 = -1$, $R_2 = 2$, $R_3 = 6$, $R_4 = 3$, and $R_5 = 2$, with $T = 5$. What are G_0, G_1, $\ldots$, G_5? Hint: Work backwards.

Solution:

$$
\begin{aligned}
G_t &= R_{t+1} + \gamma G_{t+1} \\
G_5 &= 0 \\
G_4 &= 2 + 0.5 \cdot 0 = 2 \\
G_3 &= 3 + 0.5 \cdot 2 = 4 \\
G_2 &= 6 + 0.5 \cdot 4 = 8 \\
G_1 &= 2 + 0.5 \cdot 8 = 6 \\
G_0 &= -1 + 0.5 \cdot 6 = 2
\end{aligned} \tag{9}
$$

Exercise 3.9

Suppose $\gamma = 0.9$ and the reward sequence is $R_1 = 2$ followed by an infinite sequence of 7s. What are G_1 and G_0?

Solution:

$$G_0 = 2 + \sum_{k=1}^{\infty} 0.9^k \cdot 7 = 2 + \frac{0.9 \cdot 7}{1 - 0.9} = 65$$

$$G_1 = \sum_{k=0}^{\infty} 0.9^k \cdot 7 = \frac{7}{1 - 0.9} = 70$$

Exercise 3.10

Prove the second equality in (3.10).

Solution:

The equality in question:

$$\sum_{k=0}^{\infty} \gamma^k = \frac{1}{1-\gamma}$$

Proof:

$$\begin{aligned}
\sum_{k=0}^{\infty} \gamma^k &= 1 + \gamma + \gamma^2 + \gamma^3 + \dots \\
&= 1 + \gamma(1 + \gamma + \gamma^2 + \dots) \\
&= 1 + \gamma \sum_{k=0}^{\infty} \gamma^k
\end{aligned}$$

$$(1-\gamma) \sum_{k=0}^{\infty} \gamma^k = 1$$

$$\sum_{k=0}^{\infty} \gamma^k = \frac{1}{1-\gamma}$$

Exercise 3.10

If the current state is S_t, and actions are selected according to a stochastic policy π, then what is the expectation of R_{t+1} in terms of π and the four-argument function p (3.2)?

Solution:

$$\mathbb{E}_\pi\left[R_{t+1}|S_t = s\right] = \sum_r r \sum_{s',a} \pi(a|s)p(s', r|s, a)$$

Exercise 3.12

Give an equation for v_π in terms of q_π and π.

Solution:

$$\begin{aligned} v_\pi(s) &= \mathbb{E}_\pi\left[G_{t+1}|S_t = s\right] \\ &= \sum_a \mathbb{E}_\pi\left[G_{t+1}|S_t = s, A_t = a\right]\pi(a|s) \\ &= \sum_a q_\pi(s, a)\pi(a|s) \end{aligned}$$

Exercise 3.13

Give an equation for q_π in terms of v_π and the four-argument p.

Solution:

$$\begin{aligned} q_\pi(s, a) &= \mathbb{E}_\pi\left[G_{t+1}|S_t = s, A_t = a\right] \\ &= \sum_{s',r} p(s', r|s, a)(r + \gamma\mathbb{E}_\pi\left[G_{t+2}|S_{t+1} = s'\right]) \\ &= \sum_{s',r} p(s', r|s, a)(r + \gamma v_\pi(s')) \end{aligned}$$

Exercise 3.14

The Bellman equation (3.14) must hold for each state for the value function v_π shown in Figure 3.2 (right) of Example 3.5. Show numerically that this equation holds for the center state, valued at $+0.7$, with respect to its four neighboring states, valued at $+2.3$, $+0.4$, -0.4, and $+0.7$. (These numbers are accurate only to one decimal place.)

Solution:

$$\begin{aligned} v_\pi(s) &= \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \left[r + \gamma v_\pi(s')\right] \\ &= 0.25 \cdot 0.9 \cdot (2.3 + 0.4 - 0.4 + 0.7) \\ &= 0.675 \end{aligned}$$

Exercise 3.15

In the gridworld example, rewards are positive for goals, negative for running into the edge of the world, and zero the rest of the time. Are the signs of these rewards important, or only the intervals between them? Prove, using (3.8), that adding a constant c to all the rewards adds a constant, v_c, to the values of all states, and thus does not affect the relative values of any states under any policies. What is v_c in terms of c and γ?

Solution:

$$\begin{aligned} G'_t &= \sum_{k=0}^{\infty} \gamma^k (R_{t+k+1} + c) \\ &= \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} + c \sum_{k=0}^{\infty} \gamma^k \\ &= G_t + \frac{c}{1-\gamma} \end{aligned}$$

Exercise 3.16

Now consider adding a constant c to all the rewards in an episodic task, such as maze running. Would this have any effect, or would it leave the task unchanged as in the continuing task above? Why or why not? Give an example.

Solution:

This could have an adverse effect for the task. The maze running example could be formulated as the agent getting a -1 reward every timestep while still in the maze, and a 0 reward when reaching the exit. Adding a $c > 1$ constant to every reward would mean that the agent is incentivised to stay in the maze and avoid the exit.

Exercise 3.17

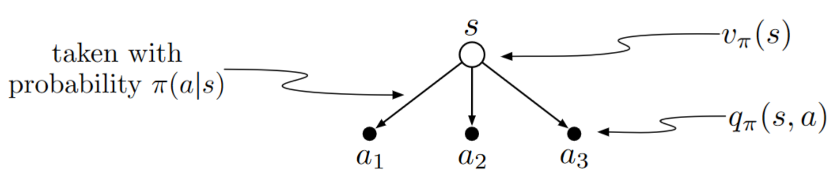
What is the Bellman equation for action values, that is, for q_π? It must give the action value $q_\pi(s, a)$ in terms of the action values, $q_\pi(s', a')$, of possible successors to the state-action pair (s, a). Hint: The backup diagram to the right corresponds to this equation. Show the sequence of equations analogous to (3.14), but for action values.

Solution:

$$\begin{aligned}
q_\pi(s,a) &= \mathbb{E}_\pi\left[G_{t+1} | S_t = s, A_t = a\right] \\
&= \sum_{s',r} p(s',r|s,a)\left(r + \gamma \sum_{a'} \pi(a'|s')\mathbb{E}_\pi\left[G_{t+2}|S_{t+1} = s', A_{t+1} = a'\right]\right) \\
&= \sum_{s',r} p(s',r|s,a)\left(r + \gamma \sum_{a'} \pi(a'|s') q_\pi(s',a')\right)
\end{aligned}$$

Exercise 3.18

The value of a state depends on the values of the actions possible in that state and on how likely each action is to be taken under the current policy. We can think of this in terms of a small backup diagram rooted at the state and considering each possible action:



Give the equation corresponding to this intuition and diagram for the value at the root node, $v_\pi(s)$, in terms of the value at the expected leaf node, $q_\pi(s, a)$, given $S_t = s$. This equation should include an expectation conditioned on following the policy, π. Then give a second equation in which the expected value is written out explicitly in terms of $\pi(a|s)$ such that no expected value notation appears in the equation.

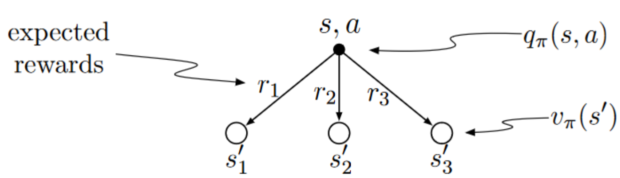
Solution:

$$v_\pi(s) = \mathbb{E}_\pi \left[q_\pi(s,a) | S_t = s \right]$$

$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)$$

Exercise 3.19

The value of an action, $q_\pi(s,a)$, depends on the expected next reward and the expected sum of the remaining rewards. Again we can think of this in terms of a small backup diagram, this one rooted at an action (state-action pair) and branching to the possible next states:



Give the equation corresponding to this intuition and diagram for the action value, $q_\pi(s,a)$, in terms of the expected next reward, R_{t+1}, and the expected next state value, $v_\pi(S_{t+1})$, given that $S_t = s$ and $A_t = a$. This equation should include an expectation but not one conditioned on following the policy. Then give a second equation, writing out the expected value explicitly in terms of $p(s', r|s,a)$ defined by (3.2), such that no expected value notation appears in the equation.

Solution:

$$q_\pi(s,a) = \mathbb{E} \left[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s, A_t = a \right]$$

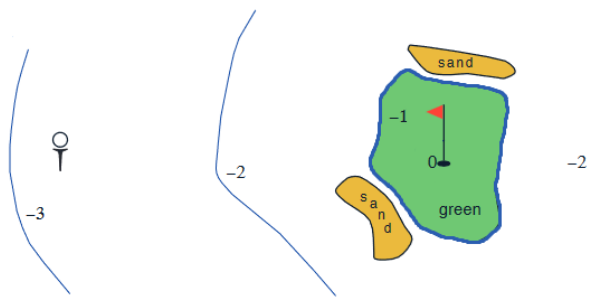
$$q_\pi(s,a) = \sum_{s',r} p(s', r|s,a) \left(r + \gamma v_\pi(s') \right)$$

Exercise 3.20

Draw or describe the optimal state-value function for the golf example.

Solution:

Use the putter when on the green, use the driver in the other cases.

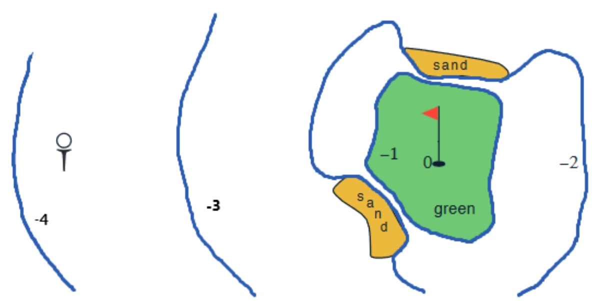


Exercise 3.21

Draw or describe the contours of the optimal action-value function for putting, $q_*(s, \text{putter})$, for the golf example.

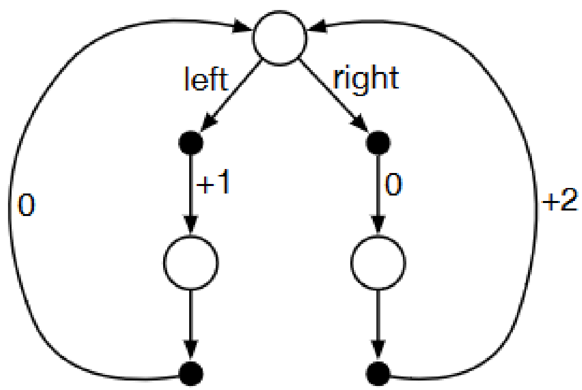
Solution:

$q_*(s, \text{putter})$ is equal to $1 + v_*(s')$ where s' is the new location of the ball. If the ball arrives at the green, the next action should be using a putter, else the agent should use a driver.



Exercise 3.22

Consider the continuing MDP shown to the right. The only decision to be made is that in the top state, where two actions are available, left and right. The numbers show the rewards that are received deterministically after each action. There are exactly two deterministic policies, π_{left} and π_{right}. What policy is optimal if $\gamma = 0$? If $\gamma = 0.9$? If $\gamma = 0.5$?



Solution:

After 2 actions the state will return to the original state, so $R_n = R_{n+2}$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 G_t$$

$$(1 - \gamma^2)G = R_{t+1} + \gamma R_{t+2}$$

$$G_t = \frac{R_{t+1} + \gamma R_{t+2}}{1 - \gamma^2}$$

π_{right}: $G_t = \frac{2\gamma}{1-\gamma^2}$
π_{left}: $G_t = \frac{1}{1-\gamma^2}$
The optimal policy maximizes the value function, and because there is no stochasticity, $G_t = v_\pi(s)$.
If $(2\gamma > 1)$, going right is the optimal policy, while in the case of $(2\gamma < 1)$, going left is optimal.
For $(\gamma = 0)$: left is the optimal policy.
For $(\gamma = 0.9)$: right is the optimal policy.
For $(\gamma = 0.5)$: both policies are optimal.

Exercise 3.23

Give the Bellman equation for q_* for the recycling robot.

Solution:

$$
\begin{aligned}
q_*(\text{high}, \text{search}) &= r_{\text{search}} + \gamma \left(\alpha \max_a q_*(\text{high}, a) + (1-\alpha) \max_a q_*(\text{low}, a) \right) \\
q_*(\text{high}, \text{wait}) &= r_{\text{wait}} + \gamma \max_a q_*(\text{high}, a) \\
q_*(\text{low}, \text{search}) &= \beta \left(r_{\text{search}} + \gamma \max_a q_*(\text{low}, a) \right) \\
&\quad + (1-\beta) \left(-3 + \max_a q_*(\text{high}, a) \right) \\
q_*(\text{low}, \text{wait}) &= r_{\text{wait}} + \gamma \max_a q_*(\text{low}, a) \\
q_*(\text{low}, \text{recharge}) &= \gamma \max_a q_*(\text{high}, a)
\end{aligned}
$$

Exercise 3.24

Figure 3.5 gives the optimal value of the best state of the gridworld as 24.4, to one decimal place. Use your knowledge of the optimal policy and (3.8) to express this value symbolically, and then to compute it to three decimal places.

Solution:

$$
G_t = 10 + 0 + 0 + 0 + 0 + 0.9^5 \cdot G_t
$$
$$
G_t = \frac{10}{1 - 0.9^5} = 24.419
$$

Exercise 3.25

Give an equation for v_* in terms of q_*.

Solution:

$$
v_*(s) = \max_a q_*(s, a)
$$

Exercise 3.26

Give an equation for q_* in terms of v_* and the four-argument p.

Solution:

$$q_*(s,a) = \sum_{s',r} p(s',r|s,a)\left(r + \gamma v_*(s')\right)$$

Exercise 3.27

Give an equation for π_* in terms of q_*.

Solution:

$$\pi_*(a|s) = \begin{cases} 1, & \text{if } a = \operatorname{argmax}_a q_*(s,a) \\ 0, & \text{else} \end{cases}$$

Exercise 3.28

Give an equation for π_* in terms of v_* and the four-argument p.

Solution:

$$\pi_*(a|s) = \begin{cases} 1, & \text{if } a = \operatorname{argmax}_a \sum_{s',r} p(s',r|s,a)\left(r + \gamma v_*(s')\right) \\ 0, & \text{else} \end{cases}$$

Exercise 3.29

Rewrite the four Bellman equations for the four value functions (v_π, v_*, q_π, q_*) in terms of the three argument function p (3.4) and the two-argument function r (3.5).

Solution:

$$q_\pi(s,a) = r(s,a) + \gamma \sum_{s',a'} p(s'|s,a)\pi(a'|s')q_\pi(s',a')$$

$$q_*(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|s,a) \max_{a'} \pi(a'|s')q_*(s',a')$$

$$v_\pi(s) = \sum_a \pi(a|s) \left(r(s,a) + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right)$$

$$v_*(s) = \max_a r(s,a) + \gamma \sum_{s'} p(s'|s,a) v_*(s')$$

Chapter 4

Exercise 4.1

Solution:

In Example 4.1, if π is the equiprobable random policy, what is $q_\pi(11, \text{down})$? What is $q_\pi(7, \text{down})$?

$$q_\pi(11, \text{down}) = r + v_\pi(\text{end state}) = -1 + 0 = -1$$

$$q_\pi(7, \text{down}) = r + v_\pi(11) = -1 - 14 = -15$$

Exercise 4.2

In Example 4.1, suppose a new state 15 is added to the gridworld just below state 13, and its actions, left, up, right, and down, take the agent to states 12, 13, 14, and 15, respectively. Assume that the transitions from the original states are unchanged. What, then, is $v_\pi(15)$ for the equiprobable random policy? Now suppose the dynamics of state 13 are also changed, such that action down from state 13 takes the agent to the new state 15. What is $v_\pi(15)$ for the equiprobable random policy in this case?

Solution:

First case:

$$v_\pi(15) = -1 + 0.25(v_\pi(12) + v_\pi(13) + v_\pi(14) + v_\pi(15))$$

$$v_\pi(15) = -\frac{4}{3} + \frac{1}{3}(v_\pi(12) + v_\pi(13) + v_\pi(14)) = -\frac{4}{3} + \frac{1}{3}(-22 - 20 - 14) = -20$$

Second case:
Using the iterative method with $v^1_\pi(15) = -20$.

$$\begin{aligned} v^1_\pi(13) &= -1 + 0.25(v_\pi(12) + v_\pi(9) + v_\pi(14) + v_\pi(15)) \\ &= -1 + 0.25(-22 - 20 - 14 - 20) = -20 \end{aligned}$$

The $v(s)$ values didn't change, so the iterative algorithm can terminate here, $v_\pi(15) = -20$.

Exercise 4.3

What are the equations analogous to (4.3), (4.4), and (4.5), but for action-value functions instead of state-value functions?

Solution:

$$\begin{aligned} q_\pi(s,a) &= \mathbb{E}\left[R_{t+1} + \gamma \sum_{a'} \pi(a' \mid S_{t+1}) q_\pi(S_{t+1}, a') \middle| S_t = s, A_t = a \right] \\ &= \sum_{s',r} p(s', r \mid s, a) \left(r + \gamma \sum_{a'} \pi(a' \mid s') q_\pi(s', a') \right) \end{aligned}$$

$$q_{k+1}(s,a) = \sum_{s',r} p(s', r \mid s, a) \left(r + \gamma \sum_{a'} \pi(a' \mid s') q_k(s', a') \right)$$

Exercise 4.4

The policy iteration algorithm on page 80 has a subtle bug in that it may never terminate if the policy continually switches between two or more policies that are equally good. This is okay for pedagogy, but not for actual use. Modify the pseudocode so that convergence is guaranteed.

Solution:

Instead of

$$\pi(s) \leftarrow \operatorname*{argmax}_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma V(s')]$$

Use:

$$\pi(s) \leftarrow \min\{a \in \underset{a}{\operatorname{argmax}} \sum_{s',r} p(s', r \mid s, a)[r + \gamma V(s')]\}$$

This way if there are multiple policies that are equally good, the agent always chooses the action with the lowest value. We could also check if $q_\pi(s, \pi(s)) == q_\pi(s, a)$.

Exercise 4.5

How would policy iteration be defined for action values? Give a complete algorithm for computing q_*, analogous to that on page 80 for computing v_*. Please pay special attention to this exercise, because the ideas involved will be used throughout the rest of the book.

Solution:

1. **Initialization**
   $Q(s, a) \in \mathbb{R}$ and $\pi(s) \in A(s)$ arbitrarily for all $s \in S$; $Q(\text{terminal}, a) \doteq 0$
2. **Policy Evaluation**
   Loop:

   $\Delta \leftarrow 0$

   Loop for each $s \in S$:

   Loop for each $a \in A(s)$:
   $q \leftarrow Q(s, a)$
   $Q(s, a) \leftarrow \sum_{s',r} p(s', r \mid s, a)[r + \gamma Q(s', \pi(s'))]$
   $\Delta \leftarrow \max(\Delta, |q - Q(s, a)|)$

   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)
3. **Policy Improvement**

   policy-stable $\leftarrow$ *true*

   For each $s \in S$:

   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \operatorname{argmax}_a Q(s, a)$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

   If *policy-stable*, then stop and return $Q \approx q_*$ and $\pi \approx \pi_*$; else go to 2

Exercise 4.6

Suppose you are restricted to considering only policies that are ε-soft, meaning that the probability of selecting each action in each state, s, is at least $\varepsilon/|A(s)|$. Describe qualitatively the changes that would be required in each of the steps 3, 2, and 1, in that order, of the policy iteration algorithm for v_* on page 80.

Solution:

1. $\pi(a \mid s) = 1/|A(s)|$
2. $V(s) \leftarrow \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a)(r + \gamma V(s'))$
3. $\pi(a \mid s) = \begin{cases} 1 - \varepsilon + \frac{\varepsilon}{|A(s)|}, & \text{if } a = \arg\max_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma V(s')] \\ \frac{\varepsilon}{|A(s)|}, & \text{else} \end{cases}$

Exercise 4.7 (programming)

Write a program for policy iteration and re-solve Jack's car rental problem with the following changes. One of Jack's employees at the first location rides a bus home each night and lives near the second location. She is happy to shuttle one car to the second location for free. Each additional car still costs \$2, as do all cars moved in the other direction. In addition, Jack has limited parking space at each location. If more than 10 cars are kept overnight at a location (after any moving of cars), then an additional cost of \$4 must be incurred to use a second parking lot (independent of how many cars are kept there). These sorts of nonlinearities and arbitrary dynamics often occur in real problems and cannot easily be handled by optimization methods other than dynamic programming. To check your program, first replicate the results given for the original problem.

Solution:

See the notebooks.

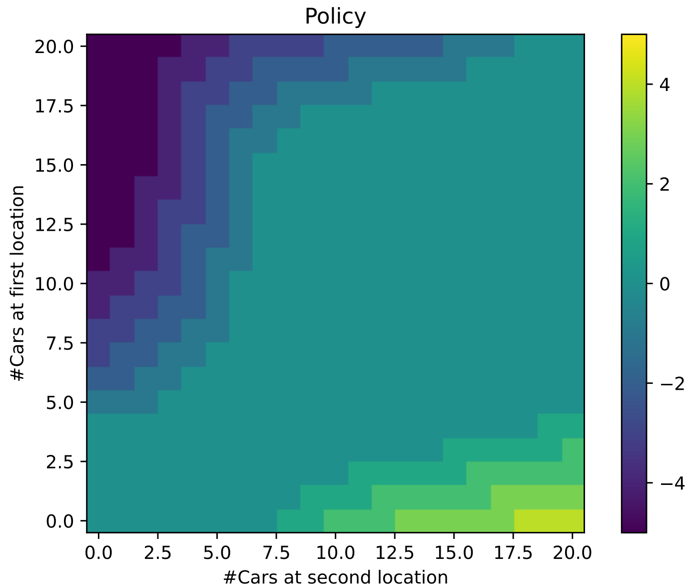


Figure 1: The original policy.

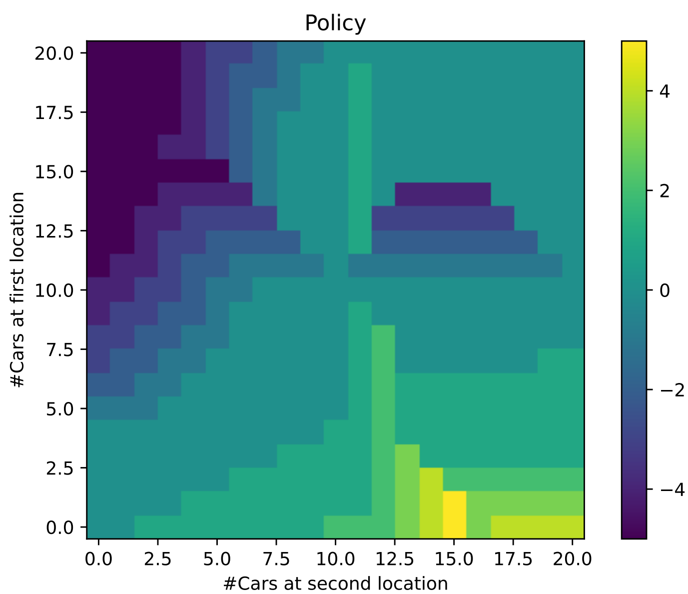


Figure 2: The modified policy.

Exercise 4.8

Why does the optimal policy for the gambler's problem have such a curious form? In particular, for capital of 50 it bets it all on one flip, but for capital of 51 it does not. Why is this a good policy?

Solution:

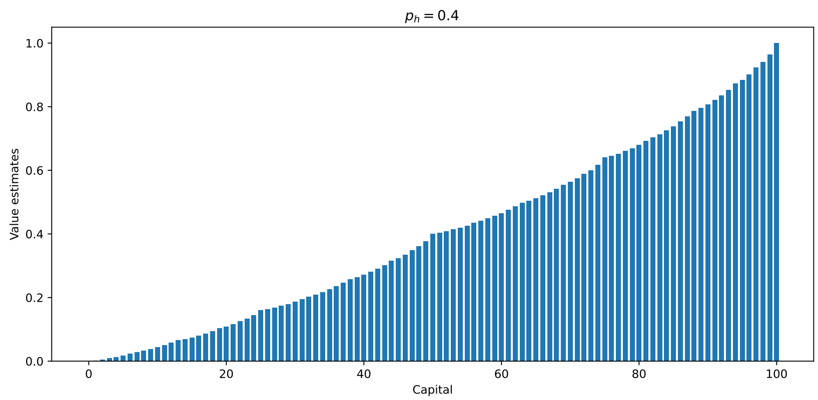
Betting 1 at 51 has two outcomes: receiving 1 coin with probability of 0.6 arriving at 52 coins, and losing 1 coin with probability of 0.4 arriving at 50 coins. At 50 coins we still have a relatively good chance of winning the game in one bet, and at 52 coins we have an even better chance. Betting more than one coin would mean that with great probability we would have less than 50 coins, in which case we would need a minimum of 2 bets to win the game, which is not good.

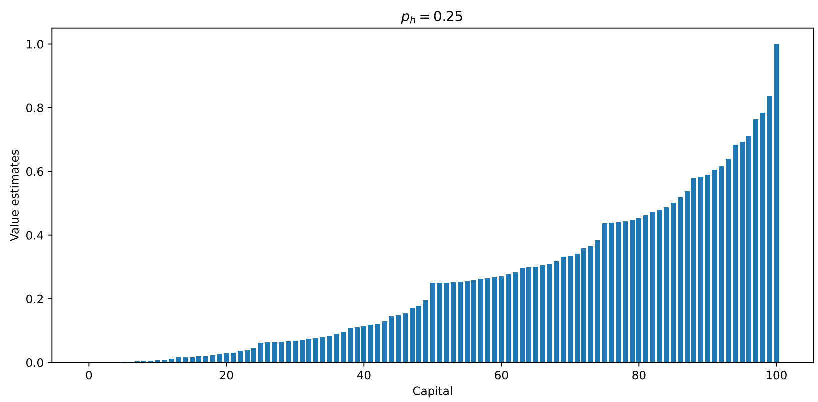
Exercise 4.9 (programming)

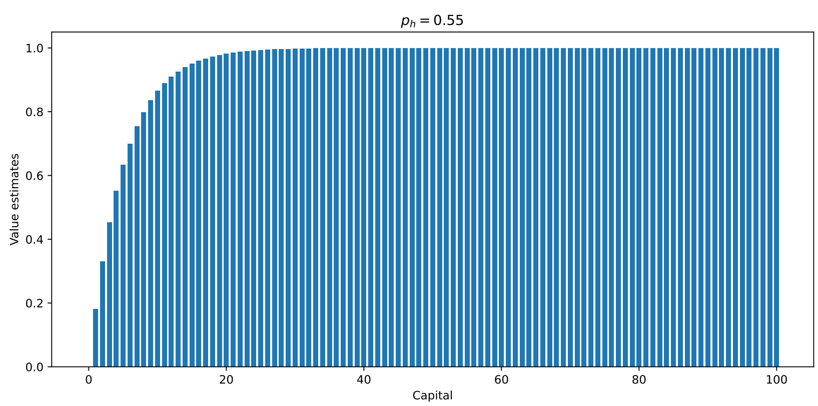
Implement value iteration for the gambler's problem and solve it for $p_h = 0.25$ and $p_h = 0.55$. In programming, you may find it convenient to introduce two dummy states corresponding to termination with capital of 0 and 100, giving them values of 0 and 1 respectively. Show your results graphically, as in Figure 4.3. Are your results stable as $\theta \to 0$?

Solution:

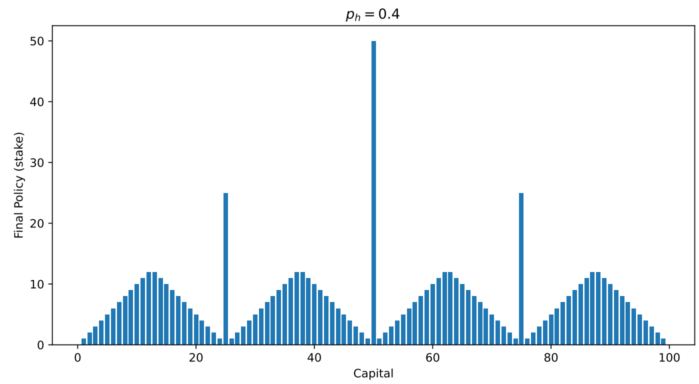
See the notebooks.
The state values for different head probabilities:

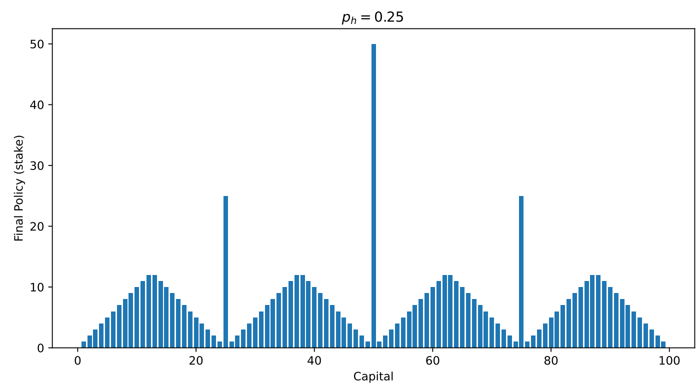


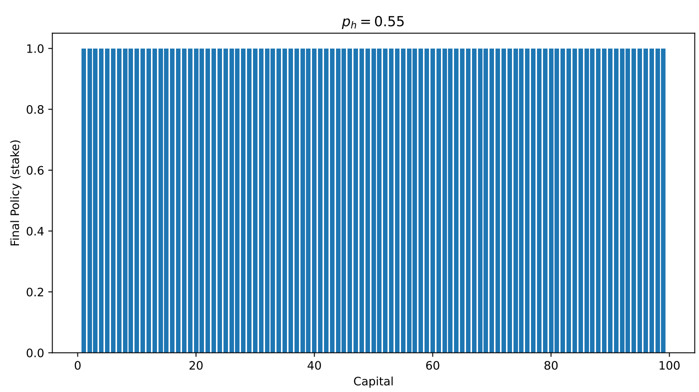




The optimal policies for different head probabilities:


$p_h = 0.4$
Final Policy (stake)
Capital
0
10
20
30
40
50
0
20
40
60
80
100


$p_h = 0.25$
Final Policy (stake)
Capital
0
10
20
30
40
50
0
20
40
60
80
100


$p_h = 0.55$
Final Policy (stake)
Capital
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100

Exercise 4.10

What is the analog of the value iteration update (4.10) for action values, $q_{k+1}(s, a)$?

Solution:

$$q_{k+1}(s, a) = \sum_{s,r'} p(s', r \mid s, a) \left[r + \gamma \max_{a'} q_k(s', a') \right]$$

Chapter 5

Exercise 5.1

Consider the diagrams on the right in Figure 5.1. Why does the estimated value function jump up for the last two rows in the rear? Why does it drop off for the whole last row on the left? Why are the frontmost values higher in the upper diagrams than in the lower?

Solution:

The estimated value function jumps up for the last two rows in the rear because the player has a high probability of winning at 20 and 21, as the dealer is likely to have lower value or go bust.
It drops off for the whole last row on the left because having an ace is a big advantage, thus if the dealer has an ace the probability of the player winning is less.
The frontmost values are higher in the upper diagrams than in the lower because having an usable ace gives an advantage, as the probability of going bust on the next draw is less.

Exercise 5.2

Suppose every-visit MC was used instead of first-visit MC on the blackjack task. Would you expect the results to be very different? Why or why not?

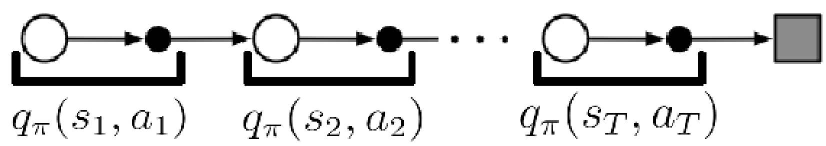
Solution:

It would be similar, as in an episode the probability of visiting the same state twice is very low, so the end results would be almost identical. One such rare case is having two aces, then getting a 10.

Exercise 5.3

What is the backup diagram for Monte Carlo estimation of q_π?

Solution:



Exercise 5.4

The pseudocode for Monte Carlo ES is inefficient because, for each state-action pair, it maintains a list of all returns and repeatedly calculates their mean. It would be more efficient to use techniques similar to those explained in Section 2.4 to maintain just the mean and a count (for each state-action pair) and update them incrementally. Describe how the pseudocode would be altered to achieve this.

Solution:

We would need to initialize a counter variable $N(s, a) = 0 \quad \forall s \in S, \forall a \in A$, we don't need Returns$(s, a)$ and $Q(s, a) = 0 \quad \forall s \in S, \forall a \in A$. In the last step we should have the following:

$$N(S_t, A_t) \leftarrow N(S_t, A_t) + 1$$

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + (G - Q(S_t, A_t))/N(S_t, A_t)$$

Exercise 5.5

Consider an MDP with a single nonterminal state and a single action that transitions back to the nonterminal state with probability p and transitions to the terminal state with probability $1-p$. Let the reward be $+1$ on all transitions, and let $\gamma = 1$. Suppose you observe one episode that lasts 10 steps, with a return of 10. What are the first-visit and every-visit estimators of the value of the nonterminal state?

Solution:

First visit:

$$V(S) = 10$$

Every visit:

$$V(S) = \frac{10 + 9 + \cdots + 1}{10} = 5.5$$

Exercise 5.6

What is the equation analogous to (5.6) for *action* values $Q(s, a)$ instead of state values $V(s)$, again given returns generated using b?

Solution:

$$Q(s, a) = \frac{\sum_{t \in \mathcal{T}(s,a)} \rho_{t+1:T(t)-1} G_t}{\sum_{t \in \mathcal{T}(s,a)} \rho_{t+1:T(t)-1}}$$

Where $\mathcal{T}(s, a)$ are the timesteps where action a was takein in state s.

Exercise 5.7

In learning curves such as those shown in Figure 5.3 error generally decreases with training, as indeed happened for the ordinary importance-sampling method. But for the weighted importance-sampling method error first increased and then decreased. Why do you think this happened?

Solution:

In the beginning there are no good samples, so the estimate is 0. After the first usable sample the estimate might be -1, 0 or 1. This increases the MSE of a single run. In the first 10 episodes more and more runs get their first usable samples, so on average the MSE also increases.

Exercise 5.8

The results with Example 5.5 and shown in Figure 5.4 used a first-visit MC method. Suppose that instead an every-visit MC method was used on the same problem. Would the variance of the estimator still be infinite? Why or why not?

Solution:

Yes, it would still be infinite. Following the proof in the book, instead of at length n only considering that episode consisting of n timesteps, we should take average of all subsamples in that episode. As the importance of the last timestep is the least, this average is greater than the value in first-visit example.

Exercise 5.9

Modify the algorithm for first-visit MC policy evaluation (Section 5.1) to use the incremental implementation for sample averages described in Section 2.4.

Solution:

Input: a policy π to be evaluated
Initialize:
    $V(s) \leftarrow 0$, for all $s \in S$
    $N(s) \leftarrow 0$, for all $s \in S$
Loop forever (for each episode):
    Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \dots, S_{T-1}, A_{T-1}, R_T$
    $G \leftarrow 0$
    Loop for each step of episode, $t = T-1, T-2, \dots, 0$:
        $G \leftarrow \gamma G + R_{t+1}$
        Unless S_t appears in $S_0, S_1, \dots, S_{t-1}$:
            $N(s) \leftarrow N(s) + 1$
            $V(S_t) \leftarrow V(S_t) + (G - V(S_t))/N(S_t)$

Exercise 5.10

Derive the weighted-average update rule (5.8) from (5.7). Follow the pattern of the derivation of the unweighted rule (2.3).

Solution:

$$\begin{aligned}
V_{n+1} &= \frac{\sum_{k=1}^{n} W_k G_k}{\sum_{k=1}^{n} W_k} \\
&= \frac{W_n G_n + \sum_{k=1}^{n-1} W_k G_k}{C_n} \\
&= \frac{W_n G_n + V_n (C_n - W_n)}{C_n} \\
&= V_n + \frac{W_n}{C_n} \left[G_n - V_n \right]
\end{aligned}$$

Exercise 5.11

In the boxed algorithm for off-policy MC control, you may have been expecting the W update to have involved the importance-sampling ratio $\frac{\pi(A_t|S_t)}{b(A_t|S_t)}$, but instead it involves $\frac{1}{b(A_t|S_t)}$. Why is this nevertheless correct?

Solution:

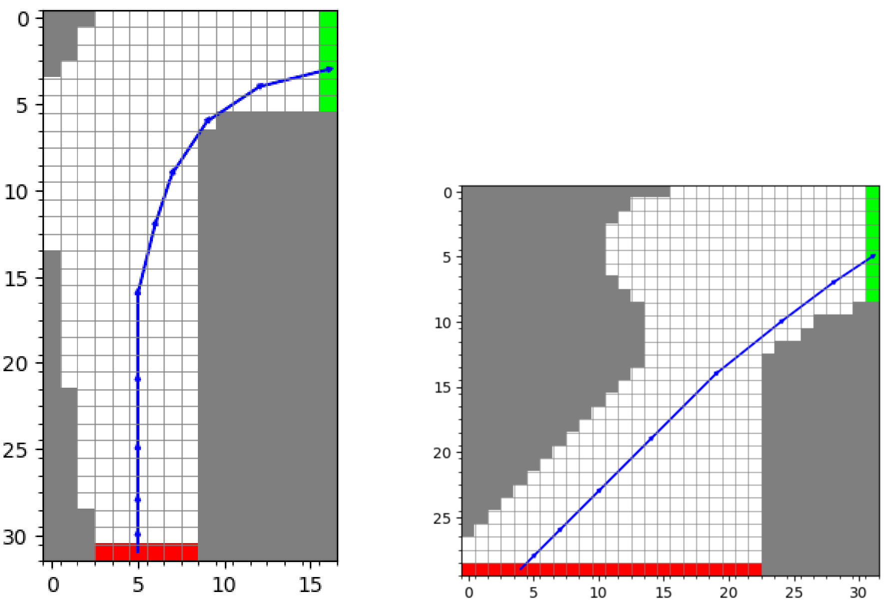
The policy π is greedy, it always chooses the action it thinks is best with probability 1, so $\pi(A_t \mid S_t) = 1$.

Exercise 5.11: Racetrack (programming)

Consider driving a race car around a turn like those shown in Figure 5.5. You want to go as fast as possible, but not so fast as to run off the track. In our simplified racetrack, the car is at one of a discrete set of grid positions, the cells in the diagram. The velocity is also discrete, a number of grid cells moved horizontally and vertically per time step. The actions are increments to the velocity components. Each may be changed by $+1$, -1, or 0 in each step, for a total of nine (3×3) actions. Both velocity components are restricted to be nonnegative and less than 5, and they cannot both be zero except at the starting line. Each episode begins in one of the randomly selected start states with both velocity components zero and ends when the car crosses the finish line. The rewards are -1 for each step until the car crosses the finish line. If the car hits the track boundary, it is moved back to a random position on the starting line, both velocity components are reduced to zero, and the episode continues. Before updating the car's location at each time step, check to see if the projected path of the car intersects the track boundary. If it intersects the finish line, the episode ends; if it intersects anywhere else, the car is considered to have hit the track boundary and is sent back to the starting line. To make the task more challenging, with probability 0.1 at each time step the velocity increments are both zero, independently of the intended increments. Apply a Monte Carlo control method to this task to compute the optimal policy from each starting state. Exhibit several trajectories following the optimal policy (but turn the noise off for these trajectories).

Solution:

See the notebooks.



*Exercise 5.13

Show the steps to derive (5.14) from (5.12).

Solution:

$$
\begin{aligned}
\mathbb{E}\left[\rho_{t:T-1}R_{t+1}\right] &= \mathbb{E}\left[\prod_{k=t}^{T-1}\frac{\pi(A_k \mid S_t)}{b(A_k \mid S_k)}R_{t+1}\right] \\
&= \sum_{A,S,R}\left(\prod_{k=t}^{T-1}\left(p(S_{k+1},R_{k+1}\mid S_k,A_k)b(A_k\mid S_t)\right)\prod_{k=t}^{T-1}\frac{\pi(A_k\mid S_k)}{b(A_k\mid S_k)}R_{t+1}\right) \\
&= \sum_{A,S,R}\left(\prod_{k=t}^{T-1}\left(p(S_{k+1},R_{k+1}\mid S_k,A_k)\right)\prod_{k=t}^{T-1}\pi(A_k\mid S_k)R_{t+1}\right) \\
&= \sum_{A,S,R}\left(R_{t+1}\prod_{k=t}^{T-1}\left(p(S_{k+1},R_{k+1}\mid S_k,A_k)\right)\prod_{k=t}^{T-1}\pi(A_k\mid S_k)\right) \\
&= \sum_{A_t,S_t,R_{t+1}} R_{t+1}p(S_{k+1},R_{t+1}\mid S_t,A_t)\pi(A_t\mid S_t) \\
&= \sum_{A_t,S_t,R_{t+1}} R_{t+1}p(S_{k+1},R_{t+1}\mid S_t,A_t)\frac{\pi(A_t\mid S_t)}{b(A_t\mid S_t)}b(A_t\mid S_t) \\
&= \mathbb{E}\left[\rho_{t:t}R_{t+1}\right]
\end{aligned}
$$

*Exercise 5.14

Modify the algorithm for off-policy Monte Carlo control (page 111) to use the idea of the truncated weighted-average estimator (5.10). Note that you will first need to convert this equation to action values.

Solution:

We want to achieve this:

$$
\begin{aligned}
Q(s,a) &= \frac{\sum_{t\in\mathcal{T}(s,a)}\left((1-\gamma)\sum_{h=t+1}^{T(t)-1}\gamma^{h-t-1}\rho_{t:h-1}\bar{G}_{t:h}+\gamma^{T(t)-t-1}\rho_{t:T(t)-1}\bar{G}_{t:T(t)}\right)}{\sum_{t\in\mathcal{T}(s,a)}\left((1-\gamma)\sum_{h=t+1}^{T(t)-1}\gamma^{h-t-1}\rho_{t:h-1}+\gamma^{T(t)-t-1}\rho_{t:T(t)-1}\right)} \\
&= \frac{\sum_{t\in\mathcal{T}(s,a)}\left((1-\gamma)A_t+W_t\bar{G}_{t:T(t)}\right)}{\sum_{t\in\mathcal{T}(s,a)}\left((1-\gamma)B_t+W_t\right)}
\end{aligned}
$$

$$
W_{t-1} = W_t\gamma\frac{1}{b(A_{t-1}\mid S_{t-1})}
$$

$$B_{t-1} = (\gamma B_t + 1)\frac{1}{b(A_{t-1} \mid S_{t-1})}$$

$$A_{t-1} = (\gamma A_t + R_t)\frac{1}{b(A_{t-1} \mid S_{t-1})} + R_{t-1}B_{t-1}$$

Initialize, for all $s \in S$, $a \in A(s)$:

$Q(s, a) \in \mathbb{R}$ (arbitrarily)

$C(s, a) \leftarrow 0$

$\pi(s) \leftarrow \arg\max_a Q(s, a)$ (with ties broken consistently)

Loop forever (for each episode):

$b \leftarrow$ any soft policy

Generate an episode using b: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$\bar{G} \leftarrow R_T$

$W \leftarrow \frac{1}{b(A_{T-1}|S_{T-1})}$

$A \leftarrow 0$

$B \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$C(S_t, A_t) \leftarrow C(S_t, A_t) + (1-\gamma)B + W$

$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{(1-\gamma)B+W}{C(S_t,A_t)}[(1-\gamma)A + W\bar{G} - Q(S_t, A_t)]$

$\bar{G} \leftarrow \bar{G} + R_t$

$W \leftarrow W\gamma\frac{1}{b(A_{t-1}|S_{t-1})}$

$B \leftarrow (\gamma B + 1)\frac{1}{b(A_t|S_t)}$

$A \leftarrow (\gamma A + R_t)\frac{1}{b(A_t|S_t)} + R_{t-1}B$

$\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken consistently)

If $A_t \neq \pi(S_t)$ then exit inner Loop (proceed to next episode)