Invalid Ranges for Taylor Approximation

(... Add later ...)
Therefore, the problem area is in the interval $[\frac{\pi}{8}, \frac{3\cdot\pi}{8}]$. The Taylor polynomial $T_n(x)$ of degree n has error $R_n(x) = f(x) - T_n(x)$. This can be determined analytically, by calculating

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Do to the possibity of mirroring the range $[\frac{\pi}{8}, \frac{\pi}{4}]$ in the same, as discribed earlier, it is adequate to restrict attention to $[\frac{\pi}{8}, \frac{\pi}{4}]$.
To obtain the best possible taylor approximation of this range the developement point should be the center of it. This leads to $a = \frac{3\cdot\pi}{16}$ and $\max_{x\in[\frac{\pi}{8},\frac{\pi}{4}]} x - a = \frac{1}{16}\pi$. To determin which taylor degree is needed to obtain a sufficiantly small error, it is required to determin the derivatives of the tangens. To obtain an accurate result we choose the following recusion.
(... Add details on the functionality of the recursion ...)

Suppose $t = tan(x)$, then the polynomial P_{n+1} is given by

$$P_{n+1}(t) = (1+t^2)\frac{d}{dt}P_n(t).$$

It is obvious, that the polynomial $P_0(t) = t$, as the 0-th derivative of the tan should be the tan. To illustrate this procedure, the first three derivatives are listed in detail, then the calculations are shorten.

The first derivative is

$$P_1(t) = (1+t^2)\frac{d}{dt}P_0(t) = (1+t^2).$$

The second derivative is

$$P_2(t) = (1+t^2)\frac{d}{dt}P_1(t) = (1+t^2)\frac{d}{dt}(1+t^2) = (1+t^2)\cdot 2t = 2t + 2t^3.$$

The thierd derivative is

$$P_3(t) = (1+t^2)\frac{d}{dt}P_2(t) = (1+t^2)\frac{d}{dt}(2t+2t^3) = (1+t^2)\cdot(2+6t^2) = 2+8t^2+6t^4.$$

To obtain the derivative values for the developement of the tangens, a short c-script was used (...write or link it...). The results are listed in the following table for the developement point $\frac{3}{16}\pi$.

Doing the same for the developement point 0, the following table is obtained.

Derivative	Derivative Value	Taylor Denominator	Taylor Coeff
0	0.66817863791929888	1	0.66817863791929888
1	1.4464626921716894	1	1.4464626921716894
2	1.932990942912723	2	0.9664954714563615
3	6.7676751503806729	6	1.1279458583967787
4	25.820027627097467	24	1.0758344844623944
5	135.23702263622559	120	1.1269751886352133
6	815.83914130228402	720	1.1331099184753946
7	5850.9611918070368	5040	1.160904998374412
8	47551.317519900986	40320	1.179348152775322
9	436443.31842397049	362880	1.2027207848985078
10	4443139.2071207101	3628800	1.2244100548723298
11	49795578.649179153	39916800	1.2474842334350236
12	608587438.15438557	479001600	1.2705332052218314
13	8059141408.449441	6227020800	1.2942210516543387
14	114922949548.23985	87178291200	1.3182519176085876
15	1755911985972.0549	1307674368000	1.3427746455393206
16	28616777966325.5	20922789888000	1.367732416160155
17	495530869503632.75	355687428096000	1.3931638578181318
18	9085375714111666	6402373705728000	1.4190636366607701
19	$1.7583154625861808 \cdot 10^{17}$	$1.21645100408832 \cdot 10^{17}$	1.4454470066420522
20	$3.5820101079484406 \cdot 10^{18}$	$2.43290200817664 \cdot 10^{18}$	1.4723199273582785
21	$7.662072342175998 \cdot 10^{19}$	$5.109094217170944 \cdot 10^{19}$	1.4996929037685105
22	$1.7169949429725245 \cdot 10^{21}$	$1.1240007277776077 \cdot 10^{21}$	1.5275745829519118
23	$4.0225084593790964 \cdot 10^{22}$	$2.5852016738884978 \cdot 10^{22}$	1.5559747233679808
24	$9.8335042453104768 \cdot 10^{23}$	$6.2044840173323941 \cdot 10^{23}$	1.5849028247700077
25	$2.504081303708718 \cdot 10^{25}$	$1.5511210043330986 \cdot 10^{25}$	1.6143687673066762
26	$6.6316542932966174 \cdot 10^{26}$	$4.0329146112660565 \cdot 10^{26}$	1.6443825204656977
27	$1.8238358520470522 \cdot 10^{28}$	$1.0888869450418352 \cdot 10^{28}$	1.6749542827671426
28	$5.2016830410933021 \cdot 10^{29}$	$3.0488834461171384 \cdot 10^{29}$	1.7060944221130627
29	$1.5365333431892238 \cdot 10^{31}$	$8.8417619937397008 \cdot 10^{30}$	1.7378135085259556
30	$4.6953000340818369 \cdot 10^{32}$	$2.6525285981219103 \cdot 10^{32}$	1.7701223042067427
31	$1.4826039361440767 \cdot 10^{34}$	$8.2228386541779224 \cdot 10^{33}$	1.8030317734506245
32	$4.8325374992598159 \cdot 10^{35}$	$2.6313083693369352 \cdot 10^{35}$	1.8365530834675867
33	$1.6243861512167602 \cdot 10^{37}$	$8.6833176188118859 \cdot 10^{36}$	1.8706976095146228
34	$5.625592899507537 \cdot 10^{38}$	$2.9523279903960412 \cdot 10^{38}$	1.9054769381341297
35	$2.0055636558286352 \cdot 10^{40}$	$1.0333147966386144 \cdot 10^{40}$	1.9409028713735235
36	$7.3542613135187288 \cdot 10^{41}$	$3.7199332678990118 \cdot 10^{41}$	1.9769874306568826
37	$2.7716659527005135 \cdot 10^{43}$	$1.3763753091226343 \cdot 10^{43}$	2.0137428609259942
38	$1.072814387515251 \cdot 10^{45}$	$5.2302261746660104 \cdot 10^{44}$	2.0511816347669867
39	$4.2617630713174557 \cdot 10^{46}$	$2.0397882081197442 \cdot 10^{46}$	2.0893164566560101
40	$1.7363984872618265 \cdot 10^{48}$	$8.1591528324789768 \cdot 10^{47}$	2.128160267264243
41	$7.2515919996323374 \cdot 10^{49}$	$3.3452526613163803 \cdot 10^{49}$	2.167726247852019
42	$3.1022926019289015 \cdot 10^{51}$	$1.4050061177528798 \cdot 10^{51}$	2.2080278247404399
43	$1.3587867973196388 \cdot 10^{53}$	$6.0415263063373834 \cdot 10^{52}$	2.2490786738680777
44	$6.0898150129041416 \cdot 10^{54}$	$2.6582715747884485 \cdot 10^{54}$	2.2908927254314801
45	$2.7913655860385158 \cdot 10^{56}$	$1.1962222086548019 \cdot 10^{56}$	2.3334841686123804
46	$1.3079003536482053 \cdot 10^{58}$	$5.5026221598120885 \cdot 10^{57}$	2.3768674563925889
47	$6.2614168951508748 \cdot 10^{59}$	$2.5862324151116818 \cdot 10^{59}$	2.4210573104584983
48	$3.061356901239919 \cdot 10^{61}$	$1.2413915592536073 \cdot 10^{61}$	2.4660687261967325
49	$1.5279535415449727 \cdot 10^{63}$	$6.0828186403426752 \cdot 10^{62}$	2.5119169777826804
50	$7.7818034864528027 \cdot 10^{64}$	$3.0414093201713376 \cdot 10^{64}$	2.5586176233636371

Derivative	Derivative Value	Taylor Denominator	Taylor Coeff
0	0	1	0
1	1	1	1
2	0	2	0
3	2	6	0.33333333333333331
4	0	24	0
5	16	120	0.13333333333333333
6	0	720	0
7	272	5040	0.053968253968253971
8	0	40320	0
9	7936	362880	0.021869488536155203
10	0	3628800	0
11	353792	39916800	0.0088632355299021973
12	0	479001600	0
13	22368256	6227020800	0.0035921280365724811
14	0	87178291200	0
15	1903757312	1307674368000	0.0014558343870513183
16	0	20922789888000	0
17	209865342976	355687428096000	0.00059002744094558595
18	0	6402373705728000	0
19	29088885112832	$1.21645100408832 \cdot 10^{17}$	0.00023912911424355248
20	0	$2.43290200817664 \cdot 10^{18}$	0
21	4951498053124096	$5.109094217170944 \cdot 10^{19}$	$9.6915379569294509 \cdot 10^{-5}$
22	0	$1.1240007277776077 \cdot 10^{21}$	0
23	$1.0154238865068524 \cdot 10^{18}$	$2.5852016738884978 \cdot 10^{22}$	$3.9278323883316833 \cdot 10^{-5}$
24	0	$6.2044840173323941 \cdot 10^{23}$	0
25	$2.4692148019020798 \cdot 10^{20}$	$1.5511210043330986 \cdot 10^{25}$	$1.5918905069328964 \cdot 10^{-5}$
26	0	$4.0329146112660565 \cdot 10^{26}$	0
27	$7.025160160394396 \cdot 10^{22}$	$1.0888869450418352 \cdot 10^{28}$	$6.4516892156554306 \cdot 10^{-6}$
28	0	$3.0488834461171384 \cdot 10^{29}$	0
29	$2.3119184187809599 \cdot 10^{25}$	$8.8417619937397008 \cdot 10^{30}$	$2.6147711512907551 \cdot 10^{-6}$
30	0	$2.6525285981219103 \cdot 10^{32}$	0
31	$8.7139627571251711 \cdot 10^{27}$	$8.2228386541779224 \cdot 10^{33}$	$1.0597268320104656 \cdot 10^{-6}$
32	0	$2.6313083693369352 \cdot 10^{35}$	0
33	$3.7294077037205298 \cdot 10^{30}$	$8.6833176188118859 \cdot 10^{36}$	$4.2949110782738063 \cdot 10^{-7}$
34	0	$2.9523279903960412 \cdot 10^{38}$	0
35	$1.7986516934508886 \cdot 10^{33}$	$1.0333147966386144 \cdot 10^{40}$	$1.7406618963571648 \cdot 10^{-7}$
36	0	$3.7199332678990118 \cdot 10^{41}$	0
37	$9.7098281078505895 \cdot 10^{35}$	$1.3763753091226343 \cdot 10^{43}$	$7.0546369464009681 \cdot 10^{-8}$
38	0	$5.2302261746660104 \cdot 10^{44}$	0
39	$5.8320332491730988 \cdot 10^{38}$	$2.0397882081197442 \cdot 10^{46}$	$2.8591366623052533 \cdot 10^{-8}$
40	0	$8.1591528324789768 \cdot 10^{47}$	0
41	$3.8763598377208303 \cdot 10^{41}$	$3.3452526613163803 \cdot 10^{49}$	$1.1587644432798853 \cdot 10^{-8}$
42	0	$1.4050061177528798 \cdot 10^{51}$	0
43	$2.8372792190743183 \cdot 10^{44}$	$6.0415263063373834 \cdot 10^{52}$	$4.6962953982309007 \cdot 10^{-9}$
44	0	$2.6582715747884485 \cdot 10^{54}$	0
45	$2.2768137912993072 \cdot 10^{47}$	$1.1962222086548019 \cdot 10^{56}$	$1.9033368339312746 \cdot 10^{-9}$
46	0	$5.5026221598120885 \cdot 10^{57}$	0
47	$1.9950025215785898 \cdot 10^{50}$	$2.5862324151116818 \cdot 10^{59}$	$7.7139336353590609 \cdot 10^{-10}$
48	0	$1.2413915592536073 \cdot 10^{61}$	0
49	$1.9016956465792844 \cdot 10^{53}$	$6.0828186403426752 \cdot 10^{62}$	$3.1263395458920874 \cdot 10^{-10}$
50	0	$3.0414093201713376 \cdot 10^{64}$	0