Interval Split

After a range reduction to $[0, \frac{\pi}{2}]$ the interval is split into quadrants.

$$\begin{array}{ll} q_0 = [0, \frac{\pi}{8}) & q_1 = [\frac{\pi}{8}, \frac{\pi}{4}) \\ q_2 = [\frac{\pi}{4}, \frac{3}{8}\pi) & q_3 = [\frac{3}{8}\pi, \frac{\pi}{2}) \end{array}$$

Utilizing basic transforms that can be found in the appendix (add later), vectors are created. This circumvents any if statements as those would require gathering the commands to an ALU register, comparing to a value and loading back int the SIMD register for further processing. This procedure is vastly time expensive and would take away all benifits.

0.0.1 Treatment of q_0

Generally speaking the smaller the values for the more feasable the Taylor approximation around 0 gets. With further explenation in appendix add reference, we are trying to reduce as much as possible before applying Taylor. Values that are not in q_0, cannot be determined with the Taylor polynomial within reasonable time.

0.0.2

Invalid Ranges for Taylor Approximation

When trying to implement a function which is infinitly often differentiable, the Taylor polynomial is in most cases the best choice. In case of the approximation of the tangens taylor is not feasable for values within $[\frac{\pi}{8}, \frac{\pi}{4}]$. The reason for this is the form of the derivative of the tangens and the error of the Taylor polynomial. The error of the n-th degree Taylor polynomial, can is described by the following theorem.

Theorem 0.1 (Taylor Remainder Estimate)**.** *Let $f\colon I \to \mathbb{R}$ be an infinitely differentiable function on an interval I containing the point a. Then, for every integer $n \geq 0$ and every $x \in I$, we have*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x),$$

where the remainder term $R_n(x)$ satisfies

$$|R_n(x)| \leq \frac{\sup\limits_{\xi \in I} |f^{(n+1)}(\xi)|}{(n+1)!} |x-a|^{n+1}.$$

It is obvious, that the minimizing $R_n(x)$ is equivalent to minimizing the error at value x. Before stating why the interval $[0, \frac{\pi}{8}]$, can be approximated by the Taylor polynomial, we investigate the reason why $[\frac{\pi}{8}, \frac{\pi}{4}]$ cannot. To develope the Taylor polynomial for the tangens within $[\frac{\pi}{8}, \frac{\pi}{4}]$, the canonical choice for the developement point would be $a = \frac{3 \cdot \pi}{16}$. Then the error dependent on the distance, can be calucalted by

$$\frac{|x-a|^{n+1}}{(n+1)!}.$$

The maximum can be written as a function of n, due to $x - a$ being maximized for $x = \frac{\pi}{8}$ or $x = \frac{\pi}{4}$ respectively.

$$E^{\text{ind}}(n) = \frac{\left(\frac{\pi}{16}\right)^{n+1}}{(n+1)!}.$$

The first 15 values of the independent error are written in Table 1.

n	Error
1	0.019276571095877652
2	0.001261648627941887
3	$6.193103220240429 \cdot 10^{-5}$
4	$2.432025947453828 \cdot 10^{-6}$
5	$7.958786301938231 \cdot 10^{-8}$
6	$2.2324343372910912 \cdot 10^{-9}$
7	$5.479218213793196 \cdot 10^{-11}$
8	$1.195379978324173 \cdot 10^{-12}$
9	$2.347123098844718 \cdot 10^{-14}$
10	$4.18960493431902 \cdot 10^{-16}$
11	$6.855225043281344 \cdot 10^{-18}$
12	$1.0354002228210309 \cdot 10^{-19}$
13	$1.4521454167588328 \cdot 10^{-21}$
14	$1.9008539055140157 \cdot 10^{-23}$
15	$2.33269869730871 \cdot 10^{-25}$

Table 1: Error values for different n.

The derivative dependend part of the remainder can be obtained through the following iterative scheme.

Lemma 0.2. *Suppose $t = tan(x)$, $P_0(t) = t$ and*

$$P_{n+1}(t) = (1+t^2)\frac{d}{dt}P_n(t).$$

Then the n-th derivative of the tangens is given by $P_n(tan(x))$.

To illustrate this calculation, the the first 3 derivatives are calculated.

$$P_1(t) = (1+t^2)\frac{d}{dt}P_0(t) = (1+t^2).$$

$$P_2(t) = (1+t^2)\frac{d}{dt}P_1(t) = (1+t^2)\frac{d}{dt}(1+t^2) = (1+t^2)\cdot 2t = 2t+2t^3.$$

$$P_3(t) = (1+t^2)\frac{d}{dt}P_2(t) = (1+t^2)\frac{d}{dt}(2t+2t^3) = (1+t^2)\cdot(2+6t^2) = 2+8t^2+6t^4.$$

It is obvious, that P_n is the sum of polynomials with maximum rank $n+1$. Due to the monotonicity of the tangens within the interval $[\frac{\pi}{8}, \frac{\pi}{4}]$, the maximum is at $t = \tan(\frac{\pi}{4}) = 1$ and the minimum is at $t = \tan(\frac{\pi}{8}) = 0.414$. Therefore,

$$\sup_{\xi \in I} |f^{(n)}(\xi)| = P_n(1).$$

For different values of n those result are written in Table 2. Comparing the results of Table 1 and Table 2, it is obvious that $\frac{3\cdot\pi}{16}$ a taylor polynomial of 15-th degree is not bit perfect. In fact to be bit perfect an error of at least 10^{-17} would be required.

The resulting errors of the n-th degree taylor polynomial developed at $a = \frac{3\cdot\pi}{16}$ can be seen within the Table 3. The needed degree would be 24, which is not feasable due to its calculation time.

0.1 Different Developement Points

The prior section showed that developement at $\frac{3\cdot\pi}{16}$, a useable polynomial. It remains unclear if other developement points have better results. As all developement poins within $[\frac{\pi}{8}, \frac{\pi}{4}]$ different from $\frac{3\cdot\pi}{16}$ have the same issues but with larger distance errors, it is obsolete to check others within this interval.

However, when developing at 0 half of the derivatives can be ignored as they are 0. Therefore, it needs to be checked if the needed degree is less then twice the one for the developement at $\frac{3\cdot\pi}{16}$. This is not the case. Furthermore, using this approach way more degrees would be needed in order to obtain a feasable error.

Table 4: Errors and factorial terms by degree (Distance Error in scientific notation)

Degree	Distance Error	Derivative Error	$1/n!$	Total Error
1	$6.169 \cdot 10^{-1}$	$2 \cdot 10^{0}$	$5 \cdot 10^{-1}$	0.6169
2	$4.845 \cdot 10^{-1}$	$4 \cdot 10^{0}$	$1.667 \cdot 10^{-1}$	0.3230

Degree	Distance Error	Derivative Error	$1/n!$	Total Error
3	$3.805 \cdot 10^{-1}$	$1.6 \cdot 10^{1}$	$4.167 \cdot 10^{-2}$	0.2537
4	$2.988 \cdot 10^{-1}$	$8 \cdot 10^{1}$	$8.333 \cdot 10^{-3}$	0.1992
5	$2.347 \cdot 10^{-1}$	$5.12 \cdot 10^{2}$	$1.389 \cdot 10^{-3}$	0.1669
6	$1.843 \cdot 10^{-1}$	$3.904 \cdot 10^{3}$	$1.984 \cdot 10^{-4}$	0.1428
7	$1.448 \cdot 10^{-1}$	$3.482 \cdot 10^{4}$	$2.48 \cdot 10^{-5}$	0.1250
8	$1.137 \cdot 10^{-1}$	$3.546 \cdot 10^{5}$	$2.756 \cdot 10^{-6}$	0.1111
9	$8.931 \cdot 10^{-2}$	$4.063 \cdot 10^{6}$	$2.756 \cdot 10^{-7}$	0.1000
10	$7.014 \cdot 10^{-2}$	$5.173 \cdot 10^{7}$	$2.505 \cdot 10^{-8}$	0.0909
11	$5.509 \cdot 10^{-2}$	$7.246 \cdot 10^{8}$	$2.088 \cdot 10^{-9}$	0.0833
12	$4.327 \cdot 10^{-2}$	$1.107 \cdot 10^{10}$	$1.606 \cdot 10^{-10}$	0.0769
13	$3.398 \cdot 10^{-2}$	$1.832 \cdot 10^{11}$	$1.147 \cdot 10^{-11}$	0.0714
14	$2.669 \cdot 10^{-2}$	$3.266 \cdot 10^{12}$	$7.647 \cdot 10^{-13}$	0.0667
15	$2.096 \cdot 10^{-2}$	$6.238 \cdot 10^{13}$	$4.779 \cdot 10^{-14}$	0.0625
16	$1.646 \cdot 10^{-2}$	$1.271 \cdot 10^{15}$	$2.811 \cdot 10^{-15}$	0.0588
17	$1.293 \cdot 10^{-2}$	$2.751 \cdot 10^{16}$	$1.562 \cdot 10^{-16}$	0.0556
18	$1.016 \cdot 10^{-2}$	$6.304 \cdot 10^{17}$	$8.221 \cdot 10^{-18}$	0.0526
19	$7.976 \cdot 10^{-3}$	$1.525 \cdot 10^{19}$	$4.11 \cdot 10^{-19}$	0.0500
20	$6.265 \cdot 10^{-3}$	$3.884 \cdot 10^{20}$	$1.957 \cdot 10^{-20}$	0.0476
21	$4.920 \cdot 10^{-3}$	$1.038 \cdot 10^{22}$	$8.897 \cdot 10^{-22}$	0.0455
22	$3.864 \cdot 10^{-3}$	$2.909 \cdot 10^{23}$	$3.868 \cdot 10^{-23}$	0.0435
23	$3.035 \cdot 10^{-3}$	$8.518 \cdot 10^{24}$	$1.612 \cdot 10^{-24}$	0.0417
24	$2.384 \cdot 10^{-3}$	$2.603 \cdot 10^{26}$	$6.447 \cdot 10^{-26}$	0.0400
25	$1.872 \cdot 10^{-3}$	$8.285 \cdot 10^{27}$	$2.480 \cdot 10^{-27}$	0.0385
26	$1.470 \cdot 10^{-3}$	$2.743 \cdot 10^{29}$	$9.184 \cdot 10^{-29}$	0.0370
27	$1.155 \cdot 10^{-3}$	$9.429 \cdot 10^{30}$	$3.280 \cdot 10^{-30}$	0.0357
28	$9.070 \cdot 10^{-4}$	$3.362 \cdot 10^{32}$	$1.131 \cdot 10^{-31}$	0.0345
29	$7.124 \cdot 10^{-4}$	$1.241 \cdot 10^{34}$	$3.770 \cdot 10^{-33}$	0.0333
30	$5.595 \cdot 10^{-4}$	$4.741 \cdot 10^{35}$	$1.216 \cdot 10^{-34}$	0.0323
31	$4.394 \cdot 10^{-4}$	$1.871 \cdot 10^{37}$	$3.800 \cdot 10^{-36}$	0.0313
32	$3.451 \cdot 10^{-4}$	$7.624 \cdot 10^{38}$	$1.152 \cdot 10^{-37}$	0.0303
33	$2.711 \cdot 10^{-4}$	$3.204 \cdot 10^{40}$	$3.387 \cdot 10^{-39}$	0.0294
34	$2.129 \cdot 10^{-4}$	$1.387 \cdot 10^{42}$	$9.678 \cdot 10^{-41}$	0.0286
35	$1.672 \cdot 10^{-4}$	$6.180 \cdot 10^{43}$	$2.688 \cdot 10^{-42}$	0.0278
36	$1.313 \cdot 10^{-4}$	$2.833 \cdot 10^{45}$	$7.265 \cdot 10^{-44}$	0.0270
37	$1.031 \cdot 10^{-4}$	$1.335 \cdot 10^{47}$	$1.912 \cdot 10^{-45}$	0.0263
38	$8.100 \cdot 10^{-5}$	$6.457 \cdot 10^{48}$	$4.902 \cdot 10^{-47}$	0.0256
39	$6.362 \cdot 10^{-5}$	$3.206 \cdot 10^{50}$	$1.226 \cdot 10^{-48}$	0.0250
40	$4.997 \cdot 10^{-5}$	$1.633 \cdot 10^{52}$	$2.989 \cdot 10^{-50}$	0.0244
41	$3.924 \cdot 10^{-5}$	$8.524 \cdot 10^{53}$	$7.117 \cdot 10^{-52}$	0.0238
42	$3.082 \cdot 10^{-5}$	$4.558 \cdot 10^{55}$	$1.655 \cdot 10^{-53}$	0.0233

Degree	Distance Error	Derivative Error	$1/n!$	Total Error
43	$2.421 \cdot 10^{-5}$	$2.496 \cdot 10^{57}$	$3.762 \cdot 10^{-55}$	0.0227
44	$1.901 \cdot 10^{-5}$	$1.398 \cdot 10^{59}$	$8.360 \cdot 10^{-57}$	0.0222
45	$1.493 \cdot 10^{-5}$	$8.011 \cdot 10^{60}$	$1.817 \cdot 10^{-58}$	0.0217
46	$1.173 \cdot 10^{-5}$	$4.692 \cdot 10^{62}$	$3.867 \cdot 10^{-60}$	0.0213
47	$9.211 \cdot 10^{-6}$	$2.808 \cdot 10^{64}$	$8.055 \cdot 10^{-62}$	0.0208
48	$7.234 \cdot 10^{-6}$	$1.716 \cdot 10^{66}$	$1.644 \cdot 10^{-63}$	0.0204
49	$5.682 \cdot 10^{-6}$	$1.071 \cdot 10^{68}$	$3.288 \cdot 10^{-65}$	0.0200

n	$\tan^{(n)}(\pi/4)$
0	1
1	2
2	4
3	16
4	80
5	$5.12 \cdot 10^2$
6	$3.904 \cdot 10^3$
7	$3.4816 \cdot 10^4$
8	$3.5456 \cdot 10^5$
9	$4.063232 \cdot 10^6$
10	$5.1733504 \cdot 10^7$
11	$7.24566016 \cdot 10^8$
12	$1.107052544 \cdot 10^{10}$
13	$1.83240753152 \cdot 10^{11}$
14	$3.266330312704 \cdot 10^{12}$
15	$6.2382319599616 \cdot 10^{13}$

Table 2: Values of $\tan^n(\pi/4)$ for $n = 0, \ldots, 15$.

Degree	Distance Error	Derivative Error	$1/n!$	Product
1	$3.8553 \cdot 10^{-2}$	2	$5.0 \cdot 10^{-1}$	$3.8553 \cdot 10^{-2}$
2	$7.5699 \cdot 10^{-3}$	4	$1.6667 \cdot 10^{-1}$	$5.0466 \cdot 10^{-3}$
3	$1.4863 \cdot 10^{-3}$	$1.6 \cdot 10^{1}$	$4.1667 \cdot 10^{-2}$	$9.9089 \cdot 10^{-4}$
4	$2.9184 \cdot 10^{-4}$	$8.0 \cdot 10^{1}$	$8.3333 \cdot 10^{-3}$	$1.9456 \cdot 10^{-4}$
5	$5.7303 \cdot 10^{-5}$	$5.12 \cdot 10^{2}$	$1.3889 \cdot 10^{-3}$	$4.0749 \cdot 10^{-5}$
6	$1.1251 \cdot 10^{-5}$	$3.904 \cdot 10^{3}$	$1.9841 \cdot 10^{-4}$	$8.7154 \cdot 10^{-6}$
7	$2.2092 \cdot 10^{-6}$	$3.4816 \cdot 10^{4}$	$2.4802 \cdot 10^{-5}$	$1.9076 \cdot 10^{-6}$
8	$4.3378 \cdot 10^{-7}$	$3.5456 \cdot 10^{5}$	$2.7557 \cdot 10^{-6}$	$4.2383 \cdot 10^{-7}$
9	$8.5172 \cdot 10^{-8}$	$4.0632 \cdot 10^{6}$	$2.7557 \cdot 10^{-7}$	$9.5369 \cdot 10^{-8}$
10	$1.6724 \cdot 10^{-8}$	$5.1734 \cdot 10^{7}$	$2.5052 \cdot 10^{-8}$	$2.1674 \cdot 10^{-8}$
11	$3.2837 \cdot 10^{-9}$	$7.2457 \cdot 10^{8}$	$2.0877 \cdot 10^{-9}$	$4.9671 \cdot 10^{-9}$
12	$6.4475 \cdot 10^{-10}$	$1.1071 \cdot 10^{10}$	$1.6059 \cdot 10^{-10}$	$1.1462 \cdot 10^{-9}$
13	$1.2660 \cdot 10^{-10}$	$1.8324 \cdot 10^{11}$	$1.1471 \cdot 10^{-11}$	$2.6609 \cdot 10^{-10}$
14	$2.4857 \cdot 10^{-11}$	$3.2663 \cdot 10^{12}$	$7.6472 \cdot 10^{-13}$	$6.2088 \cdot 10^{-11}$
15	$4.8807 \cdot 10^{-12}$	$6.2382 \cdot 10^{13}$	$4.7795 \cdot 10^{-14}$	$1.4552 \cdot 10^{-11}$
16	$9.5831 \cdot 10^{-13}$	$1.2708 \cdot 10^{15}$	$2.8115 \cdot 10^{-15}$	$3.4240 \cdot 10^{-12}$
17	$1.8816 \cdot 10^{-13}$	$2.7507 \cdot 10^{16}$	$1.5619 \cdot 10^{-16}$	$8.0844 \cdot 10^{-13}$
18	$3.6946 \cdot 10^{-14}$	$6.3042 \cdot 10^{17}$	$8.2206 \cdot 10^{-18}$	$1.9147 \cdot 10^{-13}$
19	$7.2543 \cdot 10^{-15}$	$1.5251 \cdot 10^{19}$	$4.1103 \cdot 10^{-19}$	$4.5475 \cdot 10^{-14}$
20	$1.4244 \cdot 10^{-15}$	$3.8836 \cdot 10^{20}$	$1.9573 \cdot 10^{-20}$	$1.0827 \cdot 10^{-14}$
21	$2.7968 \cdot 10^{-16}$	$1.0384 \cdot 10^{22}$	$8.8968 \cdot 10^{-22}$	$2.5838 \cdot 10^{-15}$
22	$5.4915 \cdot 10^{-17}$	$2.9087 \cdot 10^{23}$	$3.8682 \cdot 10^{-23}$	$6.1786 \cdot 10^{-16}$
23	$1.0782 \cdot 10^{-17}$	$8.5180 \cdot 10^{24}$	$1.6117 \cdot 10^{-24}$	$1.4803 \cdot 10^{-16}$
24	$2.1171 \cdot 10^{-18}$	$2.6029 \cdot 10^{26}$	$6.4470 \cdot 10^{-26}$	$3.5527 \cdot 10^{-17}$

Table 3: Distance error, derivative error, factorial term, and their product by degree.