Bending Energy Gradient and Hessian

The bending energy can be written as

$$E_b = \kappa \int_{\mathcal{M}} (H - H_0)^2 dA \tag{1}$$

In out discrete setting we write it as

$$E_b = \kappa \sum_{v_i} \left(\frac{H_i}{A_i} - H_0 \right)^2 A_i \tag{2}$$

Using this convention we can write the gradient of the total bending energy as

$$
\begin{aligned}
\frac{\nabla E_b}{\kappa} &= \sum_{v_i} \nabla \left[\left(\frac{H_i}{A_i} - H_0 \right)^2 A_i \right] && (3)\\
&= \sum_{v_i} \left(\frac{H_i}{A_i} - H_0 \right)^2 \nabla A_i + 2A_i \left(\frac{H_i}{A_i} - H_0 \right) \left[\frac{\nabla H_i}{A_i} - \frac{H_i}{A_i^2} \nabla A_i \right] && (4)\\
&= \sum_{v_i} \left[\left(\frac{H_i}{A_i} - H_0 \right) \left(\frac{H_i}{A_i} - H_0 - 2\frac{H_i}{A_i} \right) \right] \nabla A_i + 2 \left(\frac{H_i}{A} - H_0 \right) \nabla H_i && (5)\\
&= \sum_{v_i} - \left[\frac{H_i^2}{A_i^2} - H_0^2 \right] \nabla A_i + 2 \left(\frac{H_i}{A_i} - H_0 \right) \nabla H_i && (6)
\end{aligned}
$$

Now we just need to find the definitions for the gradients of the dual area and the gradient of the mean curvature and we win (:

For the dual area depends on your convention i use the barycentric dual area, if we call the area for triangle j as T_j

$$
\begin{aligned}
A_i &= \sum_{f_j \ni v_i} \frac{T_j}{3} && (7)\\
\implies A_i &= \frac{1}{3} \sum_{f_j \ni v_i} \nabla T_j && (8)
\end{aligned}
$$

And for the gradient of the triangle j. If we define $u = p_2 - p_1$, $v = p_3 - p_1$. using the comma notation for derivatives $u_{,p_1} = -id$, $u_{,p_2} = id$, $v_{,p_1} = -id$, $v_{,p_3} = id$. Then we can define z as $z = u \times v$. such that $z_{,u} = [u]_\times$ and $z_u = -[v]_\times$.

Then

$$
\begin{aligned}
A &= ||z|| && (9)\\
A_{,z} &= \frac{z^T}{||z||} && (10)\\
\implies A_{,u} &= A_{,z} z_{,u} = -\frac{1}{||z||} z^T [v]_\times = -\frac{1}{||z||}\left([v]_\times^T z\right)^T && (11)\\
&= \frac{1}{||z||}(v \times z)^T && (12)\\
A_{,v} &= A_{,z} z_{,v} = \frac{1}{||z||} z^T [u]_\times = \frac{1}{||z||}([u]_\times^T z)^T && (13)\\
&= -\frac{1}{||z||}(u \times z)^T && (14)
\end{aligned}
$$

Using this calculation we can write the derivative of the area w.r.t the 3 points.

$$
\begin{aligned}
A_{,p1} &= A_{,u} u_{,p_1} + A_{,v} v_{,p_1} = -A_{,u} - A_{,v} && (15)\\
A_{,p2} &= A_{,u} u_{,p_2} = A_{,u} && (16)\\
A_{,p3} &= A_{,v} v_{,p_3} = A_{,v} && (17)
\end{aligned}
$$

Now we 'just' need to do the gradient of the H_i

For this it will depend on the discretization of the mean curvature you use. THe one i have implemented can be written as

$$
H_i = \sum_{e_{ij} \ni v_i} \frac{\ell_{ij} \varphi_{ij}}{4} \tag{18}
$$

You can writhe the gradient of this term as

$$
\nabla H_i = \sum_{e_{ij} \ni v_i} \frac{1}{4}(\varphi_{ij} \nabla \ell_{ij} + \ell_{ij} \nabla \varphi_{ij}) \tag{19}
$$

Now the interesting thing here is that ℓ_{ij} depends on two points so the gradient is a 6×1 vector and the dihedral depends on 4 vertices so the gradient is a 12×1 vector.

The gradient of ℓ_{ij} is simple, for an edge $u = p1 - p2$

$$
\begin{aligned}
\nabla_{p_1} ||u|| &= \hat{u} && (20)\\
\nabla_{p_2} ||u|| &= -\hat{u} && (21)
\end{aligned}
$$

And this is where the fun ends.

Now we need to calculate the gradient of the dihedral angle φ_{ij}

Since my abilities to draw on LaTeXare limited. I will try to say it with words.

We start considering a triangle flap, that is, two triangles connected by an edge. We name the point p_1 as the lower one belonging to the edge. p_2 as the other point that lies on the edge connecting the flap. p_3 to the right and p_4 to the left. Found a description!

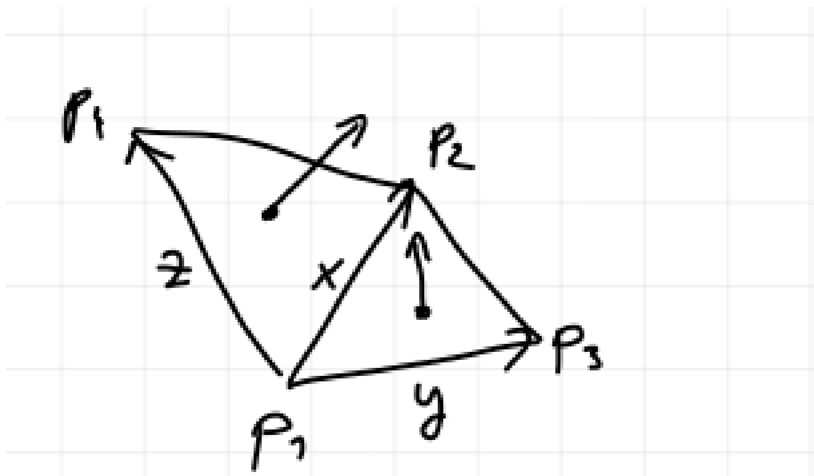

Figure 1: A triangle flap

We can define the dihedral angle as

$$\begin{aligned}
\varphi &= \operatorname{atan2}\left(\left\|\frac{x \times z}{\|x \times z\|} \times \frac{y \times x}{\times x\|}\right\|, \frac{x \times z}{\|x \times z\|} \cdot \frac{y \times x}{\|y \times x\|}\right) && (22)\\
&= \operatorname{atan2}\left(\|(x \times z) \times (x \times y)\|, (x \times z) \cdot (y \times x)\right) && (23)\\
&= \operatorname{atan2}\left(\|\det(x, y, z) \cdot x\|, (x \times z) \cdot (y \times x)\right) && (24)\\
&= \operatorname{atan2}\left(\det(x, y, z)\|x\|, (x \times z) \cdot (y \times x)\right) && (25)
\end{aligned}$$

That is out silly little expression for the dihedral. We will use u, v, w instead of x, y, z but thats how it was on the image :p.

Defining $u = p_2 - p_1$, $v = p_3 - p_1$ and $w = p_4 - p_1$.

And the functions $g := \det(u, v, w)\|u\|$, and $h = (v \times w) \cdot (v \times u)$ and $r = g^2 + h^2$. We get.

$$f = \operatorname{atan2}(g, h) \tag{26}$$

We can rewrite h as

$$\begin{aligned}
h &= u^T[w]_\times[v]_\times u && (27)\\
&= u^T[v]_\times[w]_\times u && (28)\\
&= w^T[u]_\times^2 v && (29)\\
&= v^T[u]_\times^2 w && (30)
\end{aligned}$$

Now we need to do the derivatives. Because i am lazy i will not write the coma and the subindices here means derivative w.r.t that variable.

$$\begin{aligned}
f_u = f_h h_u + f_g g_u &= -\frac{g}{r}u^T([w]_\times[v]_\times + [v]_\times[w]_\times) + \frac{h}{r}\left((v\times w)^T\|u\| + \det(u,v,w)\frac{u^T}{\|u\|}\right) && (31)\\
&= \frac{1}{r}\left(-gu^T([w]_\times[v]_\times + [v]_\times[w]_\times) + h(\|u\|(u\times w)^T + \det(u,v,w)\frac{u^T}{\|u\|})\right) && (32)
\end{aligned}$$

And the other two

$$\begin{aligned}
f_v = f_h h_v + f_g g_v &= \frac{1}{r}(-gh_v + hg_v) && (33)\\
&= \frac{1}{r}\left(-gw^T[u]_\times^2 + h\|u\|(w\times u)^T\right) && (34)
\end{aligned}$$

$$\begin{aligned}
f_w = f_h h_w f_g g_w &= \frac{1}{r}(-gh_w + hg_w) && (35)\\
&= \frac{1}{r}\left(-gv^T[u]_\times^2 + h\|u\|u(u\times v)^T\right) && (36)
\end{aligned}$$

Then we have

$$f_{p_1} = -f_u - f_v - f_w \quad f_{p_2} = f_u \quad f_{p_3} = f_v \quad f_{p_4} = f_w \qquad (37)$$