MA 106:
Mathematics of Democracy

Anders O.F. Hendrickson St. Norbert College
Mariah Birgen Wartburg College

Winter 2025

Contents

*

Preface

This is not a textbook. You will find in these printed pages almost no formulae, answers, or information. This book will not tell you how to do anything. So what is this book? It is most emphatically a *workbook*: a place to work on learning mathematics by doing it yourself.

The mathematics in this book will, for the most part, be content that you have never seen or thought about before. However, it is material that you can figure out with some patience and persistence. The point to keep in mind is that you are not expected to know beforehand. To learn the content, use the following devices:

- Try something.
- Ask yourself why what you have tried doesn't work.
- Brainstorm by yourself.
- Brainstorm with your neighbors or your team members.
- Be willing to say what you are thinking and ask questions.
- Be willing to challenge the statements of others (including the teacher).

One of the most important skills in life, a skill highly valued by employers and spouses alike, is the ability to solve problems no one has taught you how to do. Any monkey or computer can be taught to follow a rote procedure over and over again, and frankly, your previous math courses probably treated you like a monkey. It is when you

- encounter unforeseen difficulties,
- swiftly learn from experience,
- and devise a suitable plan to overcome them

that you show yourself to have a human mind. One of the chief goals of MA 106 is for you to practice this highly creative, utterly vital, and most fully human act of doing what you can't do.

There are places in this book for taking notes, and in particular for noting the meanings of mathematical terms. It has become clear over the past five years that the physical act of writing down notes by hand using a pencil or paper leads to significantly superior learning over recording audio or taking pictures with your camera. Please don't restrict your note-taking to just filling in the blanks, though! There will often be essential points covered in class that do not have a corresponding blank space in these pages. It's up to you whether to take your other notes in the margins of this workbook or in another notebook, but you should take them just as assiduously in this course as in any other course.

There are places in this book where we assume that you have a calculator to do things for you. You will need a calculator that will put any number into an exponent, so look for a button that is labeled $\wedge$ or x^y. You will need a calculator that will calculate standard deviations for you, so make sure that the calculator has the ability to perform statistical calculations.

There will be a lot of trial and error as you work in these pages, so I recommend using a pencil rather than a pen!

Chapter 1

Voting Theory

1.1 Voting Systems

1. The Bellevue Taxicab Union is preparing to elect its president. Three members are considering running for president: Alice, Bob, and Charles. Each of the five voters has a first, second, and third choice, as listed in the following table:

Voter:	Voter 1	Voter 2	Voter 3	Voter 4	Voter 5	Voter 6	Voter 7
1st choice:	Alice	Bob	Bob	Charles	Alice	Charles	Bob
2nd choice:	Charles	Alice	Charles	Alice	Charles	Alice	Alice
3rd choice:	Bob	Charles	Alice	Bob	Bob	Bob	Charles

Depending on who decides to run, the ballot the voters see in the booth will look different. For each of the following ballots, how many votes will each candidate receive? Who will win?

(a)

Who should be president?
☐ Alice
☐ Bob
☐ Charles

(b)

Who should be president?
☐ Alice
☐ Bob

(c)

Who should be president?
☐ Alice
☐ Charles

(d)

Who should be president?
☐ Bob
☐ Charles

2. The Shell Rock Gardening Club is electing a new Master of the Greensward. Four candidates are running for the position: Danielle, Edgar, Frederika, and Gustav.

   Here is the ballot:

Who should be Master of the Greensward?
☐ Danielle
☐ Edgar
☐ Frederika
☐ Gustav

   The members cast the following ballots:

Ranking	1st choice	2nd choice	3rd choice	4th choice
Voter 1	Danielle	Gustav	Edgar	Frederika
Voter 2	Danielle	Gustav	Frederika	Edgar
Voter 3	Edgar	Gustav	Frederika	Danielle
Voter 4	Frederika	Gustav	Danielle	Edgar
Voter 5	Gustav	Edgar	Frederika	Danielle
Voter 6	Danielle	Gustav	Edgar	Frederika
Voter 7	Danielle	Gustav	Frederika	Edgar
Voter 8	Edgar	Gustav	Frederika	Danielle
Voter 9	Frederika	Gustav	Danielle	Edgar
Voter 10	Gustav	Edgar	Frederika	Danielle
Voter 11	Danielle	Gustav	Edgar	Frederika
Voter 12	Edgar	Gustav	Frederika	Danielle
Voter 13	Frederika	Gustav	Danielle	Edgar
Voter 14	Edgar	Gustav	Frederika	Danielle

   (a) Create a preference table for this election by counting the number of voters who have the same preference and putting that number above the appropriate ballot in the table. You may use tally marks if you prefer.

Ranking	# of Ballots				
1st choice	D	D	E	F	G
2nd choice	G	G	G	G	E
3rd choice	E	F	F	D	F
4th choice	F	E	D	E	D

(b) Who will be elected Master? What percent of the vote will he or she have?

(c) As you look at the voters' preferences, who seems like the best choice to make the most voters happy?

(d) If your answers to parts 2b and 2c are different, can you explain why?

(e) Write down how you would tell someone else how to create a preference table.

(f) Can you think of a different voting system that would help the voters elect your choice from part (b)?

3. **Plurality voting:**

4. **A *majority* is. . .**

5. **Borda count:**

6. If the Shell Rock Gardening Club runs their election as a Borda count, who will win?

Ranking	# of Ballots				
1st choice	D	D	E	F	G
2nd choice	G	G	G	G	E
3rd choice	E	F	F	D	F
4th choice	F	E	D	E	D

This page intentionally left blank.

Name: ________________________________

Homework 1

Show all your work.

V1. Consider the following election ballots:

Voter 1	A	B	C
Voter 2	C	B	A
Voter 3	B	C	A
Voter 4	C	B	A
Voter 5	C	B	A
Voter 6	B	C	A
Voter 7	A	B	C
Voter 8	A	B	C
Voter 9	A	B	C
Voter 10	B	C	A
Voter 11	C	B	A
Voter 12	A	B	C

(a) Construct a preference table for the above ballots.

Ranking	# of Ballots		
1st choice	A		
2nd choice	B		
3rd choice	C		

(b) How many votes would be required to win a *majority* in this election?

(c) Find the winner of an election held using each of the following voting schemes. (If it is a tie, say who tied.)

i. Using plurality vote.

ii. Using Borda Count.

(over)

Name: ______________________________

V2. Consider the following preference table:

Ranking	# of Ballots 8	6	8	5	1
1st choice	B	C	D	B	A
2nd choice	D	A	C	C	B
3rd choice	C	D	A	A	C
4th choice	A	B	B	D	D

Find the winner of an election held using each of the following voting schemes. (If it is a tie, say who tied.)

(a) Using plurality vote.

(b) Using Borda Count.

Name: ______________________________

V3. Consider the following preference table:

Ranking	# of Ballots 3	7	4	1	7	8
1st choice	B	B	C	A	C	A
2nd choice	C	A	B	C	A	B
3rd choice	A	C	A	B	B	C

Find the winner of an election held using each of the following voting schemes. (If it is a tie, say who tied.)

(a) Using plurality vote.

(b) Using Borda Count.

(over)

Name: ______________________________

V4. Consider the following preference table:

Ranking	# of Ballots 3	5	4
1st choice	C	B	D
2nd choice	E	E	C
3rd choice	A	A	E
4th choice	D	C	A
5th choice	B	D	B

Find the winner of an election held using each of the following voting schemes. (If it is a tie, say who tied.)

(a) Using plurality vote.

(b) Using Borda Count.

7. The Ashwaubenon Classical League is voting for its new proconsul, and the candidates are Publius, Quintus, Rufus, and Sextus.

	# of Ballots			
Ranking	3	10	9	8
1st choice	S	P	Q	R
2nd choice	R	R	S	Q
3rd choice	P	S	R	S
4th choice	Q	Q	P	P

(a) Each League member votes for his favorite candidate. What percentage of votes did the first-place candidate receive?

(b) Since none of the candidates received more than 50% of the votes, the League holds a run-off election between the top two candidates. Who will win the run-off election?

(c) Was anyone eliminated from the run-off who perhaps shouldn't have been?

(d) Can you think of a better runoff system for the League than a runoff between just the top two candidates? What result does it give?

8. **Plurality with elimination:**

9. The Federal Engineering Board is trying to decide on a flood prevention project for the metro area of Oak Rapids. The options are

(D) A diversion through Marionville
(L) A lock-and-dam system along the Oak River
(F) Building floodwalls to 50′ throughout the city
(C) Relocating the community to higher ground in, say, Colorado
(G) Annual goat sacrifices to Poseidon, God of Waters

The eighteen members of the Board have the following preferences:

	# of Ballots					
Ranking	5	4	2	3	1	3
1st choice	F	G	L	D	C	L
2nd choice	D	D	F	G	D	F
3rd choice	L	L	C	L	F	D
4th choice	G	C	D	F	G	G
5th choice	C	F	G	C	L	C

The Board's voting rules employ *plurality with elimination*.

(a) What will the vote tallies be after the first round of voting? Is there a winner yet? If not, who should be eliminated?

(b) What will the vote tallies be after the second round of voting? Is there a winner yet? If not, who should be eliminated?

(c) What will the vote tallies be after the third round of voting? Is there a winner yet? If not, who should be eliminated?

10. If there are n candidates in an election conducted by plurality with elimination, what is the most rounds of voting that could be required?

11. The Wartburg College Science Fiction Club is voting on who is the Best Starship Captain Ever. The candidates are:

- Han Solo, *Millennium Falcon*
- James T. Kirk, *USS Enterprise*
- Malcolm Reynolds, *Serenity*
- Zaphod Beeblebrox, *Heart of Gold*

The preference table is the following:

Ranking	# of Ballots 4	2	3	2	3
1st choice	R	B	K	S	S
2nd choice	S	S	R	K	R
3rd choice	K	K	B	R	B
4th choice	B	R	S	B	K

(a) Who would win in a head-to-head race between Solo and Kirk?

(b) Who would win in a head-to-head race between Solo and Reynolds?

(c) Fill in the following table of who would win head-to-head races; if there are any ties, put both names in the box.

	Beeblebrox	Reynolds	Kirk
Solo			
Kirk			
Reynolds			

(d) Who won the most head-to-head races?
(Count any ties as $\frac{1}{2}$ a win for each contender.)

12. **Pairwise Comparison:**

13. Organizational idea for setting up the head-to-head races:

14. Our Mathematics in Democracy classes are voting on where to take a math field trip. The options are:

    (E) The mudflats of Egypt, birthplace of geometry

    (A) Athens, cradle of logical thought

    (M) Moose Lake Credit Union, dispenser of mortgages

    (K) Königsberg, for a walking tour of the bridges

    (P) Papa John's Pizza, home of yummy garlic sauce

    The preference table is the following.

	# of Ballots					
Ranking	5	4	4	3	2	1
1st choice	K	M	M	E	P	K
2nd choice	E	K	E	P	K	A
3rd choice	A	A	A	K	E	M
4th choice	P	E	K	M	M	P
5th choice	M	P	P	A	A	E

    We decide to use the pairwise comparison method to decide. Where will we go?

15. If you use the pairwise comparison method with n candidates, how many head-to-head races are there?

This page intentionally left blank.

Name: ________________________________

Homework 2

Show all your work. In particular, for plurality with elimination, show the results of each round of voting; for pairwise comparison, show the result of each head-to-head vote.

V5. Consider the following preference table:

	# of Ballots		
Ranking	5	4	3
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

(a) How many votes would be required to win a *majority* in this election?

(b) Find the winner of an election held using each of the following voting schemes. (If it is a tie, say who tied.)

i. Find the winner using plurality with elimination.

ii. Find the winner using pairwise comparison.

(over)

Name: ________________________________

V6. Consider the following preference table:

Ranking	# of Ballots 3	4	4	9	3	6
1st choice	D	D	A	E	B	A
2nd choice	C	C	B	C	E	B
3rd choice	E	E	E	B	D	C
4th choice	B	A	D	A	A	D
5th choice	A	B	C	D	C	E

This race is being decided by plurality vote.

(a) How many voters are voting in this election?

(b) How many votes would be required to win a *majority* in this election?

(c) Who will win this election? How many votes will that person get? Does that candidate have a majority?

(d) Suppose candidate B drops out of the race. Who will win this election? How many votes will that person get? Does that candidate have a majority?

(e) Suppose all the candidates drop out of the race except A and C. Who will win this election? How many votes will that person get? Does that candidate have a majority?

Name: ______________________________

V7. Consider the following preference table:

Ranking	# of Ballots 8	6	8	5	1
1st choice	B	C	D	B	A
2nd choice	D	A	C	C	B
3rd choice	C	D	A	A	C
4th choice	A	B	B	D	D

V8. Find the winner of an election held using each of the following voting schemes. (If it is a tie, say who tied.)

(a) Find the winner using plurality with elimination.

(b) Find the winner using pairwise comparison.

(over)

Name: ________________________________

V9. Consider the following preference table:

Ranking	# of Ballots					
	3	7	4	1	7	8
1st choice	B	B	C	A	C	A
2nd choice	C	A	B	C	A	B
3rd choice	A	C	A	B	B	C

Find the winner of an election held using each of the following voting schemes. (If it is a tie, say who tied.)

(a) Find the winner using plurality with elimination.

(b) Find the winner using pairwise comparison.

Name: ______________________________

V10. Consider the following preference table:

Ranking	# of Ballots 3	5	4
1st choice	C	B	D
2nd choice	E	E	C
3rd choice	A	A	E
4th choice	D	C	A
5th choice	B	D	B

Find the winner of an election held using each of the following voting schemes. (If it is a tie, say who tied.)

(a) Find the winner using plurality with elimination.

(b) Find the winner using pairwise comparison.

(over)

Name: ____________________

V11. The AP college football poll is a ranking of the top 25 college football teams in the country and is one of the key polls used for the BCS[1]. The voters in the AP poll are a group of sportswriters and broadcasters chosen from across the country. The top 25 teams are ranked using a Borda count: each first-place vote is worth 25 points, each second-place vote is worth 24 points, each third-place vote is worth 23 points, and so on. The following table shows the ranking and total points for each of the top three teams at the end of the 2006 regular season. (The remaining 22 teams are not shown here because they are irrelevant to this exercise.)

Team	Points
1. Ohio State	1625
2. Florida	1529
3. Michigan	1526

(a) Given that Ohio State was the unanimous first-place choice of all the voters, find the number of voters that participated in the poll.

(b) Find the number of second and third-place votes for Florida.

(c) Find the number of second and third-place votes for Michigan.

[1] Bowl Championship Series

1.2 Voting Paradoxes and Problems

1. Of the four voting systems we've studied, which is best? Why?

2. The Des Moines Running Club is electing a new team captain. There are three candidates: Irving, Joseph, and Karl. The preference table is the following.

Ranking	# of Ballots 4	3	2
1st choice	J	I	K
2nd choice	I	J	I
3rd choice	K	K	J

   (a) If the club uses the plurality method, who will win the election?

   (b) If it were a head-to-head race between Irving and Joseph, who would win?

   (c) If it were a head-to-head race between Irving and Karl, who would win?

   (d) If it were a head-to-head race between Joseph and Karl, who would win?

   (e) Why does the plurality method's result seem unfair in this election?

3. **Head-to-Head Criterion:**

4. The school board is voting whether to award the contract for building a new elementary school to Aardvark Architects, to Bob's Builders, to the Confident Construction Company, or to Davenfield Developers.

	# of Ballots			
Ranking	5	3	3	1
1st choice	B	B	C	A
2nd choice	C	C	A	D
3rd choice	D	A	D	C
4th choice	A	D	B	B

   (a) The school board votes using a Borda count.
   Which company will get the contract?

   (b) Does this seem fair? Why or why not?

5. **Majority Criterion:**

6. The board of regents of Western Iowa State at Sema will be voting tomorrow for the University's new president from four candidates: Zehnpfennig, Gersbach, Haider, and Rentmeester. As of this moment, the voters' preferences are as follows:

	# of Ballots			
Ranking	7	5	4	1
1st choice	Z	H	G	R
2nd choice	R	Z	H	G
3rd choice	G	G	R	Z
4th choice	H	R	Z	H

The WIS regents use the plurality with elimination method.

(a) If the vote were held today, who would be chosen as university president?

(b) After thinking it over in bed tonight, the single voter in the last column decides that Zehnpfennig really is the best candidate after all, and so his preferences change to Z,R,G,H. Thus when the election is held tomorrow, the preference table is

	# of Ballots			
Ranking	7	5	4	1
1st choice	Z	H	G	~~R~~ Z
2nd choice	R	Z	H	~~G~~ R
3rd choice	G	G	R	~~Z~~ G
4th choice	H	R	Z	~~H~~ H

Who will be elected president?

(c) What is odd about your answers to (a) and (b)?

7. **Monotonicity Criterion:**

8. Voters are trying to decide what to do with a \$50 million surplus in the state budget. Their options are:

   (T) Give the money back to taxpayers as property tax relief.

   (R) Spend it on road construction and other infrastructure.

   (H) Build a new state historical building.

   Here is the preference table:

	# of Ballots		
Ranking	70,000	50,000	40,000
1st choice	T	R	H
2nd choice	R	H	T
3rd choice	H	T	R

   (a) The election is run by pairwise comparison. What is the result?

   (b) What is odd about your results from part (a)?

9. **Irrelevant Alternatives Criterion:**

10. Approval Voting:

11. **Arrow's Impossibility Theorem:**

This page intentionally left blank.

Name: ______________________________

Homework 3

V12. Consider the following preference table:

	# of Ballots				
Ranking	4	10	2	3	5
1st choice	C	D	A	C	A
2nd choice	D	C	C	B	B
3rd choice	B	A	B	A	D
4th choice	A	B	D	D	C

(a) How many votes would be required to win a *majority* in this election?

(b) Does the Borda Count violate the Majority Criterion for this particular preference table?

(c) Does the Borda Count violate the Head-to-Head Criterion for this particular preference table?

(over)

Name: ______________________________

V13. Consider the following preference table:

Ranking	# of Ballots 11	2	5	1	8	4
1st choice	C	D	C	D	A	B
2nd choice	A	A	A	A	D	A
3rd choice	D	C	B	B	C	C
4th choice	B	B	D	C	B	D

(a) Does the Borda Count violate the Majority Criterion for this particular preference table?

(b) Does the Borda Count violate the Head-to-Head Criterion for this particular preference table?

Name: ______________________________

V14. Consider the following preference table:

Ranking	# of Ballots 9	6	8	5
1st choice	B	B	D	D
2nd choice	D	D	B	B
3rd choice	C	A	A	C
4th choice	A	C	C	A

(a) How many points will each candidate receive in a Borda count? Who will win?

(b) The five voters in the last column really do think D is the best candidate and B is the second-best. However, they decide to be sneaky and lie on their Borda count ballots, claiming they think B is the worst candidate; in other words, they say they prefer D, C, A, and B in that order. Now how many points will each candidate receive in a Borda count? Who will win?

(c) Explain in a few complete sentences how these voters manipulated the Borda count and why it is unfair.

(over)

Name: ____________________

V15. Voters go to the polls to vote on three propositions simultaneously:

Proposition 1. Spend \$200,000 to plant new flowers and trees at the zoo.

Proposition 2. Spend \$400,000 to build a children's playground at the zoo.

Proposition 3. Spend \$500,000 to add a panda exhibit to the zoo.

All the voters love the zoo, plants, children, and pandas. They're all fiscally responsible, and only differ in how much money they want to spend.

(a) Alice is one of a thousand voters who want to improve the zoo, but don't want to spend more than \$1,000,000 total. She wants to spend as close that limit as possible, without going over. Which propositions will Alice and her bloc vote for?

(b) Zeke is one of a thousand voters who want to improve the zoo, but don't want to spend more than \$800,000 total. He wants to spend as close that limit as possible, without going over. Which propositions will Zeke and his bloc vote for?

(c) Shadrach is one of a thousand voters who want to improve the zoo, but don't want to spend more than \$600,000 total. He wants to spend as close that limit as possible, without going over. Which propositions will Shadrach and his bloc vote for?

(d) Fill out the following preference table with the votes—either "Yes" or "No."

Voters Number of votes	Alice et al. 1000	Zeke et al. 1000	Shadrach et al. 1000
Proposition 1			
Proposition 2			
Proposition 3			

(e) Which of the three propositions will pass? (Only a majority vote is needed for each proposition.)

(f) How many of the 3,000 voters are happy with the result?

(g) Explain in a few complete sentences what is odd or paradoxical about this situation.

1.3 Weighted Voting Systems

1. Notes on weighted voting systems:

   - P_i

   - w_i

   - q

   - $[q|w_1, w_2, \ldots, w_n]$

2. In a weighted voting system with weights $[30, 29, 16, 8, 3, 1]$, if a two-thirds majority of votes is needed to pass a motion, what is the quota?

3. Consider the weighted voting system $[14, 9, 8, 5]$.

   (a) What is the largest reasonable quota for this system?

   (b) What is the smallest reasonable quota for this system?

4. Consider the weighted voting system $[20|7, 5, 4, 4, 2, 2, 2, 1, 1]$.

   (a) How many voters are there?

   (b) What is the quota?

   (c) What is the weight for voter P_2?

   (d) If the first 4 voters vote for a motion and the rest vote against, does the motion pass?

   (e) If P_1 and P_2 vote against a motion, will the motion pass?

5. What is peculiar about each of the following weighted voting systems?

   (a) $[20|10, 10, 9]$

   (b) $[7|4, 2, 1]$

   (c) $[51|50, 49, 1]$

   (d) $[6|6, 2, 1, 1]$

6. A *dummy* is ...

7. A *dictator* is ...

8. A voter has *veto power* if ...

9. In the weighted voting system $[12|9, 5, 4, 2]$, are there any dummies or dictators?

10. In designing a weighted voting system $[q|6, 5, 4, 3, 2, 1]$, what is the largest quota q you could pick without giving veto power to anyone?

11. In the weighted voting system $[q|8, 5, 4, 1]$, if every voter has veto power, what is the quota q?

12. A committee has four members (P_1, P_2, P_3, and P_4). In this committee, P_1 has twice as many votes as P_2; P_2 has twice as many votes as P_3: P_3 has twice as many votes as P_4. Describe the committee as a weighted voting system when the requirements to pass a motion are

    (a) at least two-thirds of the votes

    (b) more than two-thirds of the votes

    (c) at least 80% of the votes

    (d) more than 80% of the votes

This page intentionally left blank.

Name: ______________________________

Homework 4

V16. Alice, Bob, Charles, and Danielle are the stockholders in Alphabet Industries, Inc. Alice owns 252 shares, Bob owns 741 shares, Charles inherited 637 shares, and 412 shares are in Danielle's hands. As usual, each share corresponds to a vote in the stockholder's meeting.

(a) If a certain type of motion requires a majority vote, what is the smallest number of votes needed to pass the motion?

(b) A different type of motion requires a 2/3 vote to pass. What is the smallest number of votes needed to pass this motion?

(c) Using the quota you found in part (b), express the weighted voting system in the correct notation (with brackets and quota).

V17. Which voters have veto power in the system $[51|29, 21, 8, 3, 1]$?

(over)

Name: ______________________________

V18. Find all dictators, dummies, and voters with veto power in the following weighted voting systems:

(a) $[51|20, 20, 20]$

(b) $[51|36, 34, 23, 6]$

(c) $[25|27, 11, 7, 2]$

(d) $[31|15, 13, 6, 4, 2]$

V19. In 1958, the Treaty of Rome established the European Economic Community (EEC) and instituted a weighted voting system for the EEC's governance. The members at that time were France, Germany, Italy, Belgium, the Netherlands, and Luxembourg. The three largest countries (France, Germany and Italy) were each given a vote with weight 4, Belgium and the Netherlands had votes of weight 2 and Luxembourg's vote had weight 1. The quota was 12.

What is unusual or interesting about this weighted voting system?

1.4 Banzhaf Power Index

1. A *coalition* is . . .

2. (a) Consider a weighted voting system with three voters P_1, P_2, and P_3. List all the coalitions. How many are there?

   (b) Consider a weighted voting system with four voters P_1, P_2, P_3, and P_4. List all the coalitions. How many are there?

   (c) If a weighted voting system has n voters P_1, P_2, . . . , P_n, how many coalitions are there?

3. A *winning coalition* is ...

4. List all the winning coalitions in the weighted voting system $[10|5, 4, 3, 2, 1]$.

5. In the weighted voting system
$$[10|6, 4, 3, 2, 1],$$
consider **the winning coalition** $\{P_1, P_2, P_3, P_4\}$.
Which voter(s) could change their minds and vote "no" without changing the outcome of the vote? Which voter(s) *need* to keep voting "yes" in order for the motion to pass?

6. A *critical voter* in a winning coalition is ...

7. Consider the voting system $[19|11, 9, 8, 5]$.

   (a) List all the *winning* coalitions.

   (b) In each winning coalition above, circle the *critical voters*.

   (c) Count the number of times each voter is a critical voter. This is called that voter's *Banzhaf power*.

Voter	Banzhaf power
P_1	
P_2	
P_3	
P_4	

   (d) Add up all the voters' Banzhaf powers; this sum is called the *total Banzhaf power* of the voting system.

   (e) Finally, divide each voter's Banzhaf power by the total Banzhaf power. The percentage that results is called the voter's *Banzhaf power* index.

Voter	Banzhaf power *index*
P_1	
P_2	
P_3	
P_4	

8. Calculate the Banzhaf Power Index for each voter in the weighted voting system

$$[51|32, 22, 12]$$

.

9. Make up a weighted voting system with a dummy, and calculate the Banzhaf Power Index for the dummy.

10. Make up a weighted voting system with a dictator, and calculate the Banzhaf Power Index for the dictator.

11. Make up a weighted voting system in which several voters have veto power. Calculate the Banzhaf Power Index for the voters with veto power. What do you notice?

12. The U.N. Security Council consists of 15 member countries–5 permanent members and 10 non-permanent members. A motion can pass only if it has the vote of *all five* of the permanent members plus at least four of the non-permanent members.

    (a) Describe the critical players in a winning coalition.

    (b) Use your answer in (a),together with the fact that there are 210 nine-member winning coalitions and 638 winning coalitions with 10 or more members, to explain why the total number of times all players are critical is 5080.

    (c) Using the results of (a) and (b), show that the Banzhaf power index of a permanent member is given by the ratio 848/5080.

    (d) Using the results of (a), (b) and (c), show that the Banzhaf power index of a non-permanent member is given by the ratio 84/5080.

    (e) Explain why the U.N. Security Council is equivalent to the weighed voting system in which each non-permanent member has 1 vote, each permanent member has 7 votes and the quota is 39 votes.

This page intentionally left blank.

Name: ______________________________

Homework 5

V20. List all the winning coalitions in the weighted voting system $[12|7,5,4,2]$.

V21. List all the winning coalitions in the weighted voting system $[11|6,4,3,3,1]$.

V22. In the weighted voting system

$$[38|22,20,17,9,5],$$

consider the winning coalition $\{P_2, P_3, P_4, P_5\}$. Which voters are critical voters in this coalition?

V23. In the weighted voting system

$$[7|3,3,2,2,2,1],$$

consider the winning coalition $\{P_1, P_3, P_4, P_6\}$. Which voters are critical voters in this coalition?

V24. Calculate the Banzhaf Power Index for each voter in the weighted voting system

$$[34|12,10,7,6]$$

.

(over)

Name: ______________________________

V25. Calculate the Banzhaf Power Index for each voter in the weighted voting system $[27|15, 7, 5]$.

V26. Consider the voting system $[25|24, 20, 1]$.

(a) Calculate the percentage of the total weight that each voter holds.

(b) Calculate the Banzhaf Power Index for each voter.

(c) Comparing your answers to parts (a) and (b), explain in complete sentences why the weight controlled by the voter is not the same thing as the power held by each voter.

Name: ______________________

V27. Calculate the Banzhaf Power Index for each voter in the weighted voting system

$$26|15, 13, 7]$$

.

V28. Calculate the Banzhaf Power Index for each voter in the weighted voting system

$$[63|43, 35, 22, 16]$$

.

(over)

Name: ______________________________

V29. Nassau County, New York used to be governed by a Board of Supervisors. The county had six districts, each of which one delegate to vote on county issues. The delegates' votes were weighted proportionately to the districts' population in 1964:

District	Weight
Hempstead #1	31
Hempstead #2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2

A simple majority was needed to pass a motion.

(a) Express this weighted voting system in our usual notation.

(b) Calculate the Banzhaf power of each district.

(c) What percentage of the county population lived in districts that are dummies?

(d) In 1965 John F. Banzhaf III argued in court that even though the weights were proportionate to population, this system of government was unfair. He won!

1.5 Shapley-Shubik Power Index

1. A *sequential coalition* is ...

2. (a) Consider a weighted voting system with three voters P_1, P_2, and P_3. List all the sequential coalitions. How many are there?

   (b) Consider a weighted voting system with four voters P_1, P_2, P_3, and P_4. List all the sequential coalitions. How many are there?

   (c) If a weighted voting system has n voters P_1, P_2, $\ldots$, P_n, how many sequential coalitions are there?

3. A *pivotal player* is . . .

4. List all the sequential coalitions in the weighted voting system $[4|3, 2, 1]$ and determine the pivotal player.

5. In the weighted voting system $[4|3, 2, 1]$, Is there a voter who is always pivotal? Is there a voter who is never pivotal?

6. Consider the voting system $[6|4, 3, 2, 1]$.

   (a) List all the *sequential* coalitions.

   (b) In each sequential coalition above, circle the *pivotal voters*.

   (c) Count the number of times each voter is a pivotal voter. This is called that voter's *Shapley-Shubik power*.

Voter	Shapley-Shubik power
P_1	
P_2	
P_3	
P_4	

   (d) Add up all the voters' Shapley-Shubik powers; this sum is called the *total Shapley-Shubik power* of the voting system.

   (e) Finally, divide each voter's Shapley-Shubik power by the total Shapley-Shubik power. The percentage that results is called the voter's *Shapley-Shubik power index*.

Voter	Shapley-Shubik power *index*
P_1	
P_2	
P_3	
P_4	

7. Calculate the Shapley-Shubik Power Index for each voter in the weighted voting system $[51|32, 22, 12]$.

8. Make up a weighted voting system with a dummy, and calculate the Shapley-Shubik Power Index for the dummy.

9. Make up a weighted voting system with a dictator, and calculate the Shapley-Shubik Power Index for the dictator.

10. Make up a weighted voting system in which several voters have veto power. Calculate the Shapley-Shubik Power Index for the voters with veto power. What do you notice?

11. In some cities the city Council operates under what is known as the, "strong – mayor". Under this system the city Council can pass a motion under a simple majority, but the mayor has the power to veto the decision. The mayor's veto can then be overruled by a "super majority" of the council members. As an example, consider the city of Ice-n-knock. In Ice-n-knock, the city Council has four members plus a strong mayor who has a vote as well as the power to veto motion supported by a simple majority of the council members. On the other hand, the mayor cannot veto a motion supported by all four Council members. Thus, a motion can pass if the mayor +2 or more Council members supported or, alternatively, if the mayor is against it at the four council members support it.

    It makes sense that under these rules, the four council members have the same amount of power, but the mayor has more. Compute the Shapley-Shubik Power Index of this weighted voting system to figure out exactly how much more.

    For purposes of comparison, calculate the Banzhaf power distribution of Ice-n-knock.

This page intentionally left blank.

Name: ______________________________

Homework 6

V30. List all the sequential coalitions in the weighted voting system $[16|9,8,7]$.

V31. List all the sequential coalitions in the weighted voting system $[51|40,30,20,10]$.

(over)

Name: ____________________________

V32. Consider the voting system $[25|24,20,1]$.

(a) Calculate the percentage of the total weight that each voter holds.

(b) Calculate the Shapley-Shubik Power Index for each voter.

(c) Comparing your answers to parts (a) and (b), explain in complete sentences why the weight controlled by the voter is not the same thing as the power held by each voter.

V33. Calculate the Shapley-Shubik Power Index for each voter in the weighted voting system $[15|16,8,4,1]$.

$\{16,8,4,2\}$	$\{8,16,4,2\}$	$\{4,16,8,2\}$	$\{2,16,8,4\}$
$\{16,8,2,4\}$	$\{8,16,2,4\}$	$\{4,16,2,8\}$	$\{2,16,4,8\}$
$\{16,4,8,2\}$	$\{8,4,16,2\}$	$\{4,8,16,2\}$	$\{2,8,16,4\}$
$\{16,4,2,8\}$	$\{8,4,2,16\}$	$\{4,8,2,16\}$	$\{2,8,4,16\}$

Name: ______________________________

V34. Calculate the Shapley-Shubik Power Index for each voter in the weighted voting system $[24|16,8,4,1]$.

$\{16,8,4,2\}$	$\{8,16,4,2\}$	$\{4,16,8,2\}$	$\{2,16,8,4\}$
$\{16,8,2,4\}$	$\{8,16,2,4\}$	$\{4,16,2,8\}$	$\{2,16,4,8\}$
$\{16,4,8,2\}$	$\{8,4,16,2\}$	$\{4,8,16,2\}$	$\{2,8,16,4\}$
$\{16,4,2,8\}$	$\{8,4,2,16\}$	$\{4,8,2,16\}$	$\{2,8,4,16\}$

V35. Calculate the Shapley-Shubik Power Index for each voter in the weighted voting system $[28|16,8,4,1]$.

$\{16,8,4,2\}$	$\{8,16,4,2\}$	$\{4,16,8,2\}$	$\{2,16,8,4\}$
$\{16,8,2,4\}$	$\{8,16,2,4\}$	$\{4,16,2,8\}$	$\{2,16,4,8\}$
$\{16,4,8,2\}$	$\{8,4,16,2\}$	$\{4,8,16,2\}$	$\{2,8,16,4\}$
$\{16,4,2,8\}$	$\{8,4,2,16\}$	$\{4,8,2,16\}$	$\{2,8,4,16\}$

This page intentionally left blank.

1.6 Study Guide

To prepare for the exam over this chapter, you should review the in-class worksheets and homework. Be ready to do the kind of problems you faced on the homework.

As a general guide, I recommend reviewing the following topics.

1. Know how to read a preference table
2. Calculate the winner according to
   (a) Plurality
   (b) Borda count
   (c) Plurality with elimination
   (d) Pairwise comparison
3. Devise preference tables that satisfy given conditions (e.g., "Come up with a preference table where the pairwise comparison test produces no winner.")
4. Weighted voting
   (a) Know how weighted voting on yes/no motions works.
   (b) Understand the notation $[q|w_1, \ldots, w_n]$.
   (c) Given a weighted voting system, find any dictators, dummies, or voters with veto power.
5. Construct weighted voting systems that satisfy given conditions (e.g., "Come up with a weighted voting system where two people have veto power.")
6. Calculate the Banzhaf Power Index for the voters in a weighted voting system.
7. Calculate the Schaply-Shubik Power Index for the voters in a weighted voting system.

This page intentionally left blank.

Chapter 2

Financial Mathematics

2.1 Simple Interest

In this chapter, we retain the following conventions:

1 year	is 12 months
1 year	is 52 weeks
1 year	is 365 days
1 week	is 7 days
1 day	is 24 hours

All interest rates will be annual unless specified otherwise.

1. Sara rents a post-hole digger from RentAll to build a fence. The cost is $12.95 per day, and she keeps it for three days. How much will Sara pay?

2. Hailey is visiting Kentucky for work and will need to rent a furnished apartment. The rate is $100 per week. They stay for 18 days. How much will Hailey pay?

3. Sara needs to lease a car for $2500 per year. How much will Sara pay if she keeps the car for

   (a) 2 years?

   (b) 5 months?

   (c) 18 days?

   (d) 6 weeks?

4. Guido "the Organist" Landini is a small-time loan shark in downtown Metropolis. He charges clients 43% simple interest per year. What percentage interest does he charge on a loan of

   (a) 3 years?

   (b) 1 month?

   (c) 1 week?

   (d) 3 weeks?

   (e) 4 days?

5. The interest charged is...

6. Loans:

7. Calculations:

8. **Simple Interest Formula:**

9. Your friend, Aleah borrows \$1800 from you for 3 years, at an annual simple interest rate of 7.0%. How much will she repay you?

$$\begin{aligned} F &= \\ P &= \\ r &= \\ t &= \end{aligned}$$

10. Suppose you borrow \$1000 for 4 months at an annual simple interest rate of 8.0%. Find the total amount that you will have to repay.

$$\begin{aligned} F &= \\ P &= \\ r &= \\ t &= \end{aligned}$$

11. Suppose Luke borrows \$1500 from you for a period of 2 years, at an annual simple interest rate of 6.0%. Find the amount of interest you will earn from Luke.

$$\begin{aligned} F &= \\ P &= \\ r &= \\ t &= \end{aligned}$$

12. Logan took out a loan for 6 months at an annual simple interest rate of 4.0%. The total amount he paid for the loan was \$1890.46. How much did he borrow?

$$\begin{aligned} F &= \\ P &= \\ r &= \\ t &= \end{aligned}$$

13. Suppose \$2000 is borrowed for 7 months, at the end of which \$2100 is repaid. Find the annual simple interest rate r.

$$\begin{aligned} F &= \\ P &= \\ r &= \\ t &= \end{aligned}$$

14. R'moni got a loan from the city to help repair the sidewalk in front of her home. She borrowed \$1800 at an annual simple interest rate of 5%. The amount she repaid was \$2025. How long was her loan?

$$\begin{aligned} F &= \\ P &= \\ r &= \\ t &= \end{aligned}$$

This page intentionally left blank.

Name: ______________________________

Homework 7

F1. Given the principal P, the annual interest rate r, and the time t, find the amount F that must be repaid.

(a) $P = \$15{,}000$, $r = 6\%$, $t = 5$ years

(b) $P = \$5{,}300$, $r = 2\%$, $t = 3$ months

(c) $P = \$9{,}000$, $r = 4.5\%$, $t = 50$ days

F2. Of the four values F, P, r, and t in the Simple Interest Formula, three are given to you. Use the Simple Interest Formula to find the fourth one.

(a) $F = \$12{,}000$, $r = 3\%$, $t = 3$ years

(b) $F = \$8{,}500$, $t = 6$ months, $P = \$8{,}200$

(c) $F = \$4{,}250$, $r = 7\%$, $P = \$3{,}500$

F3. You borrow \$2,000 from the city to pay for sidewalk repairs. You promise to repay the loan in three years at 5% simple interest. How much will you pay the city then?

F =
P =
r =
t =

(over)

Name: ______________________________

F4. Sara Nelson knows she will inherit \$10,000 from her dying uncle within five months. The bank will lend money at 4% simple interest. What is the largest amount of money she could borrow now, if she plans to use her inheritance to repay it in five months?

F5. Sara's friend Aleah wants to borrow \$220 from her for three months. If Sara wants to earn \$20 in interest on the loan, what percent simple interest should she charge?

F6. Alisyn has borrowed \$800 from her friend Jackson, who is charging 2% simple interest. Eventually Alisyn repaid the loan, but it cost \$200 to do so. For how long did Alisyn borrow Jackson's money?

2.2 Compound Interest

How to Fill Out a Compound-Interest Balance Sheet

The ledger below grows your money *one period at a time.* Each row covers a slice of the year-one year, one quarter, one month, or one day- so the "time" t in the simple-interest formula

$$I = Prt$$

is written as a **fraction of a year**. Follow the four steps every time you move to a new row. Work with a basic calculator (or by hand) and round all money amounts to the nearest cent. Do *not* jump ahead to the compound-interest shortcut until your instructor says so.

1. **Record the Beginning Balance.** In the first row this is the principal you deposited. In every later row copy the *Ending Balance* from the line just above.

2. **Compute the Interest for this Period.** Use $I = Prt$ *exactly as written*:

$$\text{Interest} = (\text{Beginning Balance})\,(\text{Interest Rate})\,(\text{Time in years}).$$

$$\begin{aligned} 5\% \text{ interest rate, yearly period} &\Longrightarrow t = 1 & I &= P(0.05)(1) \\ 4.8\% \text{ interest rate, quarterly period} &\Longrightarrow t = \tfrac{1}{4} & I &= P(0.048)\left(\tfrac{1}{4}\right) \\ 3.6\% \text{ interest rate, monthly period} &\Longrightarrow t = \tfrac{1}{12} & I &= P(0.036)\left(\tfrac{1}{12}\right) \end{aligned}$$

   Write the dollar value for I in the "Interest" column.

3. **Find the Ending Balance.** Add the interest you just calculated to the beginning balance:

$$\text{Ending Balance} = \text{Beginning Balance} + \text{Interest}.$$

   Enter this rounded amount in the right-hand column.

4. **Carry Forward.** The Ending Balance you just found becomes next period's Beginning Balance. Repeat Steps 2–4 for as many rows as the problem requires.

1. Alison deposits \$1,000 in a savings account that pays 5% interest, compounded *once per year*. Fill in the balance sheet for three years.

Year	Beginning Balance (\$)	Interest Earned (\$)	Ending Balance (\$)
1	1000	50	1050
2	1050		
3			

2. A certificate of deposit (CD) offers 4% interest compounded *quarterly*. If R'moni places \$500 in the CD, complete the ledger for the first year (four quarters).

Quarter	Beginning Balance (\$)	Interest Earned (\$)	Ending Balance (\$)
Q1	500	5	
Q2			
Q3			
Q4			

3. A certificate of deposit starts at \$2,500 and earns 3.6% APR, compounded *monthly*. Zane wants to track the balance for the first six months.

Month	Beginning Balance (\$)	Interest Earned (\$)	Ending Balance (\$)
1			
2			
3			
4			
5			
6			

Using the Compound Interest Formula

4. **Compound Interest Formula:**

5. Notes on *nominal interest rate* and *interest per period*:

6. Suppose \$1000 is deposited into an account that yields 9% annually. Find the amount in the account at the end of the fifth year if the compounding is done

   (a) annually

   (b) semiannually

   (c) quarterly

   (d) monthly

   (e) weekly

   (f) daily

7. Luke is going to take out a short-term business loan of \$5000 for a period of 5 months. The interest rate is 4%, compounded monthly. How much will Luke have to repay at the end of the loan?

$$F =$$

$$P =$$

$$r =$$

$$n =$$

$$t =$$

8. R'moni deposits \$538 in a savings account that earns 3% interest, compounded monthly. How much interest will she earn in four years? (Your answer should be in dollars.)

9. In January 2008 Alison's uncle deposited some money into a college savings account for Alison. The account paid 5% interest, compounded quarterly. In January 2025 there was \$28,626.40 in the account. How much did Alison's uncle deposit back in 2008?

10. Alisyn takes out a loan of \$10,000.00 for four years. At the end of the four years, she pays back \$11,273.30 to the bank. If the interest was compounded monthly, what was the interest rate?

11. If the loan that Alisyn took out was only compounded yearly, what was the interest rate?

12. Moose Jaw State Bank offers an savings account paying 5.4% interest, compounded daily. The savings account at Moose Lake Credit Union pays 5.41% interest, compounded quarterly. Finally, Moose Kneecap Savings & Loan offers a CD earning 5.5% interest, compounded annually. You have \$4500 to invest.

    (a) If you deposit your money in Moose Jaw State Bank, how many dollars of interest will you *really* earn in one year? What percent is that of your deposit?

    (b) If you instead deposit your money at Moose Lake Credit Union, how many dollars of interest will you have earned one year later? What percent is that of your deposit?

    (c) Finally, how many dollars of interest would you earn if you put your money into Moose Kneecap Savings & Loan for one year? What percent is that of your deposit?

    (d) Which account is best?

13. **Annual Effective Yield Formula**[1]**:**

14. If you open a savings account at Muddy Lake Credit Union, you will earn 3% interest compounded daily. What is the annual effective yield?

15. If you open a CD at Muddy Ridge State Bank, you will earn 3.2% interest compounded quarterly. What is the annual effective yield?

16. Your financial adviser offers you two investment funds. If you invest in the AmerEquiShare Fund, you can expect a return of 9% compounded semiannually (twice a year). If you invest in the PreferredSource Fund, you can expect a return of 8.8% compounded monthly.

    (a) What is the annual effective yield for the AmerEquiShare Fund?

    (b) What is the annual effective yield for the PreferredSource Fund?

    (c) Which investment offers the higher return?

This page intentionally left blank.

Name: ______________________________

Homework 8

F7. Logan deposits \$1 000 in a savings account that pays 5% interest, compounded *once per year*.

a) Complete the three-year ledger below, rounding every interest entry to *two* decimal places (cents).

b) After finishing the ledger, verify your final balance with the compound-interest formula
$$F = P(1 + \frac{r}{n})^{nt}.$$
The two answers should agree to the cent if every ledger line is correct.

Year	Beginning Balance (\$)	Interest Earned (\$)	Ending Balance (\$)
1			
2			
3			

F8. Given the principal P, the interest rate r, the time t, and the frequency of compounding, find the future amount F.

(a) $P =$ \$15,000, $r = 6.5\%$ compounded quarterly, $t = 2$ years

(b) $P =$ \$100,000, $r = 3\%$ compounded monthly, $t = 30$ years

(c) $P =$ \$1,500, $r = 4\%$ compounded daily, $t = 60$ days

(d) $P =$ \$6,000, $r = 5\%$ compounded monthly, $t = 6$ months

(over)

Name: ______________________________

F9. When Zane Licht was born, his parents deposited \$4,000 into a bank account bearing 5% interest, compounded monthly. When he reached age 18, how much was in his account?

F10. Seventh National Bank is lending \$10,000 to Zane Licht for three years. The bank compounds interest quarterly. If the bank needs to receive \$2,300 in interest from Zane to cover its expenses, what interest rate should it charge?

F11. R'moni deposited some money in a bank account earning 3% interest, compounded daily. Twelve years later, there was \$1,720 in the account. How much money did R'moni originally deposit?

F12. How much money must Alisyn Parkhurst deposit today in order to have \$50,000 in twenty years, if her account bears

(a) 8% interest, compounded annually?

(b) 8% interest, compounded quarterly?

(c) 8% interest, compounded monthly?

(d) 8% interest, compounded daily?

This page intentionally left blank.

2.3 Savings

1. Deriving a very useful formula:

2. An *annuity* is a series of equal payments made at regular intervals. For an *ordinary annuity*, the payments are made at the *end* of the interval. If the payments are made at the *beginning* of the interval, it is called an *annuity due*. We will not be working with this type of annuity in this class.

   A good way to think about these is as a savings account. You are putting some money, R, into the account every compounding period in order to save up for something in the future.

3. **Future Value of an Annuity:**

$$F = R\left(\frac{(1+\frac{r}{n})^{nt}-1}{\frac{r}{n}}\right)$$

If we let $i = \frac{r}{n}$ and $m = nt$, we can shorten the formula to:

$$F = R\left(\frac{(1+i)^{m}-1}{i}\right)$$

4. **Example:**
You are going to graduate, get a job, and want to buy a house in two years. How much should you deposit each month into an account bearing 7% interest, compounded monthly, in order to have $15,000 for a down payment in two years?

5. How much money will you have when you retire if you save \$20 each month from graduation (age 22) until retirement (age 63), if you can average 6.6% annual interest compounded monthly?

6. Sara starts saving for a down payment on a house by depositing \$100 each month into an annuity that pays 4.8% interest, compounded monthly. How large a down payment can she afford in 3 years?

7. Hoopla Publishing Company knows its printing press is nearing the end of its life. They will need to purchase a new printing press for \$65,000 in 10 years. What payment should Hoopla make every year into a sinking fund earning 7% interest, compounded annually, in order to have the \$65,000 in ten years?

8. Suppose you wanted to compute how much money would be in an account earning 5% interest compounded monthly if you deposited \$100/month for 25 years.

   (a) How much would you have if you used $\frac{0.05}{12} \simeq 0.004$?

   (b) How much would you have if you used $\frac{0.05}{12} \simeq 0.0041$?

   (c) How much would you have if you used $\frac{0.05}{12} \simeq 0.00417$?

   (d) How much would you have if you used $\frac{0.05}{12} \simeq 0.004167$?

9. Suppose you wanted to save up \$60,000 over 20 years by depositing an amount of money in the bank account earning 5% interest compounded monthly.

   (a) How much would you have to save every month if you used $\frac{0.05}{12} \simeq 0.004$?

   (b) How much would you have to save every month if you used $\frac{0.05}{12} \simeq 0.0041$?

   (c) How much would you have to save every month if you used $\frac{0.05}{12} \simeq 0.00417$?

   (d) How much would you have to save every month if you used $\frac{0.05}{12} \simeq 0.004167$?

Name: ______________________________

Homework 9

F13. Find the future value of each of the following ordinary annuities.

(a) Deposits of \$1200 made at the end of each year for 10 years, where 7% annual interest is compounded annually.

(b) Deposits of \$300 made at the end of each quarter for 10 years, where 8% annual interest is compounded quarterly.

(c) Deposits of \$51 made at the end of each month for 20 years, where 6% annual interest is compounded monthly.

(d) Deposits of \$100 made at the end of each week for 2 years, where 8% annual interest is compounded weekly.

F14. Leo Dodd deposits \$50 each week in a bank CD earning 4% interest compounded weekly. He does this for 14 years.

(a) How much is in his bank account at the end of those 14 years?

(b) How many dollars of interest did he earn, in total, over those 14 years?

(over)

Name: ______________________

F15. Aleah is graduating from Wartburg College and wants to have \$20,000 ready for a down payment on a house in 5 years. If her investments pay 6% interest, compounded monthly, how much money should she invest every month in order to achieve her goal?

F16. (a) Sara Nelson said she would set up an ordinary annuity for her newborn niece Gloria and deposit \$100 each month, with the last payment to occur on Gloria's 18th birthday. The payments would earn 6% annual interest, compounded monthly. How much will Gloria have on her 18th birthday?

(b) Sara's spouse Zane suggested they should just deposit a lump sum of money *now* into a bank account (earning 6% annual interest, compounded monthly), so that it would grow to the same amount by Gloria's 18th birthday as you found in part (a). If they go with Zane's plan, how much money must the loving relatives deposit in the account now?

Name: ______________________________

F17. Alison Newton decides to pay \$300 at the end of each month into an ordinary annuity that pays 8% annual interest, compounded monthly, for five years. She decides to calculate the future value of this annuity at the end of five years, but she makes a mistake in her calculations. What was her mistake? Is her answer too big or too small?

$$\begin{aligned}
FV &= P \cdot \left(\frac{(1+i)^m - 1}{i} \right) \\
FV &= 300 \cdot \left(\frac{(1+.08)^{60} - 1}{.08} \right) \\
FV &= 300 \cdot 1253.213296 \\
FV &= \$375{,}963.99
\end{aligned}$$

This page intentionally left blank.

2.4 Loans

1. **Present Value of an Annuity:**

$$P = R\frac{1-(1+\frac{r}{n})^{-nt}}{\frac{r}{n}}$$

or

$$P = R\frac{1-(1+i)^{-m}}{i}$$

2. The Present Value formula states how much a series of payments is worth *as one lump sum at the beginning*. This can show up in several different ways; here are four.

   (a) **How much money do you need now to fund a series of future payments?** Example: You are planning to retire at age 68. You will need a \$50,000 payment each year to live on, and you plan to be retired for 20 years. The interest rate is 6%. How much money do you need in your retirement account when you retire?

(b) **What is a fair price to charge now in exchange for paying someone regularly in the future?**
Example: R'moni runs a retirement home, and Alison wants to pay a fixed sum now to have R'moni take care of Alison for the rest of her life. It will cost R'moni \$2000 per month to take care of Alison, and she will probably live another 15 years. R'moni can get interest rates of 5% compounded monthly. How much should she charge Alison?

(c) **What is a fair price to pay for something that will pay you regularly in the future?**
Example: Sara is an investor looking at buying a copper mine. The mine will produce an annual profit of \$1,200,000 per year for its owner for the next 20 years, but then the ore will run out. Interest rates are sitting at 4%. How much would be a fair price to pay for the mine?

(d) **How does a loan amount relate to the regular loan payment?**
Example: Sara put a \$2000 down payment on a \$12,000 car, financing the rest with a 5-year loan at 6% interest, compounded monthly.

i. How much will her monthly payments be?

ii. How much cash did she pay, total, for her \$12,000 car?

iii. How many dollars of interest did she pay for this car?

3. Hailey has a student loan of \$9,000 to be paid back over 12 years at 4% interest compounded monthly.

   (a) How much will the monthly payment be?

   (b) What will be the total amount she pays for her \$9,000 loan?

   (c) How much interest did she pay for this loan?

   (d) If she wants to pay it off in just 5 years, how much should she pay each month? How much interest will that save her?

4. You want to buy a car. You have \$1,500 saved up for a down payment, and you can get a 5-year car loan at 3% interest, compounded monthly.

   (a) If you can afford a \$200 monthly car payment, what is the most expensive car you can buy?

   (b) If you want to buy an \$18,000 vehicle, what will the monthly payment be?

   (c) If you buy that \$18,000 vehicle, how much will you pay in interest over the life of the loan?

This page intentionally left blank.

Name: ______________________________

Homework 10

F18. Find the present value of each of the following ordinary annuities.

(a) Payments of \$500 made at the end of each quarter for 8 years, where 10% annual interest is compounded quarterly.

(b) Payments of \$100 made at the end of each month for 10 years, where 6% annual interest is compounded monthly.

F19. Suppose you borrow \$16,000 from a bank to purchase a car. The bank charges 4% annual interest, compounded monthly. You are to make equal monthly payments at the end of each of the next 48 months to amortize your loan. How much are your monthly payments?

F20. You want to retire at age 65 with an \$80,000 annual income. You come from a long-lived family, so you want to be prepared in case you live to age 97. Interest rates are at 8%.

(a) How much money will you need in your retirement accounts when you retire, in order to fund annual payments of \$ 80,000?

(b) If you start working at age 22, how much money should you deposit into your retirement account each month in order to have saved up the amount from part (a) by the time you retire at age 65?

(over)

Name: ______________________________

F21. R'moni has to take out \$20,000 in student loans to get through college. The interest rate is 6.8% annually, compounded monthly, and she will make monthly payments for the next ten years. (No interest is charged or payments required until she leaves school.)

(a) How much will her monthly payment be?

(b) How much interest will she pay over those ten years? (The answer should be in dollars.)

(c) Suppose she tries to pay off the loan in just 5 years. How much will she have to pay each month to do so?

(d) If she does pay off the loan in 5 years, how much interest will she pay total?

(e) How much money does she save by paying off the loan in 5 years instead of 10 years?

Name: ______________________________

F22. Some time ago Sara Nelson took out a mortgage from First National State Local Bank; her payment was \$900 per month. FNSLB is strapped for cash these days, so it's considering "selling the mortgage" to Seventh National Bank; in other words, SNB will pay FNSLB a certain sum of money, and in return SNB will get all Sara Nelson's remaining mortgage payments.

There are 187 payments remaining on the mortgage, and interest rates are at 4% per year. How much should FNSLB charge Seventh National Bank to buy the mortgage?

F23. You win a "one million dollar" lottery prize. Hooray!

(a) Suppose the lottery rules stipulate that you will be paid \$50,000 at the end of each of the next 20 years, for a total of \$1,000,000 paid out. Assuming that annual interest rates will stay at 5%, what is the present value of this prize? In other words, how much is this prize really worth *today*?

(b) Suppose instead the lottery rules stipulate that you will be paid \$50,000 today, and then \$50,000 at the end of each year for the next 19 years. Assuming 5% interest rates, how much is *this* prize really worth *today*?

(over)

Name: ____________________

F24. Two oil wells are for sale. The well in Varmint, TX promises to yield payments of \$6,000 at the end of each year for the next 10 years. The well in Mule's Ear, TX will yield payments of \$4,000 at the end of each year for the next 20 years. (Notice that the lengths of time are different!)

(a) Assuming that annual interest rates will hold steady at 8% for the next 20 years, find the present value of each oil well. Which oil well is more valuable?

(b) Assuming that annual interest rates will stay at 6% for the next 20 years, find the present value of each oil well. Which oil well is more valuable?

(c) Why did the present values of the oil wells go up when the interest rates went down from part FF24(a) to part FF24(b)?

2.5 Student Loans and Financing

You are a college student and you or someone you know is paying for that education. Some students have families that have saved up over time to pay for their college, while others are financing their college education using loans. It is important to understand how student loans work and the impact of taking out a student loan on your immediate future[2]. First we will practice.

1. You need to borrow money from Alisyn. Alisyn agrees to lend you \$500, but you will accrue 1% interest every day.
   (a) Write an expression for the amount you owe after 10 days, and generalize it for t days as a function of t.

   (b) How much did you owe in interest alone?

2. Consult the link below to learn how the interest on student loans is calculated:

   `https://studentaid.gov/understand-aid/types/loans/interest-rates`

   Calculate and compare the interest owed on the following two loans:

Loan A 10 year loan with principal amount of \$15,000 with interest at 7%.

- [**Loan B**] 10 year loan with principal amount of \$15,000 with interest at 4%.

[2] Most of the content of this section is from Mathematics for Social Justice, Chapter 18 [1]

The Impact of Attending College Fifteen Years Apart:

The Case of Hailey and Alison

3. Imagine the following scenario, Hailey attended college between 2010–2011. Alison is planning to enroll in the fall of the current year and expects to graduate four years later. Assume that the amount borrowed consists of 50% of the tuition.

   (a) Fill in the table below

Hailey and Alison			
	Hailey	Alison	Note
Tuition			You need to find this information
Loan Amount			Hailey and Alison borrowed the same percentage (50%) of tuition
Loan Term	10 years	10 years	
Interest Rate	6.92%	4.66%	
Monthly Payment			
Total Interest Paid			
Cumulative Payments			

   (b) In referring to Hailey and Alison who pays less in interest and by how much?

   (c) After repaying their loans, what are the cumulative payments of Hailey and Alison?

   (d) In reference to the previous question, is this result in line with or contradictory to the interest rate each borrower got?

Same Tuition Different Interest Rates:

The Case of R'moni and Logan

4. Imagine the following scenario. R'moni and Logan attended college concurrently. They borrow a comparable sum of money. However, Logan received a lower interest rate on his loan. Assume that the amount borrowed constitutes 50% of the tuition.

   (a) Fill in the table below

R'moni and Logan			
	R'moni	Logan	Note
Tuition			The same
Loan Amount			R'moni and Logan borrowed the same amount
Loan Term	10 years	10 years	
Interest Rate	6.92%	3.46%	
Monthly Payment			
Total Interest Paid			
Cumulative Payments			

   (b) How much does Logan save in interest compared with R'moni? Is the amount significant?

2.6 Study Guide

To prepare for the exam over this chapter, you should the in-class worksheets and homework. Be ready to do the kind of problems you faced on the homework.

As a general guide, I recommend reviewing the following topics.

1. Calculate compound interest; use this to determine the future savings from a *one-time deposit*.
2. Find interest per period.
3. Calculate the future value of an annuity; use this to determine future savings when making *regular deposits*.
4. Calculate (a) how much you can save for the future by making a series of regular deposits, and (b) how much you must deposit regularly to have a certain amount in the future.
5. Calculate how much money you need now in order to fund a series of future payments.
6. Calculate the regular payment for a loan; conversely, given the regular payment, calculate how much you can borrow.
7. Calculate the equity held in a house, given information about the home's value and the mortgage.
8. Calculate mortgage costs involving down payments and closing costs.
9. Calculate mortgage costs involving taxes and insurance (PITI).

Chapter 3

Redistricting

According to the Supreme Court, the Constitution requires that population must be equalized across districts. The idea is that if one Iowan lives in a district with 1 million other voters while another Iowan lives in a district with only 200,000 other voters, the second one's vote is more influential in choosing a member of Congress. Of course, populations shift, growing or shrinking over time.

To prevent those shifts from leaving unbalanced districts, state legislatures must redraw their electoral districts every 10 years, after the Census Bureau releases its new population data. redistricting regularly leads to heated political and legal fights as legislators scramble to gain advantage for their parties.

Once the census has been taken and all the numbers have been added up and the apportionment problem has been solved (up to the Supreme Court if necessary), it is time to divide up each state into districts (one per representative). Some of the rules of redistricting are:

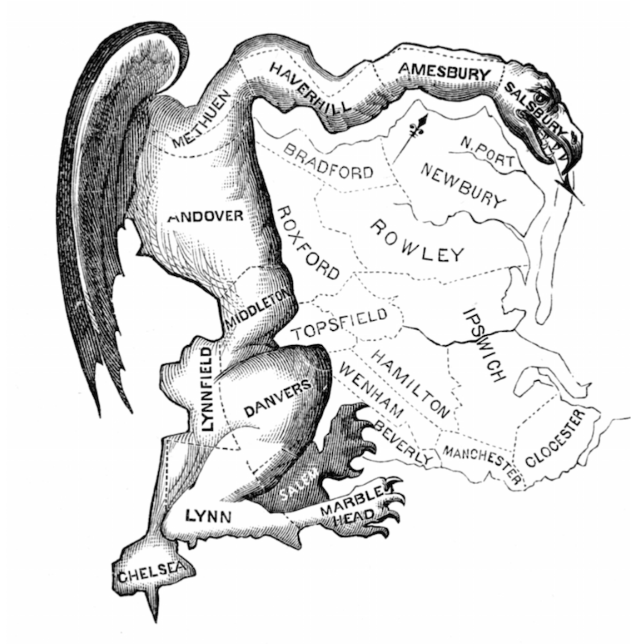
- Population must be equalized across districts.

- Each district must be (if possible) physically connected.

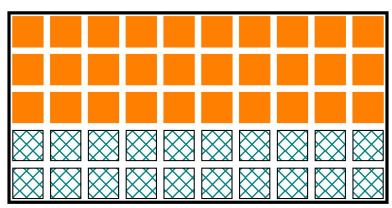
Additionally, each state has more rules that they have decided on. In most states, the party in power when the census data comes out gets to draw in district lines for Federal Representatives and all state offices that need districts. This process is done by hand often block by block. In 2011 the new field of data science made it possible for whoever was in charge to manipulate the lines in such a way that their majority was nearly guaranteed in the near future.

3.1 Gerrymandering

The process of dividing the state into the appropriate number of districts is called redistricting, but in the popular press, it is often referred to as gerrymandering. This is because the map-makers often decide to split things up to their benefit. One of the earliest, most obvious examples of this was in 1812 when Massachusetts Governor Gerry and his party drew one district in such a curvy and jagged manner that the local press thought it looked like a Salamander (see Figure 3.1) and coined the term Gerrymander.

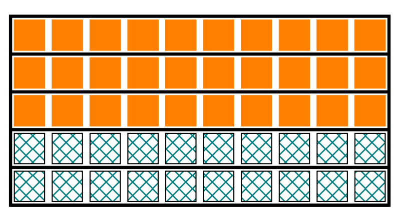


Consider the simple state[1]:



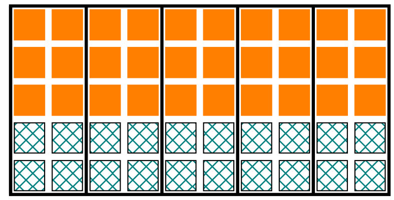
There are 50 people in this state who belong to two different political parties. The census says that they should have 5 seats, so we need to divide up these people into 5 groups.

1. Consider the following districting of this state. Who would win with a majority of votes in each district? Who has an advantage in the legislature? Does it seem fair? Are the people in each district connected to each other? Are the districts generally compact?

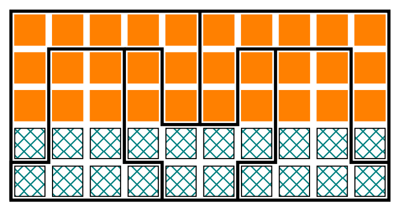


[1]Diagrams adapted from Stephen Nass, "How to Steal an Election"

2. Consider the following districting of this state. Who would win with a majority of votes in each district? Who has an advantage in the legislature? Does it seem fair? Are the people in each district connected to each other? Are the districts generally compact?



3. Consider the following districting of this state. Who would win with a majority of votes in each district? Who has an advantage in the legislature? Does it seem fair? Are the people in each district connected to each other? Are the districts generally compact?

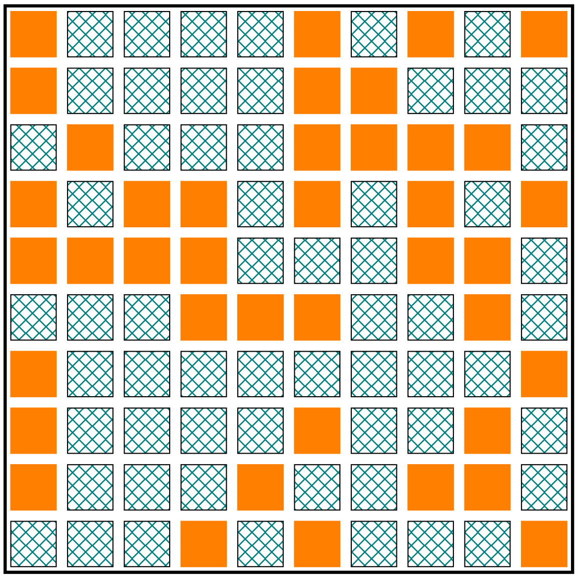


The last diagram is an example of the 'packing' and 'cracking' that are illustrative of modern gerrymandering.

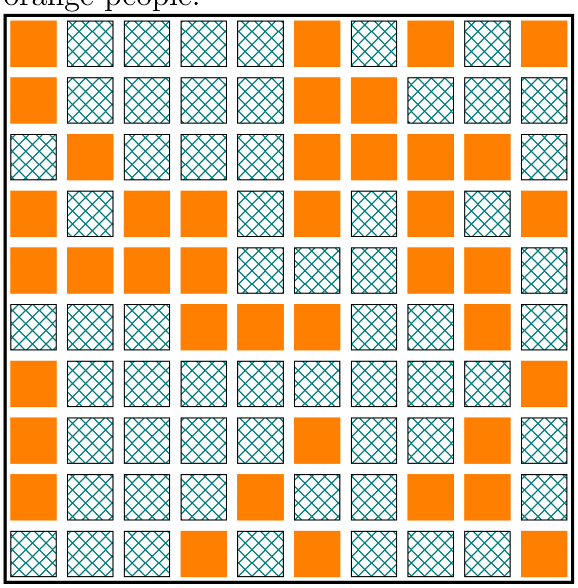
packing Putting a very large number of voters who all vote the same way in a single district. In this case, the election is won by these voters, but by a very large margin.

cracking Putting a slight majority of voters for one party in a single district. In this case, the election is won by these voters by a slim margin. There are a lot of minority voters in the district, but not enough to ever win an election.

4. The state below has approximately 40% orange (●) people and 60% teal (⊗) people (each color was selected randomly). Divide the state into 10 districts in such a way so that in an election 40% of the seats will be won by the orange people and 60% by the teal people.



5. Now, divide the same state in such a way so that 60% of the seats will be won by the orange people.

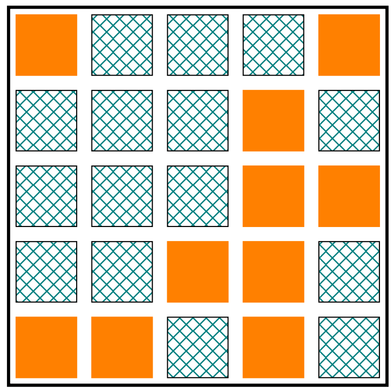


You will need these districting plans for the next two homework sets.

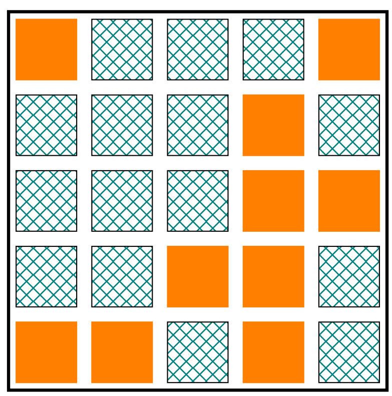
Name: ______________________________

Homework 11

R1. Consider the following map. The state has approximately orange ( ) people and teal ( ) people (each color was selected randomly). Put 5 districts on the map so that teal ( ) receives the most seats.



R2. Put 5 districts on the map so that orange ( ) receives the most districts.



Add your districts to the next two homeworks as well. Make sure you have 5 districts with 5 squares in each district.

This page intentionally left blank.

3.2 Fairness

The big question of the day is

> Justice Kennedy suggested, as he did in another redistricting case 13 years ago, that courts perhaps could be involved in placing limits on extremely partisan electoral maps. 13 years ago he wrote "If courts refuse to entertain any claims of partisan gerrymandering, the temptation to use partisan favoritism in districting in an unconstitutional manner will grow,"

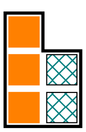
3.2.1 Geometric Measures

Although most states have a requirement that their districts be compact, there is no good definition for how compact a set of districts would be. We will look at two different ways of measuring compactness. To make measurement easier, we will be using our example districts from before.

Area vs. Perimeter

One way of measuring how compact a district is comes from dividing the area of the district by its perimeter squared and then multiplying by 16[2]. In general, this type of calculation is known as an Isoperimetric measure.

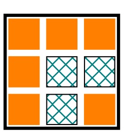
1. Example:

   Area = 5
   Perimeter = 10

   Ratio = $\frac{16\cdot 5}{10^2} = 0.8$

2. Calculate the Isoperimetric measure for a square.

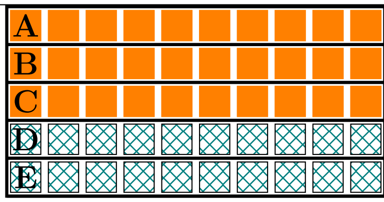
   

   Area =
   Perimeter =

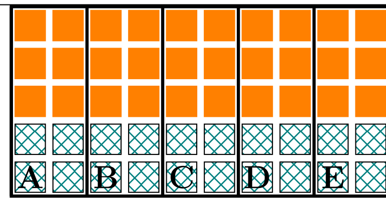
   Ratio =

3. **Notes.**

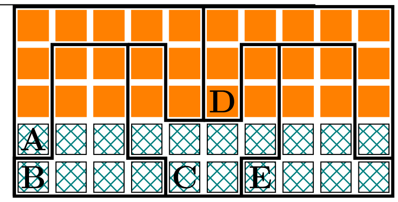
[2] You multiply by 16 because the Isoperimetric measure for a square is $\frac{1}{16}$ and we would like our ratio to be about 1.



Redistricting Plan 1



Redistricting Plan 2



Redistricting Plan 3

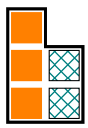
4. Calculate the area and perimeter of each district for each plan. Fill in the table below.

Plan 1	$16\frac{\text{Area}}{\text{Perimeter}^2}$	Plan 2	$16\frac{\text{Area}}{\text{Perimeter}^2}$	Plan 3	$16\frac{\text{Area}}{\text{Perimeter}^2}$
District A		District A		District A	
District B		District B		District B	
District C		District C		District C	
District D		District D		District D	
District E		District E		District E	
Average		Average		Average	

5. Does the calculation differentiate between the plans? Does it show one plan as being "better" than the others?

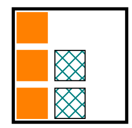
Enclosing Square

Another way of measuring how compact a district is comes from thinking of placing the district into the smallest square that contains the district. Then you calculate the area of the district and its' enclosing square and divide the first by the second. Generally, this type of calculation is known as a Reock measure. For example:



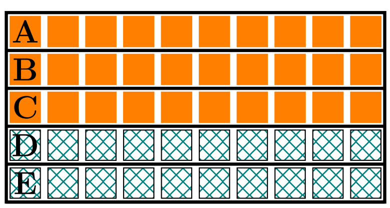
Area = 5

Ratio $= \frac{5}{9} = 0.56$

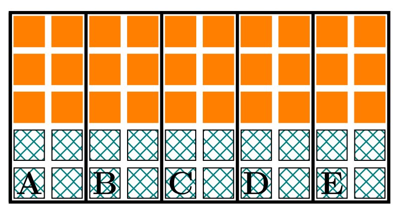


Area = 9

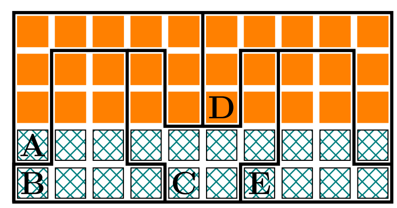
6. What would be the Enclosing Square Ratio for a district that was a square?



Redistricting Plan 1



Redistricting Plan 2



Redistricting Plan 3

7. Calculate the area and perimeter of each district for each plan. Fill in the table below.

Plan 1	$\frac{\text{Area District}}{\text{Area square}}$	Plan 2	$\frac{\text{Area District}}{\text{Area square}}$	Plan 3	$\frac{\text{Area District}}{\text{Area square}}$
District A		District A		District A	
District B		District B		District B	
District C		District C		District C	
District D		District D		District D	
District E		District E		District E	
Average		Average		Average	

8. Does the calculation differentiate between the plans? Does it show one plan as being "better" than the others?

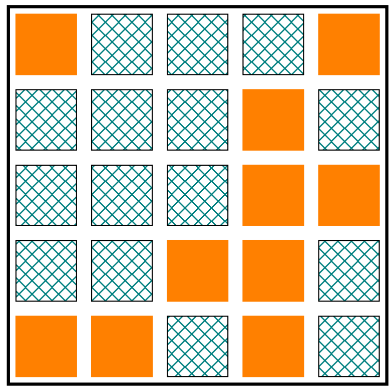
9. If you were asked to chose which method better differentiates your notion of compact, which would you use and why?

10. If you were to argue for a method of measurement that would be able to identify whether or not a district was gerrymandered and it wasn't either of the ones we have looked at, what would it be? What would you be trying to identify?

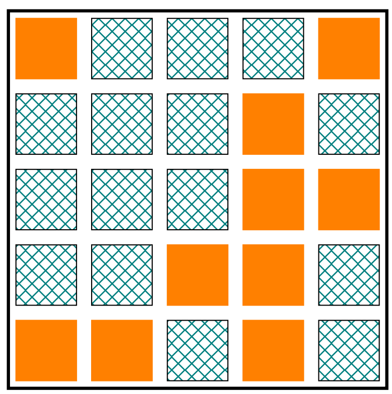
Name: ____________________________

Homework 12

R3. Go back to the 5 districts you drew on the map (or draw new ones) so that teal receives the most seats. What is your average calculation for $\frac{16\text{area}}{\text{perimeter}^2}$?



R4. Go back to the 5 districts you drew on the map (or draw new ones) so that orange receives the most seats. What is your average calculation for $\frac{16\text{area}}{\text{perimeter}^2}$?

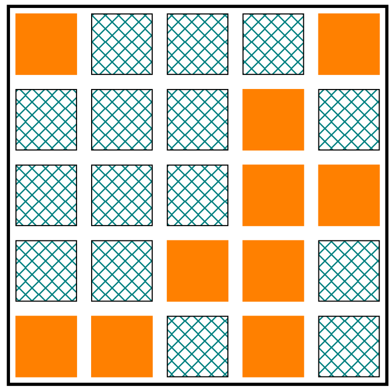


R5. Does the average of $\frac{16\text{area}}{\text{perimeter}^2}$ demonstrate unfairness in either of your plans? That is, can you tell from the average isoperimetric measure that you have Gerrymandered one of your districts?

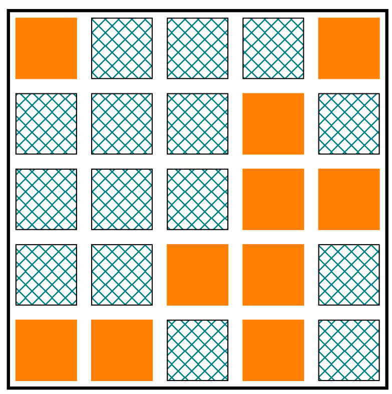
(over)

Name: ______________________

R6. Go back to the 5 districts you drew on the map (or draw new ones) so that teal receives the most seats. What is your average calculation for $\frac{\text{area of district}}{\text{area of enclosing square}}$?



R7. Go back to the 5 districts you drew on the map (or draw new ones) so that orange receives the most seats. What is your average calculation for $\frac{\text{area of district}}{\text{area of enclosing square}}$?



R8. Does the average of $\frac{\text{area of enclosing square}}{\text{area of district}}$ demonstrate unfairness in either of your plans? That is, can you tell from the average enclosing square measure that you have Gerrymandered one of your districts?

3.2.2 Arithmetic Measures

Okay, so maybe just looking geometrically at how the districts are drawn is not a good way of testing how fair a redistricting is. Maybe there is something about the election and where the people who live in the district are placed that would help us measure the fairness of the redistricting plan.

Efficiency Gap

A state has been divided into 5 districts of 100 voters each. Each district votes to elect a representative from one of two parties, A or B. In any election, many voters will end up feeling that voting was a waste of their time. A voter could feel this way for two reasons:

- Her party lost, so what was the point of showing up to vote?
- Her party won by a landslide, so what was the point of showing up to vote?

Corresponding to this, we will count:

- All of the votes for party A as wasted if party A loooses.
- All of the votes above 50% for party A as wasted if party A wins.

11. Complete the following table (we will use W_A as a short cut for the votes wasted by party A.):

District	A votes	B votes	Winner	W_A	W_B	W_A+W_B	W_A-W_B
1	95	5	A	45	5	50	40
2	40	60					
3	75	25					
4	45	55					
5	45	55					
TOTAL	300	200	A: B:				

The **Efficiency Gap** is defined to be the fraction of the difference of wasted votes divided by the total number of votes:

$$\text{EG} = \frac{W_A - W_B}{\text{Votes}} =$$

The V in the denominator normalized EG, so that its magnitude does not depend on the population of the state. The idea is that when EG is much larger than 0, the districting plan may be unfair to party A, because A is wasting more votes. When EG is much smaller than 0, then B is wasting more votes, so the plan may be unfair to B. We do have to be careful because if a candidate is running unopposed, the efficiency gap for that election is meaningless. We can either throw out that district in our calculations or use data from other elections to estimate what the election would look like if the seat were contested.

12. Consider the following election

District	A votes	B votes	Winner	W_A	W_B	$W_A + W_B$	$W_A - W_B$
1	81	19					
2	45	55					
3	80	20					
4	49	51					
5	45	55					
TOTAL	300	200	A: B:				

Fill out the table.

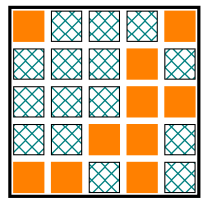
Which districts show signs of **packing**, that is putting all the voters of one party together so that, although they win the district, they have a lot of wasted votes.

Which districts show signs of **cracking**, that is putting just enough voters of the opposing party in the district so that they are confident of winning the election and therefore the first party has a lot of wasted votes?

Name: ______________________________

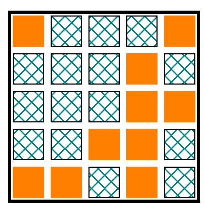
Homework 13

R9. Go back to the 5 districts you drew on the map (or draw new ones) so that teal receives the most seats. Assuming that each square represents 100 voters, what is the efficiency gap for your plan [3]?



District	O votes	T votes	W_O	W_T	$W_O - W_T$
1					
2					
3					
4					
5					
TOTAL					

R10. Go back to the 5 districts you drew on the map (or draw new ones) so that orange receives the most seats. Assuming that each square represents 100 voters, what is the efficiency gap for your plan?



District	O votes	T votes	W_O	W_T	$W_O - W_T$
1					
2					
3					
4					
5					
TOTAL					

R11. Does the efficiency gap calculation demonstrate unfairness in either of your plans? That is, can you tell from the efficiency gap that you have Gerrymandered one of your districts?

[3]It is okay for your calculations here to be negative. That just means that teal has more wasted votes than orange.

This page intentionally left blank.

Chapter 4

Apportionment and Division

4.1 Apportionment

Another mathematical challenge in most Democratic systems is that of Apportionment. Essentially the problem is one of representation. Seats in the House of Representatives are given to the states based on population. There are 435 seats for 2323.1 million people. That works out to $323100000/435 = 742758.6207$ or about $742,759$ people per seat in the House. However, even though North Dakota, Vermont, and Wyoming all have smaller populations than that they are guaranteed a seat in the House. Also, if you look at the other states, it is very rare that their population is exactly an even multiple of $742,759$. This section is all about how to fairly appoint seats in a house of representatives, as well as a variety of other tasks that turn out to be essentially the same thing.

1. The Republic of Awoi is a small country consisting of four states North-West (population 69,000), South-West (population 267,000), South-East (population 133,000) and North-East (population 331,000). Suppose that there are 160 seats in the Awoi Congress to be apportioned among the four states based on their representative populations.

   (a) On average, how many people per congress seat should there be?

   (b) How would you give out the 160 seats to the four states and why?

	NW	SW	SE	NE
Seats				

2. Notes on Apportionment

Standard Divisor (SD):

Standard quotas ($q_1, q_2, \dots q_N$):

Upper and lower quotas:

4.1.1 Hamilton's Method

3. **Hamilton's Method:**

4. Consider the Republic of Awoi:

   (a) Find the standard divisor.

	Population	Standard quota	LQ	UQ	App
NW	69,000				
SW	267,000				
SE	133,000				
NE	331,000				

   In the table above:

   (b) Find each state's standard quota.

   (c) Find each state's lower and upper quota.

   (d) Find the apportionment as described by Hamilton's Method.

5. The Republic of Bananarama is a small country consisting of five states (A, B, C, D, and E). The total population of Bananarama is 23.8 million. According to the Bananarama constitution, the seats in the legislature are apportioned to the states according to their populations. The following table shows each state's standard quota:

State	A	B	C	D	E
Standard quota	40.50	29.70	23.65	14.60	10.55

In the Table below:

(a) Find the number of seats in the Bananarama legislature.

(b) Find the Standard Divisor.

(c) Find the population of each state.

State	Pop.	Q.	LQ	Additional Seats	Apportionment
A					
B					
C					
D					
E					
Totals					

(d) Find each state's standard lower quota.

(e) Find the apportionment as described by Hamilton's method.

Worksheet for calculating Hamilton's Method:

1. What is the total population? ____________________
2. How many seats are you apportioning[1]? ____________________
3. Calculate the Standard Divisor (divide your population by the number of seats): ____
4. Calculate all the Standard Quotas (Q.) (divide the population of each state by your Standard Divisor). The total of your standard quotas should be the same as the number of seats. If that is not the case, you have rounded off too much.:

State	Pop.	Q.	LQ	Additional Seats	Apportionment
Totals					

5. Write down all the Lower Quotas (LQ) by cutting off the decimal places. The total of your lower quotas should be less than the number of seats. The difference between the two is the additional seats you get to give out.
6. Give out your additional seats by giving one seat to each state ranked by the stuff to the right of the decimal point on the standard quotas.

[1]If you aren't given this number, it is the total of the Standard Quotas

This page intentionally left blank.

Name: ______________________________

Homework 14

Show all your work.

D1. Waterloo General Hospital has a nursing staff of 175 nurses working in four shifts: A (7:00 AM to 1:00 PM), B (1:00 PM to 7:00 PM), C (7:00 PM to 1:00 AM), D (1:00 AM to 7:00 AM). The number of nurses apportioned to each shift is based on the average number of patients treated in that shift, given in the following table:

Shift	Patients	Standard Quota	Lower Quota	Additional	Apportionment
A	871				
B	1029				
C	610				
D	190				
Total		175			

(a) Find the standard divisor.

(b) Explain what the standard divisor represents in this problem.

(c) Find the standard quotas.

(d) Using Hamilton's Method, assign extra seats to the lower quotas, one at a time, until you have apportioned all your nurses.

(e) Find the apportionment based on Hamilton's method.

(over)

Name: ______________________________

D2. Southern Iowa University is made up of five different schools: Agriculture, Business, Education, Humanities, and STEM (A, B, E, H, and S for short). The 250 faculty positions at SIU are apportioned to the various schools based on the schools' representative enrollments. The following table shows each school's enrollments:

School	Enrollment	S Q	L Q	Additional	Apportionment
A	3292				
B	1524				
E	4162				
H	2132				
S	13890				
Total	25,000	250			

(a) Find the standard divisor.

(b) Explain what the standard divisor represents in this problem.

(c) Find the standard quotas.

(d) Find the apportionment based on Hamilton's method.

4.1.2 Paradoxes

1. The small country of Amabala consists of three states: Eno, Owt, and Eerht. With a total population of 20,000 and 200 seats in the House of Representatives the apportionment of the 200 seats under Hamilton's method is shown below:

State	Population	Quota	Lower quota	Additional	Apportionment
Eno	940	9.4	9	1	10
Owt	9030	90.3	90	0	90
Eerht	10,030	100.3	100	0	100
Total	20,000	200.0	199	1	200

What was the Standard Divisor that was used to apportion Amabala's seats? ______

Now, imagine that the number of seats is suddenly **increased to 201**, but **nothing else changes**. Since there is one more seat to give out, the apportionment has to be recomputed.

(a) What is the new Standard Divisor? It has to change because the number of seats has changed. ______

State	Population	SQ	LQ	Additional	Apportionment
Eno	940				
Owt	9030				
Eerht	10,030				
Total	20,000	201			

(b) Which state received an extra seat? ______

(c) Which state lost a seat? ______

(d) What do you find odd about this situation? ______

2. **Alabama Paradox:**

3. Consider the following apportionment made using Hamilton's method. Populations are given in millions:

State	Population	Quota	Lower quota	Additional	Apportionment
Alpha	150	$8.\overline{3}$	8	0	8
Beta	78	$4.\overline{3}$	4	0	4
Gamma	173	$9.6\overline{1}$	9	1	10
Delta	204	$11.3\overline{3}$	11	0	11
Epsilon	295	$16.3\overline{8}$	16	1	17
Total	900	50	48	2	50

Ten years later a new census was taken which showed only a few changes in state populations – an 8 million increase in the population of Gamma and a 1 million increase in the population of Epsilon. *The populations of the other states remained unchanged.* Use Hamilton's method to calculate the new apportionment.

(a) What is the new Standard Divisor? It has to change because your total population has changed. ____________________

State	Population	Q	LQ	Additional	Apportionment
Alpha	150				
Beta	78				
Gamma	181				
Delta	204				
Epsilon	296				
Total	**909**	50.00			50

(b) Which state received an extra seat? ____________________

(c) Which state lost a seat? ____________________

(d) What do you find odd about this situation? ____________________

4. **Population Paradox:**

5. The W-CF Garbage Company has a contract to provide garbage collection and recycling services to the two towns of Waterloo (with 89,550 homes) and Cedar Falls (with 10,450 homes). The company runs 100 little garbage trucks, which are apportioned under Hamilton's method according to the number of homes in the district. A quick calculation shows that the standard divisor is $SD = 1000$ homes, a nice, round number which makes the rest of the calculations easy.

State	Population	Q	Apportionment
Waterloo	89,550	89.55	90
Cedar Falls	10,450	10.45	10
Total	100,000	100.00	100

Now, the W-CF Garbage Company is bidding to expand its territory by adding the town of Evansdale (5250 homes) to its service area. In the bid, the company promises to buy five additional garbage trucks for the Evansdale run so that its service to the other two towns is not affected. Calculate the new apportionment.

(a) What is your new Standard Divisor? It will probably change because you have changed both your total population and the number of seats. ______

State	Population	Q	Apportionment
Waterloo	89,550		
Cedar Falls	10,450		
Evansdale	5250		
Total	105,250	105.00	

(b) Which city received an extra garbage truck? ______

(c) Which state lost a garbage truck? ______

(d) What do you find odd about this situation? ______

6. **New-States Paradox**

This page intentionally left blank.

Name: ______________________________

Homework 15

The following problems are based on the following story: Your professor found a stash of fun, math toys in her office. She decides to apportion the toys among her three first-year advisees according to the number of minutes each student spent doing homework during the week.

D3. (a) Suppose that there were 11 toys in the office. Given that Aleah did homework for a total of 54 minutes, R'moni did homework for a total of 243 minutes and Luke did homework for a total of 703 minutes, apportion the 11 toys among the students using Hamilton's method.

(b) Suppose that before the professor hands out the toys, each student decides to spend a "little" extra time on homework. Aleah puts in an extra 2 minutes (for a total of 56 minutes), R'moni put in an extra 12 minutes (for a total of 255 minutes), and Luke an extra 86 minutes (for a total of 789 minutes). Using these new totals, apportion the 11 toys among the students using Hamilton's method.

(c) These results illustrate one of the paradoxes of Hamilton's method. Which one? Explain?

(over)

Name: ____________________

D4. (a) Suppose there were only 10 toys in the office Given that Aleah did homework for a total of 54 minutes, R'moni did homework for a total of 243 minutes and Luke did homework for a total of 703 minutes, apportion the 10 toys among the students using Hamilton's method.

(b) Suppose that, just before she hands out the toys, the professor finds one additional math toy. Using the same total minutes as above, apportion now the 11 math toys among the students using Hamilton's method. [This is a repeat of Homework D3(a).]

(c) These results illustrate one of the paradoxes of Hamilton's method. Which one? Explain?

Name: ____________________________

D5. (a) Suppose that there were 11 toys in the office. Given that Aleah did homework for a total of 54 minutes, R'moni did homework for a total of 243 minutes and Luke did homework for a total of 703 minutes, apportion the 11 toys among the students using Hamilton's method. [This is a repeat of Homework D3(a).]

(b) Suppose that before the 11 toys are given out, a frantic student, Jackson, shows up at the office with a new "Declaration of Major" form. Jackson wants to be included in the game. Jackson did homework for 580 minutes during the previous week. To be fair, the professor goes to the next office and finds 6 more math toys to be added to the original 11. Apportion now the 17 toys among the four students using Hamilton's method.

(c) These results illustrate one of the paradoxes of Hamilton's method. Which one? Explain?

(over)

Name: ______________________________

D6. A professor wants to apportion 15 math toys among her three second-year advisees, Kathryn, Lacey, and Jill based on the number of minutes each student spent studying. The only information we have is that the professor will use Hamilton's method and that Kathryn's standard quota is 6.53

(a) Explain why it is impossible for all three students to end up with five toys each.

(b) Explain why it is impossible for Kathryn to end up with nine toys.

(c) Explain why it is impossible for Lacey to end up with nine toys.

4.1.3 Jefferson's Method

1. **Modified Divisor:**

2. **Modified Lower Quota:**

3. Consider the Republic of Awoi with 160 seats:

   (a) Find the standard divisor (SD).

	Pop.	LQ					Apportionment
MD							
NW	69,000						
SW	267,000						
SE	133,000						
NE	331,000						
Total		160					

   (b) Find each state's standard lower quota.

   (c) Find a modified divisor that will raise the LQ by enough so that you have exactly enough seats. If you want to raise the LQ you should make your divisor a bit smaller.

   (d) Find the apportionment as described by Jefferson's Method.

4. The Republic of Bananarama is a small country consisting of five states (A, B, C, D, and E). The total population of Bananarama is 23.8 million. According to the Bananarama constitution, the seats in the legislature are apportioned to the states according to their populations. The following table shows each state's standard quota:

State	A	B	C	D	E
Standard quota	40.50	29.70	23.65	14.60	10.55

Since we have seen this Republic before, you may want to find the information you have already calculated.

(a) Find the number of seats in the Bananarama legislature.

(b) Find the Standard Divisor.

(c) Find the population of each state.

(d) Find each state's standard lower quota.

(e) Find a modified divisor that will raise the LQ by enough so that you have exactly enough seats. If you want to raise the LQ you should make your divisor a bit smaller.

(f) Find the apportionment as described by Jefferson's method.

State	Pop.	LQ					Apportionment
Divisor							
A							
B							
C							
D							
E							
Totals							

Name: ______________________________

Homework 16

D7. Waterloo General Hospital has a nursing staff of 175 nurses working in four shifts: A (7:00 AM to 1:00 PM), B (1:00 PM to 7:00 PM), C (7:00 PM to 1:00 AM), D (1:00 AM to 7:00 AM). The number of nurses apportioned to each shift is based on the average number of patients treated in that shift, given in the following table:

Shift	Patients	LQ					Apportionment
Divisor							
A	871						
B	1029						
C	610						
D	190						
Total		175					

(a) Find the standard divisor.

(b) Find each shift's standard lower quota.

(c) Find a modified divisor that will raise the LQ by enough so that you have exactly enough seats. If you want to raise the LQ you should make your divisor a bit smaller.

(d) Find the apportionment as described by Jefferson's Method.

(over)

Name: ______________________________

D8. Southern Iowa University is made up of five different schools: Agriculture, Business, Education, Humanities, and STEM (A, B, E, H, and S for short). The 250 faculty positions at SIU are apportioned to the various schools based on the schools' representative enrollments. The following table shows each school's enrollments:

School	Enrollment	LQ					Apportionment
Divisor							
A	3292						
B	1524						
E	4162						
H	2132						
S	13890						
Total							

(a) Find the standard divisor.

(b) Find each school's standard lower quota.

(c) Find a modified divisor that will raise the LQ by enough so that you have exactly enough seats. If you want to raise the LQ you should make your divisor a bit smaller.

(d) Find the apportionment as described by Jefferson's Method.

(e) Which state violates the Upper Quota.

4.1.4 Adam's Method

1. **Adam's Method:**

2. **Modified Upper Quota:**

3. Consider the Republic of Awoi with 160 seats:

   (a) Find the standard divisor (SD).

	Pop.					Apportionment	
MD							
NW	69,000						
SW	267,000						
SE	133,000						
NE	331,000						
Total		160					

   (b) Find each state's standard upper quota.

   (c) Find a modified divisor that will lower the UQ by enough so that you have exactly enough seats. If you want to lower the UQ you should make your divisor a bit larger.

   (d) Find the apportionment as described by Adam's Method.

   (e) Compare this apportionment with that of Jefferson's Method. What stayed the same? What changed?

4. The Republic of Bananarama is a small country consisting of five states (A, B, C, D, and E). The total population of Bananarama is 23.8 million. According to the Bananarama constitution, the seats in the legislature are apportioned to the states according to their populations. The following table shows each state's standard quota:

State	A	B	C	D	E
Standard quota	40.50	29.70	23.65	14.60	10.55

   (a) Find the number of seats in the Bananarama legislature.

   (b) Find the Standard Divisor.

   (c) Find the population of each state.

State	Pop.	UQ					Apportionment
MD							
A							
B							
C							
D							
E							
Totals							

   (d) Find each state's standard upper quota.

   (e) Find a modified divisor that will lower the UQ by enough so that you have exactly enough seats. If you want to lower the UQ you should make your divisor a bit larger.

   (f) Find the apportionment as described by Adam's method.

   (g) Compare this apportionment with that of Jefferson's Method. What stayed the same? What changed?

4.1.5 Webster's Method

1. **Webster's Method:**

2. Consider the Republic of Awoi with 160 seats:

   (a) Find the standard divisor (SD).

	Pop.	Q					Apportionment
MD							
NW	69,000						
SW	267,000						
SE	133,000						
NE	331,000						
Total							

   (b) Find each state's standard rounded quota.

   (c) Find a modified divisor that will change the Quota by enough so that you have exactly enough seats. If you want to raise Q, you should make your divisor a bit smaller. If you want to lower Q, you should make your divisor a bit larger.

   (d) Find the apportionment as described by Webster's Method.

   (e) Compare this apportionment with that of Jefferson's and Adam's Methods from the earlier sections. What stayed the same? What changed?

3. The Republic of Bananarama is a small country consisting of five states (A, B, C, D, and E). The total population of Bananarama is 23.8 million. According to the Bananarama constitution, the seats in the legislature are apportioned to the states according to their populations. The following table shows each state's standard quota:

State	A	B	C	D	E
Standard quota	40.50	29.70	23.65	14.60	10.55

   (a) Find the number of seats in the Bananarama legislature.

   (b) Find the Standard Divisor.

   (c) Find the population of each state.

State	Pop.	Q					Apportionment
MD							
A							
B							
C							
D							
E							
Totals							

   (d) Find each state's standard rounded quota.

   (e) Find a modified divisor that will change the Quota by enough so that you have exactly enough seats. If you want to raise Q, you should make your divisor a bit smaller. If you want to lower Q, you should make your divisor a bit larger.

   (f) Find the apportionment as described by Webster's Method.

   (g) Compare this apportionment with that of Jefferson's and Adam's Methods from the earlier sections. What stayed the same? What changed?

Name: ______________________________

Homework 17

D9. Waterloo General Hospital has a nursing staff of 175 nurses working in four shifts: A (7:00 AM to 1:00 PM), B (1:00 PM to 7:00 PM), C (7:00 PM to 1:00 AM), D (1:00 AM to 7:00 AM). The number of nurses apportioned to each shift is based on the average number of patients treated in that shift, given in the following table:

Shift	Patients	SQ					Adam's Method Apportionment
MD							
A	871						
B	1029						
C	610						
D	190						
Total		175					

Shift	Patients	SQ					Webster's Method Apportionment
MD							
A	871						
B	1029						
C	610						
D	190						
Total		175					

(a) Find the standard divisor.

(b) Find each shift's standard upper quota.

(c) Find each shift's standard rounded quota.

(d) Find a modified divisor that will lower the Upper Quota by enough so that you have exactly enough seats. If your UQ is too big, you should make your divisor a bit larger.

(e) Find the apportionment as described by Adam's Method.

(f) Find a modified divisor that will change the Rounded Quota by enough so that you have exactly enough seats. If your SQ is too small, you should make your divisor a bit smaller. If your SQ is too large, you should make your divisor a bit larger. If your SQ works, leave it be.

(g) Find the apportionment as described by Webster's Method.

(over)

Name: ______________________________

D10. Southern Iowa University is made up of five different schools: Agriculture, Business, Education, Humanities, and STEM (A, B, E, H, and S for short). The 250 faculty positions at SIU are apportioned to the various schools based on the schools' representative enrollments. The following table shows each school's enrollments:

School	Enrollment	SUQ					Apportionment
MD							
A	3292						
B	1524						
E	4162						
H	2132						
S	13890						
Total							

(a) Find the standard divisor.

(b) Find each school's standard upper quota.

(c) Find a modified divisor that will lower the UQ by enough so that you have exactly enough seats.

(d) Find the apportionment as described by Adam's Method.

(e) Which state violates the Lower Quota?

Name: ______________________

D11. Western Iowa University is made up of five different schools: Agriculture, Business, Education, Humanities, and STEM (A, B, E, H, and S for short). The 200 faculty positions at WIU are apportioned to the various schools based on the schools' representative enrollments. The following table shows each school's enrollments:

School	Enrollment	SQ					Apportionment
MD							
A	2100						
B	952						
E	2601						
H	1458						
S	18681						
Total							

(a) Find the standard divisor.

(b) Find each school's standard rounded quota.

(c) Find a modified divisor that will change the Quota by enough so that you have exactly enough seats. If your SQ is too small, you should make your divisor a bit smaller. If your SQ is too large, you should make your divisor a bit larger. If your SQ works, leave it be.

(d) Find the apportionment as described by Webster's Method.

This page intentionally left blank.

4.2 Fair Division

1. My friend and I decide have pizza for dinner. We order pizza which is half red sauce with pepperoni and half garlic sauce with ham and pineapple. To my friend, pizza is pizza. He has no preference for one type of pizza over another. On the other hand, I have an allergy to oregano and cannot eat any red sauce. It gives me a horrible stomach ache and I'm grumpy for about eight hours.

   (a) What is a fair way to divide this pizza so that my friend and I both believe we are getting a value of at least 50% of the pizza.

   (b) My friend never remembers that I do not eat red sauce. Therefore, he divides the pizza so that each half has exactly half red sauce and half garlic sauce. Am I able to choose a slice where I receive 50% of the value of the pizza?

   (c) Suppose instead, I slice the pizza. When I do so, because I value the red sauce part as nothing, I divide the pizza so that one slice is all garlic sauce and the other slice is all red sauce. Have I divided the pizza in such a way that each slice is worth 50% to me?

   (d) Is there a way that I could divide the pizza in order to have each slice worth 50% to me and have all the red sauce on one of the slices?

   (e) In the case above, which of the slices with my friend choose?

The basic issue at all fair division problems can be stated in a reasonably simple terms: How can something that must be shared by a set of competing parties be divided among them in a way that ensures that each party receives a fair share? Of course, part of the answer to this question will involve defining what we mean by a fair share.

We are going to start with the most common method for fair division, that of the divider-chooser. This is the problem where there are two people and one thing to divide. One person has chosen to do the cutting, and then the other person gets to choose which part they want.

Rationality

Cooperation

Privacy

Symmetry

2. **Fair Share:**

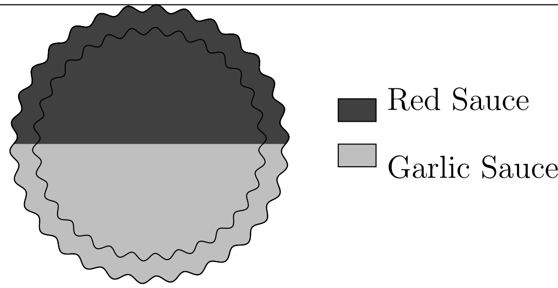


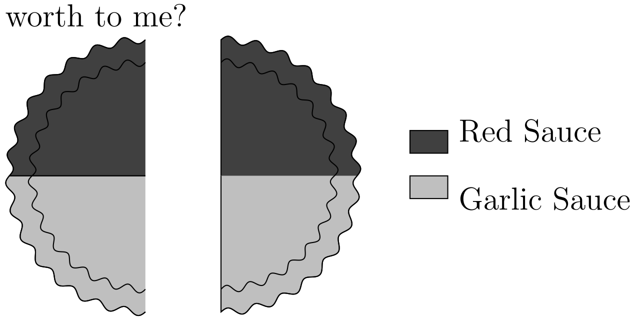
Figure 4.1: Pizza which is half Red Sauce and half Garlic Sauce

4.2.1 Divider-Chooser

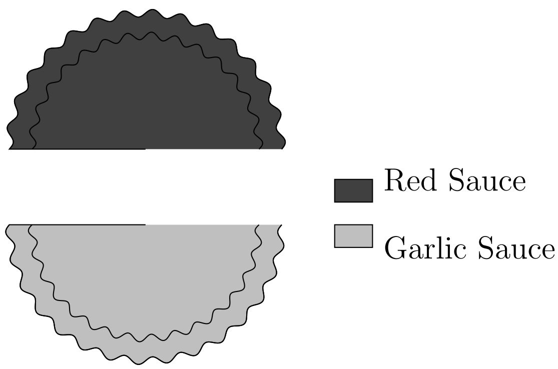
3. **Divider-Chooser:**

4. Sometimes it is easier to work with fair division problems when we put a monetary value on the object being divided because then we can quantify how much each piece is worth to each player. Suppose the cost of the pizza which is half red sauce and half white sauce is \$10.

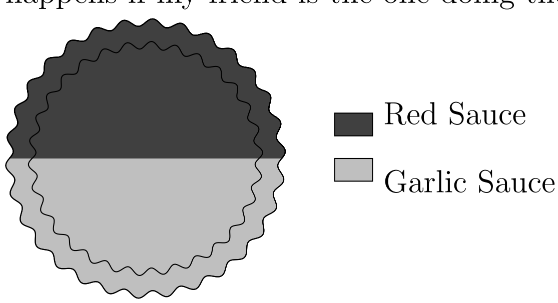
    (a) My friend never remembers that I do not eat red sauce. Therefore, he divides the pizza so that each half has exactly half red sauce and half garlic sauce. How much is each slice of pizza worth to my friend? How much is each slice of pizza worth to me?



(b) Suppose instead, I slice the pizza. When I do so, because I value the red sauce part as nothing, I divide the pizza so that one slice is all garlic sauce and the other slice is all red sauce. How much is each slice of pizza worth to me? How much of each slice of pizza worth to my friend? If I do the division, why is this not a fair division of the pizza? What happens if my friend is the one doing the division? Is it fair?



(c) Now, divide the pizza in such a way that all the red sauce is in one slice yet both slices are worth five dollars to me. How much is each slice of the pizza worth to my friend? If I do the division, why is this a fair division of the pizza? What happens if my friend is the one doing the division? Is it fair?



4.2.2 Lone Divider

5. Three players (R'moni, Luke, and Hailey) must divide cake among themselves. Suppose the cake is divided into three slices (s_1, s_2, and s_3). The values of the entire cake and at each of the three slices in the eyes of each of the players are shown in the following table.

	Whole cake	s_1	s_2	s_3
R'moni	\$12.00	\$3.00	\$5.00	\$4.00
Luke	\$15.00	\$4.00	\$4.50	\$6.50
Hailey	\$13.50	\$4.50	\$4.50	\$4.50

(a) Indicate which of the three slices are fair shares to R'moni.

(b) Indicate which of the three slices are fair shares to Luke.

(c) Indicate which of the three slices are fair shares Hailey.

(d) How do you know which player divided the cake into three equal pieces?

(e) Describe a fair division of the cake.

6. **Lone Divider:**

7. Three partners (David, Carol, and Charlotte) are dividing the plot of land among themselves using the loan divider method. Using a map, the divider (David) divides the property into three parcels s_1, s_2, and s_3. When the choosers bid lists are open, Carol's bid list is $\{s_1, s_2, s_3\}$ and Charlotte's bid list is $\{s_1\}$.

   (a) Describe the fair division where David's fair share is s_2

   (b) Describe the fair division where David's fair share is s_3

8. What should you do if both Carol and Charlotte bid $\{s_1, s_2\}$?

9. How would you run a fair division if there were four people trying to divide an item?

10. We have one divider, Demon and three choosers, Chuck, Charlie, and Charles. Demon divides the cake into four shares, s_1, s_2, s_3 and s_4. The following table shows how each of the players values each of the four shares. Remember that this information is private and not known to the other players.

	s_1	s_2	s_3	s_4
Demon	25%	25%	25%	25%
Chuck	30%	20%	35%	15%
Charlie	20%	20%	40%	20%
Charles	25%	20%	20%	35%

(a) What are each player's bidding lists?

Demon	
Chuck	
Charlie	
Charles	

(b) What is the distribution

Demon

Chuck

Charlie

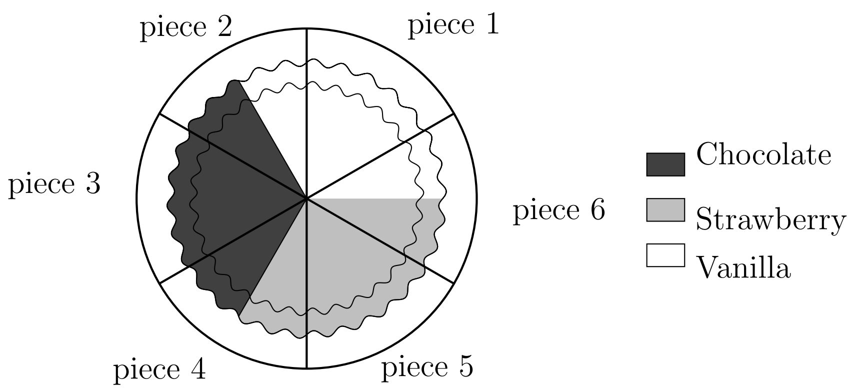
Charles

This page intentionally left blank.

Name: ______________________________

Homework 18

D12. R'moni buys a chocolate-strawberry-vanilla cake for \$14.00. She cuts the cake into six 60^o wedges as shown. R'moni likes chocolate twice as much as vanilla and likes vanilla twice as much as strawberry. Calculate the value of each wedge to R'moni.



D13. Three partners (David, Carol, and Charlotte) are dividing the plot of land among themselves using the loan divider method. Using a map, the divider (David) divides the property into three parcels s_1, s_2, and s_3. When the choosers bid lists are open, Carol's bid list $\{s_2, s_3\}$ and Charlotte's bid list is $\{s_1, s_3\}$.

(a) Describe the fair division where David's fair share is s_1.

(b) Describe the fair division where David's fair share is s_2.

(c) Describe the fair division where David's fair share is s_3.

(over)

Name: ______________________________

D14. Four partners (Luke, Aleah, Zane, and Alisyn) are dividing a piece of land valued at \$480,000 among themselves using the lone-divider method. Using a map, the divider divides the property into four parcels, s_1, s_2, s_3 and s_4. The following table shows the value of each of the four parcels in the eyes of each partner, but some of the information in the table is missing.

	s_1	s_2	s_3	s_4
Luke	\$80,000	\$85,000		\$195,000
Aleah		\$100,000	\$135,000	\$120,000
Zane	\$120,000		\$120,000	
Alisyn	\$95,000	\$100,000		\$110,000

(a) Who was the divider? Explain.

(b) Describe the choosers' respective bid lists.

(c) Described a fair division of property.

(d) Explain why your answer above is the only possible fair division of the property.

4.2.3 Lone-Divider Case 2

There is one complication that can happen with the lone-divider method we have not yet discussed. It happens when two players want one and only one of the shares. This can be solved, but requires a little more analysis.

1. Dale divides the cake into three pieces s_1, s_2, and s_3. The table below shows the value of the three pieces in the eyes of each of the players.

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	20%	30 %	50%
Cher	10%	20%	70%

   (a) Describe the choosers' respective bid lists.

   (b) What problem do you run into when you try to divide the cake?

   What we are going to do is to give Dale one of the pieces that Cindy and Cher do not want, say slice s_2.

   (c) How much are s_1 and s_3 worth together to Cindy? to Cher?

   (d) What you are going to to is combine s_1 and s_3 and give them to Cindy and Cher to perform a Divider-Chooser. Explain why this will result in both Cindy and Cher receiving at least $33\frac{1}{3}\%$ of the cake in their eyes.

   (e) In fact, what is the minimum percentage Cindy will receive? What is the minimum percentage Cher will receive?

2. Three partners (David, Carol, and Charlotte) are dividing the plot of land among themselves using the loan divider method. Using a map, the divider (David) divides the property into three parcels s_1, s_2, and s_3. When the choosers bid lists are open, Carol's bid list $\{s_1\}$ and Charlotte's bid list is $\{s_1\}$.

   (a) Considering that Carol only bid on the first parcel, explain how we know that parcel 2 is worth less than 1/3 of the value of the land to Carol. Explain how we know that parcel 3 is worth less than 1/3 of the value of the land to Carol.

   (b) Using the above information, explain how the combination of the first two parcels is worth more than 2/3 of the value of the land to Carol.

   (c) Using similar logic, explain how the combination of the first two parcels is worth more than 2/3 of the value of the land to Charlotte.

Name: ______________________________

Homework 19

D15. Six players (D,C_1, C_2, C_3, C_4 and C_5) are dividing a cake among themselves using the lone-divider method. D cuts the cake into six slices s_1, s_2, s_3, s_4, s_5, and s_6. When the choosers' bid lists are opened, C_1's bid list is $\{s_1\}$, C_2's bid list is $\{s_2, s_3\}$, C_3's bid list is $\{s_4, s_5\}$, C_4's bid list is $\{s_4, s_5\}$, and C_5's bid list is $\{s_1\}$. Describe how to proceed to obtain a fair division of the cake.

D16. We have one divider, Demon and three choosers, Chuck, Charlie, and Charles. Demon divides the cake into four shares, s_1, s_2, s_3 and s_4. The following table shows how each of the players values each of the four shares. Remember that this information is private and not known to the other players.

	s_1	s_2	s_3	s_4
Demon	25%	25%	25%	25%
Chuck	15%	20%	50%	15%
Charlie	20%	20%	40%	20%
Charles	25%	20%	20%	35%

Demon	
Chuck	
Charlie	
Charles	

(a) What are each player's bidding lists?

(a) How are you going to make a fair division?

This page intentionally left blank.

4.2.4 Method of Sealed Bids

Sometimes you want to split up items without physically cutting them apart. For example, suppose R'moni is living with two roommates and everyone is graduating. Over the years, they have purchased some appliances together and now they need to fairly divide the appliances.

In the method of sealed bids, each roommate is going to write down what they believe is a fair value for each appliance and seal the "bids" in an envelope. When the envelopes are opened, we get the following results:

	R'moni	Alison	Logan
T.V.	$75	$100	$120
Microwave	$35	$30	$40
Crock Pot	$5	$10	$7
Toaster Oven	$7	$10	$5
Blender	$20	$25	$20
Total	$	$	$
Fair Share	$	$	$

1. How much does each person think the whole property to be split is worth? What would be their fair share?

2. How would you propose splitting up the property?

3. Method of Sealed Bids

(a) Assign each item to the highest bidder. Who gets what?

Roommate	Items
R'moni	
Alison	
Logan	

(b) What is the total value of the items to the winners? Use the prices they have assigned themselves.

Roommate	Value	Fair Share	Cash Paid	Cash Received
R'moni	\$			
Alison	\$			
Logan	\$			

(c) Which of the roommates have received more than their fair share of the property? How much more did they receive?

(d) Which of the roommates have received less than their fair share of the property? How much less did they receive?

(e) Have the roommates who received too much pay cash for the extra value of the property over and above their fair share. Pay the roommates who did not receive enough the shortfall between what they see as a fair share and the value of the property they have received.

(f) How much cash do you have left?

(g) Divide the extra cash by 3 and give each roommate one third of the extra cash.

(h) Explain how each roommate has received a fair share.

(i) What are some of the drawbacks of the method of sealed bids?

4. Five heirs (A, B, C, D, and E) are dividing an estate consisting of six items using the method of sealed bids. The heirs bids on each of the times are given in the following table.

	A	B	C	D	E
Item 1	\$352	\$295	\$395	\$368	\$324
Item 2	\$98	\$102	\$98	\$95	\$105
Item 3	\$460	\$449	\$510	\$501	\$476
Item 4	\$852	\$825	\$832	\$817	\$843
Item 5	\$513	\$501	\$505	\$505	\$491
Item 6	\$725	\$738	\$750	\$744	\$761
Total					
Fair Share					

(a) Describe the first settlement of this estate and compute the surplus cash.

(b) Describe the final settlement of this estate.

This page intentionally left blank.

Name: ______________________

Homework 20

D17. Aden, Ben, and Charles are dividing four pieces of furniture using the method of sealed bids. Their bids on each of the items are given in the following table.

	Aden	Ben	Charles
Dresser	\$150	\$300	\$275
Desk	\$180	\$150	\$165
TV Cabinet	\$170	\$200	\$260
Tapestry	\$400	\$250	\$500
Total			
Fair Share			

(a) Describe the first settlement of the items and compute the Surplus.

(b) Describe the final settlement of the items.

D18. Kathy, Susie, and Judy are equal partners in a small bookstore. They can't get along anymore, but they don't want to sell the bookstore to an outsider so they decide to settle the matter using the method of sealed bids. Kathy bids \$240,000 for the business, Susie bids \$210,000, and Judy bids \$225,000. Describe the final settlement of this bookstore.

(over)

Name: ______________________________

D19. Three women (Carrie, Jeri, and Violet) share an apartment and wish to divide the chores: bathrooms, cooking, dishes, laundry, and vacuuming. For each chore, they privately write the least they are willing to receive monthly (their negative valuation) in return for doing that chore. The results are shown in the following table.

	Carrie	Jeri	Violet
Clean bathrooms	\$-20	\$-30	\$-40
Do cooking	\$-50	\$-10	\$-25
Wash dishes	\$-30	\$-20	\$-15
Mow the lawn	\$-30	\$-20	\$-10
Vacuum and dust	\$-20	\$-40	\$-15

Divide the chores using the method of sealed bids. Who does which chores? Who gets paid, and how much? Who pays, and how much?

4.3 Study Guide

To prepare for the exam over this chapter, you should review the in-class worksheets and homework. Be ready to do the kind of problems you faced on the homework.

As a general guide, I recommend reviewing the following topics.

1. Know what it means for a division of assets to be fair.
2. Calculate the fair division by method of
   (a) Divider-Chooser
   (b) Lone-Divider
   (c) Sealed Bids
3. Know what to do when two players want one and only one piece of the division.
4. Apportionment
   (a) Know how to find the Standard Divisor and the Standard Quota.
   (b) Understand the quota rule.
   (c) Be able to understand when a modification of the apportionment problem violates the Population Paradox.
   (d) Be able to understand when a modification of the apportionment problem violates the New States Paradox.
   (e) Be able to understand when a modification of the apportionment problem violates the Alabama Paradox.
5. Given an apportionment problem, be able to find the apportionment using Hamilton's Method.
6. Given an apportionment problem, be able to find the apportionment using Jefferson's Method.
7. Given an apportionment problem, be able to find the apportionment using Adam's Method.
8. Given an apportionment problem, be able to find the apportionment using Webster's Method

This page intentionally left blank.

Chapter 5

Beyond Numbers

5.1 Mathematical Stories

[1]

5.1.1 Damsel in Distress

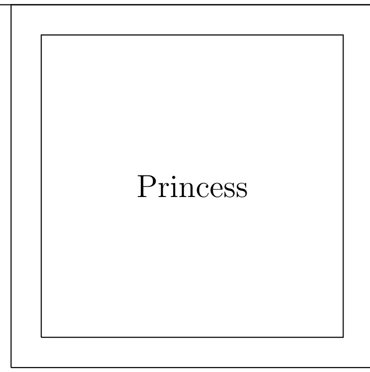
Long ago, knights in shining armor battled dragons and rescued damsels in distress on a daily basis. Although it is not often stressed, many of the surviving stories of chivalry, frequently the rescue involved logical thinking and creative problem-solving, and often the damsel provided the solution. Here then is a typical knightly encounter:

Once upon a time, a damsel was captured by a notorious knight and imprisoned in a castle surrounded by square moat. The moat was infested with extraordinarily hungry alligators 25 feet below for whom the prospect of a luncheon damsel brought enormous smiles to their green faces. The moat was 20 feet across, 30 feet deep, and no drawbridge existed because the evil knight took it with him (giving his horse a major hernia).

After a time, a good knight and his squire rode up and said, "Hail sweet damsel, for I am here; and thou art there; now what are we going to do?"

The knight, though good, was not too bright and the consequently paced back and forth along the moat looking anxiously at the alligators and trying feebly to think of a plan. Then, on the shore the knight found two sturdy beams of wood suitable for walking across but lacking sufficient length. Alas, the moat was 20 feet across, and the beams were each only 19 feet long and 8 inches wide. He tried to stretch them and tried to think. Neither effort proved successful. He had no nails, rope, screws, saws, superglue, or any other method of joining the two beams to extend their length.

[1] All stories are quoted from <u>The Heart of Mathematics</u> by Edward Burger and Michael Starbird.



What to do? What to do? Fortunately, the damsel, after a suitable time to allow the good knight to attempt to solve the puzzle herself, was able to give the knight a few hints that enabled him to rescue her and carried her home to her own castle. How did the maiden advise the knight to accomplish the rescue?

5.1.2 That's a Meanie Genie

On an archaeological dig in the Highlands of Tibet, Allie discovered an ancient oil lamp. Just for laughs she rubbed the lamp. She quickly stopped laughing when the huge puff of a magenta smoke sprouted from the lamp, and an ornery genie in Murray appeared. Murray, looking at the stunned Allie, exclaimed, "Well, what are you staring at? Okay, okay, you've found me; you get your three wishes. So what will they be?" Allie, although in shock, realized what an incredible opportunity she had. Thinking quickly, she said, "I'd like to find the Rama Nujan, the jewel that was first discovered by Hardy, the High Lama."

"You got it," replied Murray, and instantly nine identical looking stones appeared. Allie looked at the stones and was unable to differentiate anyone from the others.

Finally, she said to Murray, "So where is the Rama Nujan?" Murray explained, "It is embedded in one of the stones. You said you wished to find it. So now you have to find it. Oh, by the way, you may take only one of the stones with you, so you had best be careful how you choose!" "But they look identical to me. How will I know which one has the Rama Nujan in it?" Allie questioned. "Well, eight of the stones weigh the same, but the stone containing the jewel weighs slightly more than the others," Murray responded with a devilish grin.

Allie, now getting annoyed, whispered under her breath, "Gee, I wish I had a balance scale." Suddenly a balance scale appeared. "That was wish two!" declared Murray. "Hey, that's not fair!" Allie cried. "You want to talk fair? You think it's fair to be locked in a lamp for 1729 years? You know you can't get internet in there, and there's no room for a satellite dish! So don't talk to me about fair," Murray explained. Realizing he had gone a bit overboard, Murray proclaimed, "Hey, I want to help you out, so let me give you a tip: that balance scale may only be used once." "What? Only once?" she said, thinking out loud. "I wish I had another balance scale." ZAP! Another scale appeared. "Okay, kiddo, that was wish three." Murray snickered. "Hey, just one minute," Allie said now regretting not having asked for $1 million or something more standard. "Well at least this new scale works correctly, right?" "Sure, just like the other one. You may use it only once." "Why?" Allie inquired. "Because it is a 'wished' balance scale. That means that you can only use it once since it was only one wish. It's just like you cannot wish for 100 more wishes." "You are a very obnoxious genie." "Hey, I don't make up the rules, lady, I just follow them."

So, Allie may use each of the two balance scales exactly once. Is it possible for Allie to select the slightly heavier stone containing the Rama Nujan stone from among the nine identical looking stones? Please explain why or why not.

5.1.3 The Fountain of Knowledge

During an incredibly elaborate hazing stunt during pledge week, Trey Sheik suddenly found himself alone in the Sahara desert. His desire to become a fraternity brother was now overshadowed by his desire to find something to drink (these desires, of course, are not unrelated). As he wandered aimlessly through the desert sands, he began to regret his involvement in the whole frat scene. Both hours and miles had passed and Trey was near dehydration. Only now did Trey appreciate the advantages of sobriety. Suddenly, as though it were a mirage, Trey came upon an oasis.

There, sitting in a shaded kiosk beside a small pool of mango nectar, was an old man named Al Donte. Big Al, not only ran the mango bar but was also a travel agent and could book Trey on a two-humped camel back to Michigan. At the moment, however, Trey desired nothing but a large drink of that beautifully translucent and refreshing mangoade. Al informed Trey that the juice was sold only in 8 ounce servings and that the cost for one serving was \$3.50. Trey frantically searched on his pockets and found some change and much sand. Trey counted and discovered that he had exactly \$3.50.

Trey's jubilation at the thought of liquid coating his dried and chapped throat was quickly shattered when Al casually announced that there were no 8- ounce glasses available. Al had only a 6- ounce glass and a 10- ounce glass – neither of which would have any markings on them. Al, being a man of his word, would not hear of selling any more or any less than an 8- ounce serving of his libation. Trey, in desperation, wondered whether it was possible to use two glasses to produce exactly 8- ounces of mango juice in the 10- ounce glass. Trey thought and thought. Do you think it is possible to use only the unmarked 6- and 10- ounce glasses to produce exactly 8 ounces in the 10 ounce glass? If so, explain how, if not, explain why not.

5.1.4 Dodge Ball

Directions for the game:

Player One begins by filling in the first horizontal row of her table with X's and O's. Then Player Two places either one X or one O in the first box of his board. The game continues with Player One writing down a run of six X's and/or O's in the second horizontal row of her board, followed by Player Two writing an X or O in the second box of his board. The game continues until all boxes on both boards are filled.

Player One wins if any horizontal row she wrote down is identical to the row that Player Two created (Player One matches Player Two). Player Two wins if his row is not one of the six rows made by Player One (Player Two dodges Player One).

Player One's board:

1						
2						
3						
4						
5						
6						

Player Two's board:

1	2	3	4	5	6

Would you rather be Player One or Player Two? Who has the advantage? Why? Can you devise a strategy for either side that will always result in victory? This little game holds within it the key to understanding the sizes of infinity.

Player One's board:

1						
2						
3						
4						
5						
6						

Player Two's board:

1	2	3	4	5	6

Player One's board:

1						
2						
3						
4						
5						
6						

Player Two's board:

1	2	3	4	5	6

5.1.5 Dot of Fortune

One day three college students were selected at random from the studio audience to play the ever-popular TV game show, "Dot of Fortune." One of the students already had discovered the power and beauty of mathematical thinking, while the other two were not nearly so fortunate. The stage had no mirrors, reflecting surfaces, or television monitors. The three students were led blindfolded to their places around a small round table. As the rules of the game were explained by Pat, Vanna affixed to each of the three youthful foreheads a conspicuous but small colored dot.

"So, contestants," Pat explained, "at the bell your blindfolds will be removed. You will see your two companions sitting quietly at the table, each with a dot on his or her forehead. Each dot is either red or white. You cannot, of course, see the dot on your own forehead. After you have observed the dots on your companions' foreheads, you will raise your hand if you see at least one red dot. If you do not see a red dot, you will keep your hands on the table. The object of the game is to deduce the color of your own dot. As soon as you know the color of your dot, you are to hit the buzzer in front of you. Do you understand the rules of the game?" All the students understood the rules, although the math fan understood them better.

"Are you ready?" asked Vanna after affixing three red dots to the foreheads of the three students. After the three contestants nodded, Vanna instructed them to simultaneously remove their blindfolds as the studio audience quivered with anticipation. The three students looked at one another's dots, and all raised their hands. After some time, the math fan hit her buzzer knowing what color dot she had. Please explain how she knew this. Why did the other students not know?

5.2 What does "2" mean?

Our journey to infinity begins with 2. We are all familiar with 2, and we will use that intuitive intimacy to develop a concept of size that will take us well beyond what we know.

If we have two apples and two hands, we could put one apple in each hand, and no hand would be un-appled, or any apple un-handed. If we have two socks and we put one sock on each hand, then our socks and our hands correspond exactly. This observation also implies that we can put one apple in each sock and demonstrate that the socks and apples also correspond; however, we do not recommend this last experiment unless the socks are clean.

One-to-one Correspondence:

Grappling with infinity requires us to imagine physical scenarios that are not quite possible but can be clearly conceived in the mind. Suppose we have a huge barrel full of Volvos and a barrel full of soccer balls, and we want to know if there are more Volvos than soccer balls, more soccer balls than Volvos, or the same number of each. How would we decide?

We could take one Volvo from the first barrel and one soccer ball from the second, pair them up and put them aside (probably putting the ball in the car's trunk). Then we could pair up another Volvo from the first barrel with another soccer ball from the second. If we continue pairing in this way and every car has a ball and every ball has a car, we know there are the same number of each.

This simple idea is important. We have just described a method for determining when two collections have the same number of objects in them without actually naming that number. Two collections whose objects can be paired together evenly-one from one collection with one from the other collection-are said to have a **one-to-one correspondence**.

1. **The same, but unsure how much**. We have used a method of checking whether two sets of objects have the same number of things by pairing up and removing one object from each set until we run out. If we run out of objects from both sets at the same time, then we know that the sets contain the same number of things. Otherwise, we know that one set is larger than the other. Describe several events whereby we are able to compare the size of two collections without computing the individual sizes-for example, people filling all seats in an auditorium.

2. **Don't count on it**. The following are two collections of the symbols @ and ©.

   @ @ @ @ @ @ @ @ @ @ @ @ @ @ @ @ @ @ @
   © © © © © © © © © © © © © © © © © © ©

   Are there more @'s than ©'s? Describe how you can quickly answer the question without counting, and explain the connection with the notion of a one-to-one correspondence.

3. **Here's looking @ ®** Below are two collections of the symbols @ and ®.

   @ @ @ @ @ @ @ @ @ @ @ @ @ @ @ @ @ @ @
   ® ® ® ® ® ® ® ® ® ® ® ® ® ® ® ®

   Are there more @'s than ®'s? Describe how you can quickly answer the question without actually counting, and explain the connection with the notion of a one-to-one correspondence.

4. **Hair counts**. Do there exist two nonbald people on Earth having the property that there is a one-to-one correspondence between the collection of hairs on one person's body and the collection or hairs on the other person's body?

5. **Social Security number**. A social security number is a nine-digit number. Suppose that all nine-digit numbers are allowable social security numbers. Is there a one-to-one correspondence between allowable social security numbers and U.S. residents? You may assume that the U.S. population is about 250 million. Explain your answer.

6. **Dorm life**. Every student at a certain college is assigned to one and only one dorm room. Does this imply that there is a one-to- one correspondence between dorm rooms and students? Explain your answer.

5.3 Familiar, but Infinite

What is familiar and concrete to one person may be foreign and abstract to another, but as far as numbers are concerned we probably all agree on which are the most familiar. In 1886 Leopold Kronecker, a number theorist, made a statement about what is basic in the world of mathematics: "God created the positive integers; all the rest is human creation."

One, two, three, .. . these are the positive integers. For every positive integer there is a next bigger one. Although we may think of these positive integers successively, we may also think of all of them at once, that is, think about the collection of all positive integers. The collection (also referred to as the set) of positive integers is so basic and natural to our way of thinking that it is called the set of natural numbers.

The set of all natural numbers is our first infinite set, and it has a comfortable feel about it. Among infinite sets, the natural numbers seem the most natural. By examining this and related collections of numbers, we will begin to develop a better and more precise idea of infinity.

Question: Are there as many natural numbers as there are integers starting with 2, 3, 4, .. . , we are asking neither more nor less than the question: Is there a one-to-one correspondence between the elements of the set

$$2, 3, 4, ...$$

and the set of natural numbers

$$1, 2, 3, ...?$$

1. **Naturally even**. Let E stand for the set of all even natural numbers (so $E = (2, 4, 6, 8, ...))$. Show that the set E and the set of all natural numbers have the same cardinality by describing an explicit one-to-one correspondence between the two sets.

2. **5's take over**. Let **EIF** be the set of all natural numbers ending in 5 (**EIF** stands for "ends in five"). That is,

$$\mathbf{EIF} = (5, 15, 25, 35, 45, 55, 65, 75, ...).$$

   Describe a one-to-one correspondence between the set of natural numbers and the set **EIF**.

3. **6 times as much**. If we let $\mathbb{N}$ stand for the set of all natural numbers, then we write $6\mathbb{N}$ for the set of natural numbers all multiplied by 6 (so $6\mathbb{N} = (6, 12, 18, 24, ...))$. Show that the sets $\mathbb{N}$ and $6\mathbb{N}$ have the same cardinality by describing an explicit one-to-one correspondence between the two sets.

4. **Any times as much**. If we let $\mathbb{N}$ stand for the set of all natural numbers, and a stand for any particular natural number, then we write $a\mathbb{N}$ for the set of natural numbers all multiplied by a. Do the sets $\mathbb{N}$ and $a\mathbb{N}$ have the same cardinality? If so, describe an explicit one-to-one correspondence between the two sets.

5. **Reciprocals**. Suppose R is the set defined by

$$R = 1/1, 1/2, 1/3, 1/4, 1/5, 1/6, ...$$

   Describe the set R in words. Show that it has the same cardinality as the set of natural numbers.

6. **Half way**. Suppose you take the line below,

   cut it in half, and then take the left piece and cut it in half, and then take the leftmost piece and cut it in half, and so on, without ever stopping. How many different pieces of the line would you have? Does the set of all pieces have the same cardinality as the set of all natural numbers? Justify your answer.

5.4 Rational Numbers

We might feel more or less stymied in our quest for an infinite set bigger than the set of natural numbers. But let's not forget that lots of numbers are not integers. For example, how about the rational numbers? Recall that the set of rational numbers is the set of all ratios of integers (fractions). Between any two integers there are infinitely many rationals. Within the set of rational numbers, we actually have infinitely many different infinite sets. The set of rational numbers must be huge.

Unfortunately, in our search for different sizes of infinity, the rationals are still not numerous enough. We rely once again on the definition of same cardinality; namely, two sets have the same cardinality if there is a one-to-one correspondence between the elements of one set and the elements of the other. The trick here is to write down the rational numbers in a convenient and systematic way so that we know we have listed them all. The idea is to put all the rational numbers with numerator 1 in one column, all those with numerator 2 in another column, and so on as shown in the following diagram. Notice that all the rationals in the same row have the same denominator. For example, to find 37/112 in the diagram, we just go 37 spaces to the right of 0 and 112 spaces up. So, we can see that all the rational numbers appear somewhere in the pattern. In fact, each rational number appears many times, because the fractions are not all reduced to lowest terms. For example, 1/2 appears and so does 2/4, 3/6, and so on, but that redundancy is okay, because at this point we merely want a systematic method of writing down every rational number without leaving any out. Notice that all the positive rational numbers appear in the upper-right part of the diagram, all the negative rationals appear in the lower left, and 0 is right in the middle. So far, then, we have described a way of writing down all the rational numbers.

$$
\begin{array}{ccccccccccccc}
 & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & & & & & & & 1/5 & 2/5 & 3/5 & 4/5 & 5/5 & \cdots \\
 & & & & & & & 1/4 & 2/4 & 3/4 & 4/4 & 5/4 & \cdots \\
 & & & & & & & 1/3 & 2/3 & 3/3 & 4/3 & 5/3 & \cdots \\
 & & & & & & & 1/2 & 2/2 & 3/2 & 4/2 & 5/2 & \cdots \\
 & & & & & & & 1/1 & 2/1 & 3/1 & 4/1 & 5/1 & \cdots \\
 & & & & & & 0 & & & & & & \\
\cdots & -5/1 & -4/1 & -3/1 & -2/1 & -1/1 & & & & & & & \\
\cdots & -5/2 & -4/2 & -3/2 & -2/2 & -1/2 & & & & & & & \\
\cdots & -5/3 & -4/3 & -3/3 & -2/3 & -1/3 & & & & & & & \\
\cdots & -5/4 & -4/4 & -3/4 & -2/4 & -1/4 & & & & & & & \\
\cdots & -5/5 & -4/5 & -3/5 & -2/5 & -1/5 & & & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & &
\end{array}
$$

To show the one-to-one correspondence between the rational numbers and the natural numbers, we will thread a single rectangular spiral through all the rationals, starting in the middle at 0 and moving counterclockwise outward. To see the one-to-one correspondence with the natural numbers, we will just count the rational numbers as we encounter them along the spiral and make them bold-face to remind us that we have paired that rational with some natural number. We start with the rational O corresponding to the natural number l ; then, moving to the nght and up, the rational $1/1 = 1$ corresponds to the natural number 2, the rational $-1/1 = -1$ corresponds to 3, the rational $2/1 = 2$ corresponds to 4. We next come to 2/2, which has already been counted, so we skip it and move to 1/2, which corresponds to 5, then -2/1 = -2 corresponds to 6. We skip -2/2 since that equals - 1, which already corresponds to 3, and move to -1/2, which corresponds to 7, and so on. Notice that eve1y rational number will eventually be reached and put in correspondence with some natural number. This one-to-one correspondence shows that the set of all rational numbers has the same cardinality as the set of the natural numbers.

Threading the spiral and counting along it provides an important insight into sets with the same cardinality as the set of natural numbers. If we can write a set out as an infinite list, we can make a one-to-one correspondence with the natural numbers.

We now see that the rational numbers did not provide us with an infinity larger than that of the natural numbers. Our quest for an even grander infinity has thus far failed. But perhaps when we are chasing sets larger than an infinite set, we should expect to have to go a long, long way.

1. **A grand union**. Suppose you have two sets, each having the same cardinality as the set of natural numbers. Take the elements of both sets and put them together to make one huge set. Prove that this new huge set has the same cardinality as the set of natural numbers.

2. **Unnoticeable pruning**. Suppose you have any infinite set. Is it always possible to remove some things from that set such that the collection of remaining things has the same cardinality as the original set? Explain why or why not, and illustrate your answer with an example.

5.5 The Missing Member

Around 1872 the German mathematician Georg Cantor shook the foundations of infinity when he showed that the set of real numbers has more elements than the set of natural numbers. In other words, he proved that infinity is not one size but that some infinities are more infinite than others. At first such a notion seems almost nonsensical. Once we have reached infinity, surely we cannot climb farther. But Cantor showed that there were yet higher mountains to scale.

To show that the real numbers are more numerous than the natural numbers, Cantor focused intently on what it would mean for the real numbers and the natural numbers to have the same cardinality. It would mean that the real and natural numbers could be put in one-to-one correspondence. Writing down what such a correspondence might look like gives us a visual clue how to demonstrate conclusively that any attempted correspondence between the natural numbers and the real numbers could not include every real; some real is missing-*the missing member*.

To figure out Cantor's argument, we need to recall that each real number can be expressed as an infinitely long decimal expansion. For example,

$$243.4766668754468008876728758493457884453321\ldots$$

is a real number. Before moving forward, we must first make an easy observation about real numbers. Suppose we examine two decimal numbers, but we cover up all the digits in the numbers with ?s except for the digit that is in, say, the fifth place after the decimal point. So, we have a piece of paper with two funny looking numbers on 1't : ??.????2????... and ??.????4????.... We do not know what these numbers are because we can read only the fifth digit after the decimal point. But one thing we do know is that these two numbers are different. If they were the same, we could not have a 2 in the fifth place after the decimal point of one number and a 4 in the fifth place in the other. Likewise, if we have two numbers and one has a 2 and the other has a 4 in the 87th place after the decimal, then the two numbers must be different. This observation is not hard to understand, but it is a key to Cantor's reasoning.

Cantor proved that there are more real numbers than natural numbers through a clever, yet simple idea. If the set of real numbers and the set of natural numbers had the same cardinality, then there would be a one-to- one correspondence between the set of natural numbers and the set of real numbers. So his idea was to list the natural numbers down the left-hand side of a page, list reals in the right-hand column, and then show how to construct a real number that could not appear on the list. He showed that, once we commit ourselves to a list of reals in the right-hand column, one real number corresponding to each natural number, then we can describe a real decimal number that does not appear anywhere on that infinite list. So, we *could not* have listed all the real numbers in the right-hand column. Thus, the natural numbers and the real numbers could not be put in one-to-one correspondence, and so there are more real numbers than natural numbers. Cantor's basic strategy was to attempt an impossible task in order to understand why it couldn't be done.

We are going to write down a particular real number that we will call M, for "missing." We will w1ite it in its decimal expansion. Our number M will be between 0 and 1, so its decimal expansion begins with 0.???.... Now we must decide what the digits "???..." are. Each digit will be one of two possibilities: a 2 or a 4. We will decide on the digits of our number M one at a time, successively, so we must be patient. We now describe the criterion by which we choose each digit of our number M.

1	0.76206658
2	0.910928721
3	0.055783247
4	0.615411668
5	0.520502311
6	0.782172298
7	0.874793025
8	0.334273456

	1	2	3	4	5	6	7	8
0.								

Is M on our list of real numbers?

1. **Don't dodge the connection**. Explain the connection between the Dodge Ball game and Cantor's proof that the cardinality of the reals is greater than the cardinality of the natural numbers.

2. **Cantor with 3's and 7's**. Rework Cantor's proof from the beginning but this time, if the digit under consideration is 3, then make the corresponding digit of M a 7, and if the digit is not 3, make the associated digit of M a 3.

3. **Think positive**. Prove that the cardinality of the positive real numbers is the same as the cardinality of the negative real numbers. (Caution: You need to describe a one-to-one correspondence; however, remember that you cannot list the elements in a table .)

4. **Diagonalization**. Cantor's proof is often referred to as "Cantor's diagonalization argument." Explain why this is a reasonable name.

5. **No Vacancy**. Recall the Hotel Cardinality, described in the previous section. Create a collection of people so that it would be impossible for the (new) night manager to give each of them a room. Thus, for a really big group of people, a No Vacancy sign (or actually a Not Enough Room sign) might actually be necessary. Explain why it is not possible to give each person from your group a room.

6. **Just guess**. This is just a "guessing question." Do you think there are sets whose cardinality is actually larger than that of the set of real numbers? Or do you think the infinity of reals is the largest infinity? Just make a guess and informally explain it.

Appendix A

Answers to Selected Exercises

6((c))i. $\boxed{\text{A.}}$

6((c))ii. $\boxed{\text{B.}}$

V2(a). $\boxed{\text{B.}}$

V2(b). $\boxed{\text{C.}}$

V3(a). $\boxed{\text{C.}}$

V3(b). $\boxed{\text{Tied between A and B.}}$

V5(a). 7 votes

V6(a). $3 + 4 + 4 + 9 + 3 + 6 = \boxed{29.}$

V6(b). $29/2 = 14.5$, so it takes $\boxed{\text{15 votes.}}$

V6(c). Candidate A wins with 10 votes, but that is not a majority.

V6(d). Now candidate E wins (with 12 votes, not a majority).

V6(e). A gets 13 votes and C gets 16 votes, so C wins with a majority.

V9(a). $\boxed{\text{B.}}$

V9(b). $\boxed{\text{A.}}$

V11(a). $\boxed{65}$

V11(b). Hint: Since Ohio State was the unanimous first-place choice and we are told the other 22 teams are irrelevant, then we can assume that all the second and third place votes went either to Florida or Michigan. This also means that the voters who didn't put Florida second placed them third. This is all the information you need to solve the problem.

V12(b). The Borda count tally is

A	58
B	46
C	69
D	67

, while a plurality vote yields

A	7
B	0
C	7
D	10
total	24

Thus $\boxed{\text{no,}}$ the majority criterion is not violated since no candidate had a majority of the 24 votes. The majority criterion would only be violated if *both* (a) some candidate had a majority, and (b) the Borda count elected someone different.

V12(c). $\boxed{\text{Yes.}}$ Candidate D would beat each of A, B, and C in head-to-head contests, but the Borda count elects someone else.

V15(a). Alice et al. will vote for $\boxed{\text{Propositions 2 and 3.}}$

V15(b). Zeke et al. will vote for $\boxed{\text{Propositions 1 and 3.}}$

V15(c). Shadrach et al. will vote for $\boxed{\text{Propositions 1 and 2.}}$

V15(e). $\boxed{\text{All three}}$ propositions will pass with 2000 of the 3000 votes.

V15(f). Since this will cost taxpayers \$1,100,000, $\boxed{\text{none of the voters}}$ will be happy.

V15(g). Each of the voters deliberately voted to limit spending, yet they collectively ended up spending more than any of them wanted.

V16(a). $\boxed{1022.}$

V16(c). $\boxed{[1362|741, 637, 412, 252].}$

V18(b). No dictators, P_4 (the 6) is a dummy, no one has veto power.

V18(c). P_1 (the 27) is a dictator (and so has veto power); the rest are dummies.

V20. For convenience, we list them both by P-number and by their weights:

$\{P_1, P_2, P_3, P_4\}$	$\{7, 5, 4, 2\}$
$\{P_1, P_2, P_3\}$	$\{7, 5, 4\}$
$\{P_1, P_2, P_4\}$	$\{7, 5, 2\}$
$\{P_1, P_3, P_4\}$	$\{7, 4, 2\}$
$\{P_1, P_2\}$	$\{7, 5\}$

V22. P_2 and P_3 are critical voters (the 20 and the 17).

V24. All 4 voters have equal Banzhaf power: $\boxed{0.25, 0.25, 0.25, 0.25.}$

V26(a). 53.3%, 44.4%, 2.2%

V26(b). 0.60, 0.20, 0.20

V27. 0.5, 0.5, 0

V30. For convenience, we list them both by P-number and by their weights:

$\{P_1, P_2, P_3\}$	$\{9, 8, 7\}$
$\{P_1, P_3, P_2\}$	$\{9, 7, 8\}$
$\{P_2, P_1, P_3\}$	$\{8, 9, 7\}$
$\{P_2, P_3, P_1\}$	$\{8, 7, 9\}$
$\{P_3, P_1, P_2\}$	$\{7, 9, 7\}$
$\{P_3, P_2, P_1\}$	$\{7, 8, 9\}$

V32(a). 53.3%, 44.4%, 2.2%

V32(b). 0.60, 0.20, 0.20

V34. 0.5, 0.5, 0, 0

V35. 0.33, 0.33, 0.33, 0

F1(a). \$19,500.00

F1(c). \$9,055.48

F2(a). \$11,009.17

F2(b). 7.317%

F2(c). 3.06 years

F3. \$2,300

F4. \$9,836.07

F5. 36.36%

F6. 12.5 years

F7. \$1,157.63

F8(a). \$17,064.58

F8(c). \$1,509.89

F9. \$9,820.03

F10. 6.96%

F11. \$1200.02

F12(a). \$10,727.41

F12(c). $\boxed{\$10,148.57}$

F13(b). $\$300\dfrac{(1+\frac{.08}{4})^{4\cdot 10}-1}{\frac{.08}{4}} = \boxed{\$18,120.59.}$

F13(d). $\$100\dfrac{(1+\frac{.08}{52})^{52\cdot 2}-1}{\frac{.08}{52}} = \boxed{\$11268.83.}$

F14(a). \$48,769.22

F14(b). \$12,369.22

F15. \$286.66

F16(a). \$38,735.32

F16(b). \$13,189.79

F17. The answer is too big.

F18(a). $\$500\dfrac{1-(1+\frac{.10}{4})^{-4\cdot 8}}{\frac{.10}{4}} = \boxed{\$10924.59.}$

F18(b). \$9007.35

F19. \$361.26

F20(a). You need $97 - 65 = 32$ years of retirement income, so you need $80000\dfrac{1-(1+\frac{0.08}{1})^{-1\cdot 32}}{\frac{0.08}{1}} = \boxed{\$914,799.95.}$

F20(b). You have $65 - 22 = 43$ working years, so you need P such that\$914,799.95 $= P\dfrac{(1+\frac{0.08}{12})^{12\cdot 43}-1}{\frac{0.08}{12}}$. (This is the *future value* formula, because you are *saving* money.) Solving, you find $P = \boxed{\$204.43.}$

F21(a). \$230.16

F21(b). \$7619.20

F21(c). \$394.14

F21(d). \$3648.40

F21(e). \$3970.80

F22. \$125,088.21

F23(a). You get an annuity of \$50,000 per year for 20 years.The annuity is worth $\$50{,}000\dfrac{1-(1+\frac{.05}{1})^{-20\cdot 1}}{\frac{.05}{1}} = \boxed{\$623{,}110.52.}$

D1(a). 15.42857143

D1(b). One nurse will be assigned for each 15.4 patients.

D1(e).

A	56
B	67
C	40
D	12

D3(c). This is the Population Paradox. Be sure to explain how you know.

D6(c). Consider closely who will have the largest fractional part.

D9(b). A 57
B 67
C 40
D 13

D9(c). A 56
B 67
C 40
D 12

D12. $\boxed{\text{piece } 1 = \$2,\ \text{piece } 2 = \$3}$ If you look at the whole cake, to our student, the strawberry part will be worth \$2, the vanilla part is worth \$4 and the chocolate part is worth \$8.

D13(a). $\boxed{\text{David - } s_1,\ \text{Carol - } s_2,\ \text{Charlotte - } s_3}$ Because David has a fair share of s_1, the only other option for Charlotte is s_3. Since Carol will take s_2 or s_3, and s_3 is taken, Carol gets s_2.

D13(b). $\boxed{\text{David - } s_2,\ \text{Carol - } s_3,\ \text{Charlotte - } s_1}$ Because David has a fair share of s_1, the only other option for Charlotte is s_3. Since Carol will take s_2 or s_3, and s_3 is taken, Carol gets s_2.

D17(a). Aden gets the Desk and \$120, Ben gets the Dresser, Charles gets the TV Cabinet and the Tapestry, but pays in \$360. There is a surplus of \$240.

D17(b). Aden gets the Desk and \$200, Ben gets the Dresser and \$80, Charles gets the TV Cabinet and the Tapestry, but pays \$280.

D19. Carrie cleans the bathrooms and pays \$12. Jeri does the cooking and pays \$12. Violet washes the dishes, mows the lawn, vacuums and dusts and gets paid \$23 (or \$24 with rounding).

Appendix B

Calculator Tips

Chapter 2 requires the use of a calculator to evaluate some rather complicated formulas. This appendix contains some advice on how to avoid pitfalls that sometimes afflict students. Most of the advice applies to all calculators, but some is geared specifically to Texas Instruments calculators like the TI-84 and TI-30.

B.1 Parentheses

One of the greatest challenges to students is using parentheses correctly to help a calculator interpret a formula correctly. For example, consider the expression

$$\frac{1+7}{2}.$$

We know from elementary school that you compute the whole numerator of the fraction before dividing by the denominator, so this expression equals 4. If you type it into the calculator without parentheses, however, we get

```
1+7/2
                4.5
```

That's because the calculator, following the order of operations we learned as children, first does the division and then the addition: $\mathtt{7/2} = 3.5$, so $\mathtt{1+7/2} = 4.5$.

We human beings can tell, by looking at $\frac{1+7}{2}$, that the $1+7$ has to be done first, but the calculator doesn't know that unless we tell it. So we should really type

```
(1+7)/2
                  4
```

One way to make this process easier, especially in long complicated formulas, is to draw in the extra parentheses on your paper before you begin typing into the calculator.

$$\text{Given } \frac{1+7}{2}, \text{ draw in } \frac{(1+7)}{2} \text{ and then type } \mathtt{(1+7)/2}.$$

Here are some locations that typically need extra parentheses:

- The numerator of a fraction:

$$\frac{1+7}{2} \text{ becomes } \frac{(1+7)}{2}.$$

Incorrect:

```
1+7/2
                4.5
```

Correct:

```
(1+7)/2
                  4
```

- The denominator of a fraction:

$$\frac{8}{\frac{6}{3}} \text{ becomes } \frac{8}{\left(\frac{6}{3}\right)}.$$

Incorrect:

```
8/6/3
           0.444444
```

Correct:

```
8/(6/3)
                  4
```

- Exponents:[1]

$$5^{6-4} \text{ becomes } 5^{(6-4)}.$$

Incorrect:

```
5^6-4
              15621
```

Correct:

```
5^(6-4)
                 25
```

Of course, one complicated expression may need all of these!

$$500\frac{\left(1+\frac{.07}{52}\right)^{8\cdot 52}-1}{\frac{.07}{52}} \text{ becomes } 500\frac{\left(\left(1+\frac{.07}{52}\right)^{(8\cdot 52)}-1\right)}{\left(\frac{.07}{52}\right)}.$$

Incorrect:

```
500*(1+.07/52)^8*52-1/
.07/52
        26281.04806
```

Correct:

```
500*((1+.07/52)^(8*52)
-1)/(.07/52)
        278576.3860
```

[1]Some more recent Texas Instrument calculators help you out by making exponents actually look like exponents. On such a calculator, if you press the keys 5 ^ 6 − 4, it will display as

$$5^{6-4}$$
25

On this kind of calculator, if you need to type more *after* the exponent, press the right arrow key ▶ to get "back down" to the main line. For example, to evaluate $5^{6-4}+7$ on such a calculator, you would press the buttons 5 ^ 6 − 4 ▶ + 7 to get

$$5^{6-4}+7$$
32

B.2 Rounding Errors

We all know that $\frac{1}{3} \times 300 = 100$, but sometimes when working with a calculator we can get wrong answers:

```
1/3
                .3333333333
.3333333333*300
                99.99999999
```

The calculator is not at fault here; it has not made any mistakes. The decimal equivalent of $\frac{1}{3}$ is $0.33333333333333333333\ldots$; it goes on forever. The calculator gives us as many digits as it can, but `.3333333333` is just a *rounded-off approximation* of $\frac{1}{3}$. When we then typed `.3333333333` back into the calculator and multiplied by 300, the calculator accurately told us the answer was 99.99999999, not 100. This is called an *rounding error*.

In this example, the difference between what the calculator told us and the real answer is only 0.00000001, which is not terribly bad. Often students round off long decimals to only a few decimal places, which is okay for a final answer, but then they run into more problems:

```
1/3
                .3333333333
.33*300
                         99
```

Now we got an answer of 99 instead of 100, which seems more significant. In the more complicated expressions of the financial math chapter, we can end up with errors of several dollars (possibly hundreds of dollars) if we're not careful. For example, if we need to calculate $\$600 \cdot \frac{1-(1+\frac{.07}{12})^{-12\cdot 25}}{\frac{.07}{12}}$,a common student error would be this:

```
(1-(1+.07/12)^(-12*25))/
(.07/12)
              141.486903386
600*141.49
                      84894
```

Whereas this student answered \$84,894.00, the correct answer (to the nearest penny) is \$84,892.14; that's \$1.86 too high, enough to lose points on a test.

Thus when the calculator gives you numbers you will use again later in the problem, *never round them off*. Always type back in *all* the digits. Only round off at the very end of the problem!

However, this can *still* cause small errors (as we saw in the first example), and it's annoying to punch `141.486903386` into the calculator. There are at least three better solutions to the rounding error problem, all of which involve *keeping the numbers in the calculator until the end*.

- One solution is to combine your whole formula into one step in the calculator.

Example 1:

```
(1/3)*300
                               100
```

Example 2:

```
600*(1-(1+.07/12)^(-12*
25))/(.07/12)
                  84892.1420313
```

The downside of this approach is that writing the formula all in one line can be long, error-prone, and just plain unpleasant.

- An alternative is to break up your calculation into steps using `Ans`. Most calculators have a feature to remember the most recent answer they computed; on a TI machine, this is represented by the symbol `Ans`. The `Ans` automatically appears if you start a line by pressing an operation like [+] or [÷], or you can also get it by pressing the [2ND] key and then the [(-)] key.

Example 1:

```
1/3
                  0.3333333333
Ans*300
                           100
```

Example 2:

```
(1-(1+.07/12)^(-12*25))/
(.07/12)
                  141.486903386
600*Ans
                  84892.1420313
```

In Example 1, the keystrokes used were [1][/][3][ENTER] and then [*][3][0][0][ENTER]; the `Ans` was automatically generated by the calculator. In Example 2, the keys punched in the second line were [6][0][0][*][2ND][(-)][ENTER].

- The calculator's `Ans` ability remembers just one number, the result of the most recent calculation. There is also a more powerful way to use the calculator's memory, using *variables*, which are letters that stand for numbers.

  Most keys on your TI calculator have a letter printed above them and to the right; on a TI-84 Plus Silver Edition calculator, for example, one key looks like this:

  TEST A
  [MATH]

  To get the letter, you first press the [ALPHA] key, and then the key with the desired letter; for example, [ALPHA][MATH] produces an "`A`" on the screen.

  You can make the calculator remember any number at all by *storing* it to a variable, whose name is a letter. You use the [STO▶] key, which writes a little "→" symbol on the screen. Then you choose which letter you want to represent your number, and press [ENTER]. The first line of this example was created on my TI-84 Plus Silver Edition by pressing [5][STO▶][ALPHA][A][ENTER].

```
4+1→A
                    5
5-2→B
                    3
A*B
                   15
```

The use of variables can be immensely helpful, especially if you are doing lots of calculations with the same numbers; they also sometimes mean you need fewer parentheses. Take a look at our two examples one last time.

Example 1:

```
1/3→F
         0.3333333333
F*300
                  100
```

Example 2:

```
.07/12→I
         .005833333333
12*25→M
                   300
(1-(1+I)^-M)/I→A
          141.486903386
600*A
          84892.1420313
```

B.3 Subtraction versus Negatives

On Texas Instruments calculators, there are two keys with horizontal lines. The main `-` key is for *subtraction*, like $5-4$. When you type in a *negative number* like -6, however, you must use the smaller `(-)` key, usually located in the bottom row of your calculator. If you use the wrong button, you will probably see a screen like this:

```
ERR:SYNTAX
1:Quit
2:Goto
```

B.4 Correcting Mistakes

At some point in this course you will probably type in a long, complex expression wrong, and you'll have to fix it. Instead of retyping the whole thing, you might be able to just edit your earlier attempt! One of the following may work on your calculator:

- pressing the up arrow key ▲ to scroll up to the earlier line
- pressing `2ND` `ENTER` to access the latest "`ENTRY`"

If you do this, you might need to insert new symbols, such as parentheses, into your earlier wrong entry. Be sure to change into insert mode ("`INS`") by pressing `2ND` `DEL`; this will let you insert new characters. Unfortunately, the calculator will switch back to normal "overtype" mode as soon as you press an arrow key to move to another location, so you'll need to press `2ND` `DEL` again.

Bibliography

[1] Gizem Karaali, Lily S. Khadjavi, and Reem Jaafar. *Chapter 18*, pages 221–236. MAA Press, an imprint of the American Mathematical Society, 2019.