Rotation matrix clarification

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1 Derivation of rotation matrix

Let $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ be a unit vector[1] and orthogonal to each others. We call them **basis vectors** in frame K. Similarly, $\vec{g}_1$, $\vec{g}_2$, $\vec{g}_3$ for frame G. Given a vector $\vec{r}$ and we expressed it into the frame K and G (they share same origo) in the following:

$$\begin{aligned}\vec{r} &= [\vec{k}_1, \vec{k}_2, \vec{k}_3] \begin{bmatrix} r_1^k \\ r_2^k \\ r_3^k \end{bmatrix} = [\vec{g}_1, \vec{g}_2, \vec{g}_3] \begin{bmatrix} r_1^g \\ r_2^g \\ r_3^g \end{bmatrix} \\ &= r_1^k \vec{k}_1 + r_2^k \vec{k}_2 + r_3^k \vec{k}_3 = r_1^g \vec{g}_1 + r_2^g \vec{g}_2 + r_3^g \vec{g}_3.\end{aligned}$$

We want to find the relation between $[r_1^k, r_2^k, r_3^k]^T$ and $[r_1^g, r_2^g, r_3^g]^T$. We proceed as follows:

$$\begin{aligned}[\vec{k}_1, \vec{k}_2, \vec{k}_3] \begin{bmatrix} r_1^k \\ r_2^k \\ r_3^k \end{bmatrix} &= [\vec{g}_1, \vec{g}_2, \vec{g}_3] \begin{bmatrix} r_1^g \\ r_2^g \\ r_3^g \end{bmatrix} \\ \begin{bmatrix} \vec{k}_1 \\ \vec{k}_2 \\ \vec{k}_3 \end{bmatrix} \cdot [\vec{k}_1, \vec{k}_2, \vec{k}_3] \begin{bmatrix} r_1^k \\ r_2^k \\ r_3^k \end{bmatrix} &= \begin{bmatrix} \vec{k}_1 \\ \vec{k}_2 \\ \vec{k}_3 \end{bmatrix} \cdot [\vec{g}_1, \vec{g}_2, \vec{g}_3] \begin{bmatrix} r_1^g \\ r_2^g \\ r_3^g \end{bmatrix} \\ \begin{bmatrix} r_1^k \\ r_2^k \\ r_3^k \end{bmatrix} &= \mathbf{R}_{K,G} \begin{bmatrix} r_1^g \\ r_2^g \\ r_3^g \end{bmatrix},\end{aligned}$$

where $\cdot$ is the dot product between vectors, and the definition of rotation $\mathbf{R}_{K,G}$

$$\begin{aligned}\mathbf{R}_{K,G} &= \begin{bmatrix} \vec{k}_1 \\ \vec{k}_2 \\ \vec{k}_3 \end{bmatrix} \cdot [\vec{g}_1, \vec{g}_2, \vec{g}_3] && (1) \\ &= \begin{bmatrix} \vec{k}_1 \cdot \vec{g}_1 & \vec{k}_1 \cdot \vec{g}_2 & \vec{k}_1 \cdot \vec{g}_3 \\ \vec{k}_2 \cdot \vec{g}_1 & \vec{k}_2 \cdot \vec{g}_2 & \vec{k}_2 \cdot \vec{g}_3 \\ \vec{k}_3 \cdot \vec{g}_1 & \vec{k}_3 \cdot \vec{g}_2 & \vec{k}_3 \cdot \vec{g}_3 \end{bmatrix}. && (2)\end{aligned}$$

From above equation, we can see the rotation matrix is actually specified by inner product between two basis vectors and not coordinate frames are involved.

[1] In this document, we use the term vector without properly define vector space.

To derive $\mathbf{R}_{\vec{k}_1}$, we rotate the frame K w.r.t vector $\vec{k}_1$ with θ degree and denote the rotated frame as G. Again, positive rotation follows the right-hand rule. fig. 1 depicts the rotation, we can see that:

$$\begin{aligned}\vec{k}_1 \cdot \vec{g}_1 &= 1\\ \vec{k}_1 \cdot \vec{g}_2 = \vec{k}_1 \cdot \vec{g}_3 = \vec{k}_2 \cdot \vec{g}_1 = \vec{k}_3 \cdot \vec{g}_1 &= 0\\ \vec{k}_2 \cdot \vec{g}_2 = \vec{k}_3 \cdot \vec{g}_3 &= \cos(\theta)\\ \vec{k}_3 \cdot \vec{g}_2 &= \cos(\frac{\pi}{2} + \theta)\\ \vec{k}_2 \cdot \vec{g}_3 &= \cos(\frac{\pi}{2} - \theta).\end{aligned}$$

Since $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$, and $\cos(\frac{\pi}{2} + \theta) = -\sin(\theta)$, the rotation matrix $\mathbf{R}_{\vec{k}_1}$ in eq. (2) becomes:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

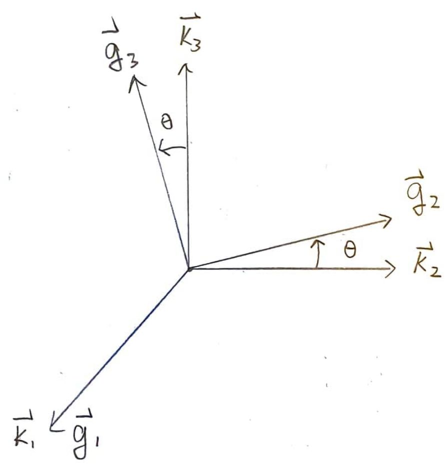

Figure 1: Rotation w.r.t $\vec{k}_1$

To derive $\mathbf{R}_{\vec{k}_2}$, we rotate the frame K w.r.t vector $\vec{k}_2$ with ψ degree and denote the rotated frame as G. The positive rotation follows the right-hand rule. fig. 2 depicts the rotation, we can see that:

$$\begin{aligned}\vec{k}_2 \cdot \vec{g}_2 &= 1\\ \vec{k}_1 \cdot \vec{g}_2 = \vec{k}_3 \cdot \vec{g}_2 = \vec{k}_2 \cdot \vec{g}_1 = \vec{k}_2 \cdot \vec{g}_3 &= 0\\ \vec{k}_1 \cdot \vec{g}_1 = \vec{k}_3 \cdot \vec{g}_3 &= \cos(\psi)\\ \vec{k}_3 \cdot \vec{g}_1 &= \cos(\frac{\pi}{2} + \psi)\\ \vec{k}_1 \cdot \vec{g}_3 &= \cos(\frac{\pi}{2} - \psi).\end{aligned}$$

Since $\cos(\frac{\pi}{2} - \psi) = \sin(\psi)$, and $\cos(\frac{\pi}{2} + \psi) = -\sin(\psi)$, the rotation matrix $\mathbf{R}_{\vec{k}_2}$ in eq. (2) becomes:

$$\begin{bmatrix} \cos\theta & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\psi \end{bmatrix}.$$

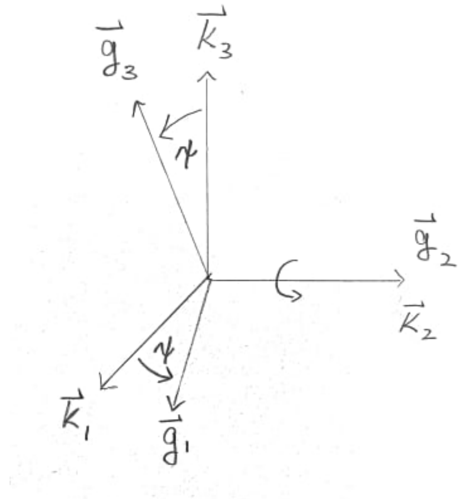

Figure 2: Rotation w.r.t $\vec{k}_2$

To derive $\mathbf{R}_{\vec{k}_3}$, we rotate the frame K w.r.t vector $\vec{k}_3$ with ϕ degree and denote the rotated frame as G. Again, positive rotation follows the right-hand rule. fig. 3 depicts the rotation, we can see that:

$$\begin{aligned} \vec{k}_3 \cdot \vec{g}_3 &= 1 \\ \vec{k}_3 \cdot \vec{g}_1 = \vec{k}_3 \cdot \vec{g}_2 = \vec{k}_2 \cdot \vec{g}_1 = \vec{k}_1 \cdot \vec{g}_3 &= 0 \\ \vec{k}_1 \cdot \vec{g}_1 = \vec{k}_2 \cdot \vec{g}_2 &= \cos(\phi) \\ \vec{k}_1 \cdot \vec{g}_2 &= \cos(\frac{\pi}{2} + \phi) \\ \vec{k}_2 \cdot \vec{g}_1 &= \cos(\frac{\pi}{2} - \phi). \end{aligned}$$

Since $\cos(\frac{\pi}{2} - \phi) = \sin(\phi)$, and $\cos(\frac{\pi}{2} + \phi) = -\sin(\phi)$, the rotation matrix $\mathbf{R}_{\vec{k}_3}$ in eq. (2) becomes:

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

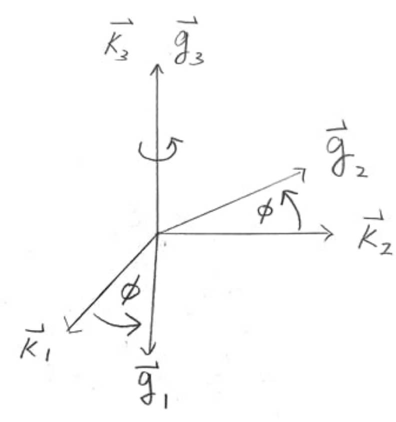

Figure 3: Rotation w.r.t $\vec{k}_3$

$$\mathbf{R}_{\vec{k}_1}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

$$\mathbf{R}_{\vec{k}_2}(\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix},$$
$$\mathbf{R}_{\vec{k}_3}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2 Intrinsic (mobile frame) rotation and extrinsic (fixed frame) rotation

The statement "We first rotate around $\vec{k}_1$, then rotate around $\vec{k}_2$" can means two things and it is ambiguous. If we first rotate around $\vec{k}_1$, then $\vec{k}_2$ and $\vec{k}_3$ will change. Then, by rotating around $\vec{k}_2$, it can mean rotate around vk_2 before it gets changed (extrinsic rotation) or rotate around $\vec{k}_2$ after it gets changed (intrinsic rotation). In this doc, we only use extrinsic rotation. To interactively see the difference between intrinsic and extrinsic rotation, you check the website: https://www.mecademic.com/en/how-is-orientation-in-space-represented-with-euler-angles.

3 Interpretation of rotation matrix

The complete consecutive rotation is the following

$$\mathbf{R}_{K,G}(\theta,\psi,\phi) = R_{\vec{k}_3}(\phi)R_{\vec{k}_2}(\psi)R_{\vec{k}_1}(\theta). \tag{3}$$

Below shows what we can do with the rotation matrix $\mathbf{R}_{K,G}$. The rotation matrix $\mathbf{R}_{K,G}$ transfer the basis vectors of frame K to the basis vectors of frame G in the following way.

$$[\vec{g}_1,\vec{g}_2,\vec{g}_3] = [\vec{k}_1,\vec{k}_2,\vec{k}_3]\begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} := \mathbf{R}_{K,G}. \tag{4}$$

We can write them even more explicitly in the following way:

$$\begin{aligned} \vec{g}_1 &= r_{1,1}\vec{k}_1 + r_{2,1}\vec{k}_2 + r_{3,1}\vec{k}_3 \\ \vec{g}_2 &= r_{1,2}\vec{k}_1 + r_{2,2}\vec{k}_2 + r_{3,2}\vec{k}_3 \\ \vec{g}_3 &= r_{1,3}\vec{k}_1 + r_{2,3}\vec{k}_2 + r_{3,3}\vec{k}_3 \end{aligned},$$

which shows the direct relation basis vectors of frame K and basis vectors of frame G. Beware that the square bracket $[\vec{k}_1,\vec{k}_2,\vec{k}_3]$ is a row-array of vectors, not vector's components. We refer eq. (4) as **basis vectors transformation** and no coordinates are involved.

Now, we have point p, and it is expressed in the basis vectors of G with a vector $\vec{p} = p_1^g\vec{g}_1 + p_2^g\vec{g}_2 + p_3^g\vec{g}_3$. If we multiply the column array $[p_1^g, p_2^g, p_3^g]^T$ on both side of equation 4 from the right. We have:

$$[\vec{g}_1,\vec{g}_2,\vec{g}_3]\begin{bmatrix} p_1^g \\ p_2^g \\ p_3^g \end{bmatrix} = [\vec{k}_1,\vec{k}_2,\vec{k}_3]\begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}\begin{bmatrix} p_1^g \\ p_2^g \\ p_3^g \end{bmatrix}. \tag{5}$$

We obtain

$$\begin{aligned} \vec{p} &= (p_1^g r_{1,1} + p_2^g r_{1,2} + p_3^g r_{1,3})\vec{k}_1 \\ &\quad + (p_1^g r_{2,1} + p_2^g r_{2,2} + p_3^g r_{2,3})\vec{k}_2 \\ &\qquad + (p_1^g r_{3,1} + p_2^g r_{3,2} + p_3^g r_{3,3})\vec{k}_3 \\ &= p_1^k\vec{k}_1 + p_2^k\vec{k}_2 + p_3^k\vec{k}_3, \end{aligned}$$

where the vector $\vec{q}$ is expressed in the basis vectors of K. In another word, the vector's components are calculated in the following way:

$$\begin{bmatrix} p_1^k \\ p_2^k \\ p_3^k \end{bmatrix} = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}\begin{bmatrix} p_1^g \\ p_2^g \\ p_3^g \end{bmatrix}.$$

A more compact expression can be:

$$\mathbf{p}^K = \mathbf{R}_{K,G}\mathbf{p}^G, \tag{6}$$

where the $\mathbf{p}^K$ is the vector components of p in the frame K, likewise for $\mathbf{p}^G$. Equation 6 is very common in varies text book. We refer eq. (6) as **coordinate transformation**.

Remark: The **basis vectors transformation** (eq. (4)) is a transformation that transform from the basis vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ to the basis vectors $\vec{g}_1, \vec{g}_2, \vec{g}_3$. In contrary, the **coordinate transformation** (eq. (6)) is a transformation of changing coordinate frame of the point p from frame G to frame K. From eq. (5) and eq. (4) shows us that the rotation matrix $\mathbf{R}_{K,G}$ transform the basis vectors in a way that from K to G, but $\mathbf{R}_{K,G}$ transform coordinate (vector's component) in a opposite way that from G to K. This is well known relation in tensor calculus because vector is co-variant object and vector's components is contra-variant object.

Here, we summarize the meaning of the subscript of the rotation $\mathbf{R}_{K,G}$. It can have two meaning:

1. $\mathbf{R}_{K,G}$ transforms the coordinate of a point **from** G **to** K. (**coordinate transformation**)
2. $\mathbf{R}_{K,G}$ transforms the set of basis vectors **from** K **to** G. (**basis vectors transformation**)

We mainly use the **coordinate transformation** to interpret the rotation matrix $\mathbf{R}_{K,G}$ in this document.