Notes for Week 1 Discussion 3D
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Strategy for Solving Linear Differential Equations.

$$\frac{dy}{dt} + P(t)y = q(t)$$

Let $\mu = e^{\int P(t)\,dt}$.

$$\Rightarrow \frac{d}{dt}(\mu(t) \cdot y) = q(t) \cdot \mu(t)$$

$$\Rightarrow \mu(t) \cdot y = \int q(t) \cdot \mu(t)\,dt$$

Steps to Solve:

1. Decide if the equation is linear. Identify $P(t)$ and $q(t)$.
2. Calculate $\mu(t)$.
3. Solve $\frac{d}{dt}(\mu(t) \cdot y) = q(t) \cdot \mu(t)$.
4. Solve for $y(t)$.

Solutions to problems during discussion.

Problem 1

Solve $y' = \frac{1}{y^4}$, $y(0) = 1$:

$$\begin{aligned} y^4 y' &= 1 \\ y^4\,dy &= dx \\ \int y^4\,dy &= \int dx \\ \frac{y^5}{5} &= x + C. \end{aligned}$$

Using $y(0) = 1$:

$$\frac{1^5}{5} = 0 + C \implies C = \frac{1}{5}.$$

The solution is:

$$y = (5x + 1)^{1/5}.$$

Problem 2

Solve $x' = \frac{1}{\sin(x)}$, $x(0) = 0$:

$$\begin{aligned} \sin(x)\,dx &= dt \\ \int \sin(x)\,dx &= \int dt \\ -\cos(x) &= t + C. \end{aligned}$$

Using $x(0) = 0$:

$$-\cos(0) = 0 + C \implies C = -1.$$

The solution is:

$$\cos(x) = 1 - t \implies x = \arccos(1 - t).$$

Problem 3

Solve $y' = (y + 2)(y + 3)$, $y(0) = 3$:

$$\frac{1}{(y+2)(y+3)}\,dy = dx.$$

Using partial fractions:

$$\frac{1}{(y+2)(y+3)} = \frac{1}{y+2} - \frac{1}{y+3}.$$

Integrating both sides:

$$\begin{aligned} \int \left(\frac{1}{y+2} - \frac{1}{y+3} \right) dy &= \int dx \\ \ln|y+2| - \ln|y+3| &= x + C. \end{aligned}$$

Using $y(0) = 3$:

$$\ln\left|\frac{5}{6}\right| = C.$$

The solution is:

$$\ln\left|\frac{y+2}{y+3}\right| = x + \ln\left|\frac{5}{6}\right|.$$

Problem 4

Solve $y' = y^n$, $y(0) = 1$:

Case 1: $n \neq 1$

$$\frac{1}{y^n}\,dy = dx$$
$$\int y^{-n}\,dy = \int dx$$
$$\frac{y^{1-n}}{1-n} = x + C \implies y = [(1-n)(x+C)]^{\frac{1}{1-n}}.$$

Case 2: $n = 1$

$$y' = y \implies \frac{dy}{y} = dx$$
$$\ln|y| = x + C \implies y = Ce^x.$$

Consider $y' = 3y^{2/3}$, $y(2) = 0$. Does it have a unique solution?

Picard's theorem requires $f(y) = 3y^{2/3}$ to be Lipschitz continuous. Near $y = 0$, $f(y)$ is not Lipschitz. Therefore, the solution is **not unique**.

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Problem 5

Solve $y' - \frac{3}{x+1}y = (x+1)^4$:

$$\mu(x) = e^{-\int \frac{3}{x+1}\,dx} = (x+1)^{-3}$$
$$\frac{d}{dx}[(x+1)^{-3}y] = (x+1)$$
$$(x+1)^{-3}y = \frac{(x+1)^2}{2} + C$$
$$y = (x+1)^3\left(\frac{(x+1)^2}{2} + C\right).$$

Problem 6

Solve $y' + \frac{2}{x}y = \frac{\sin x}{x^2}$:

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = x^2$$
$$x^2y' + 2xy = \sin x$$
$$\frac{d}{dx}(x^2y) = \sin x$$
$$x^2y = -\cos x + C$$
$$y = \frac{-\cos x + C}{x^2}.$$