Stability and Fragility in the Mammalian Cochlea: A Structured Stochastic Uncertainty Approach

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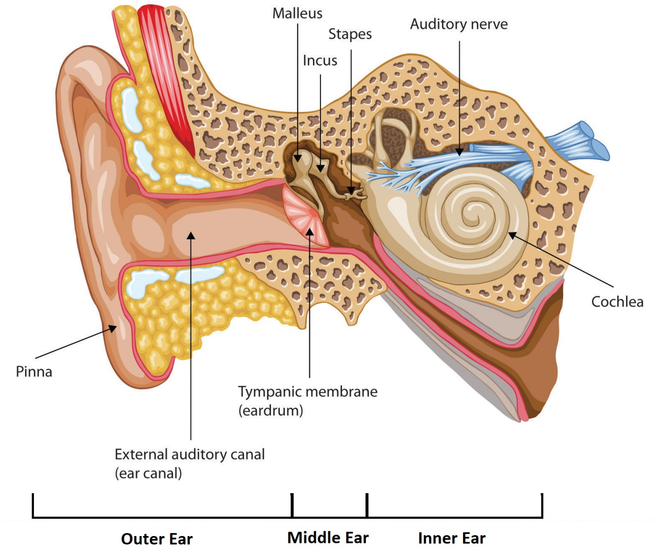
https://maurice-filo.github.io/

MTNS 2024, Cambridge

Overview

Brief Physiology: The Ear



Source: Introduction to Psychology 1.0.1 — FlatWorld

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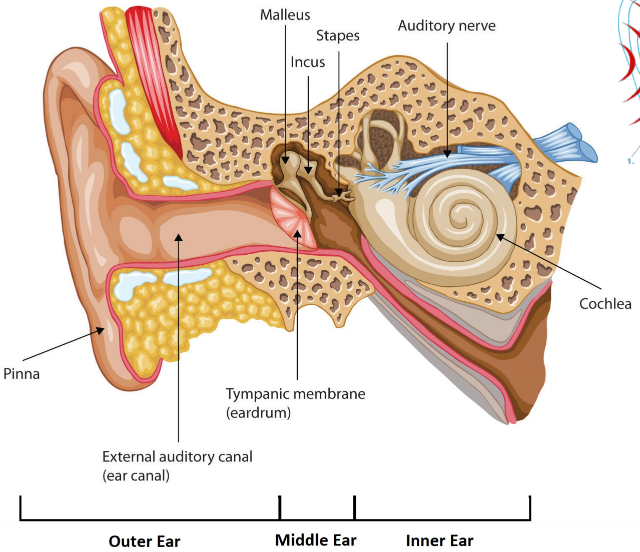

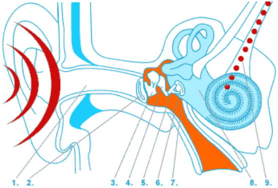

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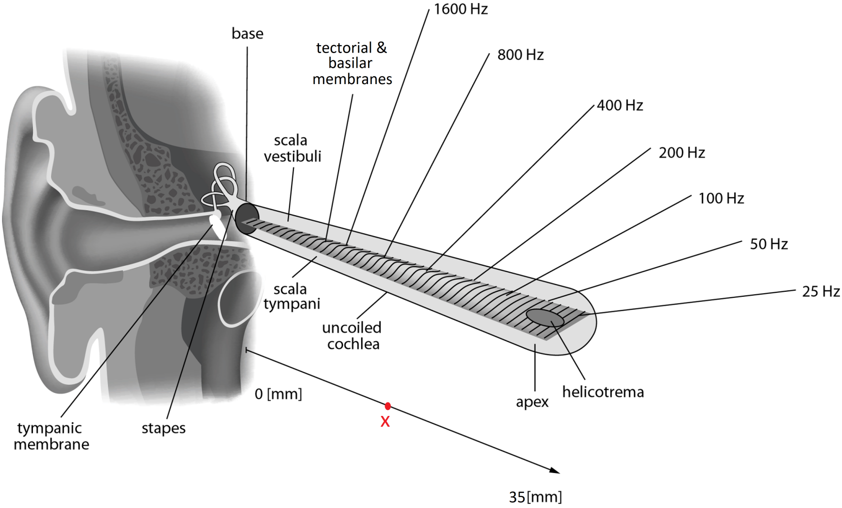
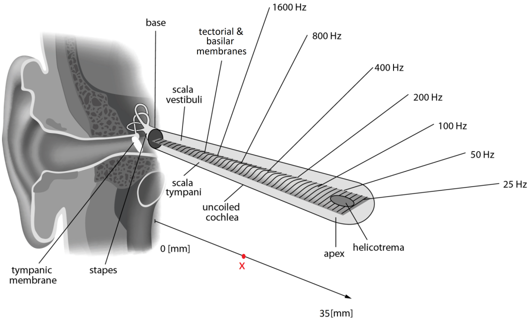
Source: http://www.byronshvhearing.com/

Brief Physiology: The Cochlea



Source: Biophysical Parameters Modification Could Overcome Essential Hearing Gaps

Brief Physiology: The Cochlea



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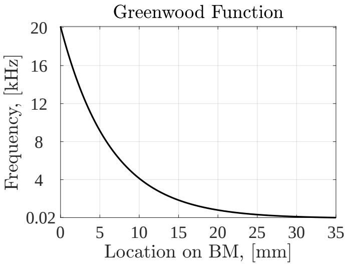
Cochlea is simply a mechanical spectrum analyzer

Cochlear Response, Frequency-Location Mapping



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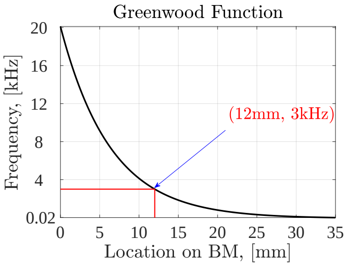

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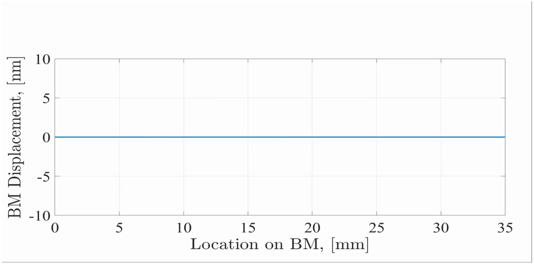

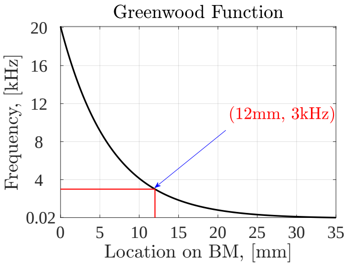

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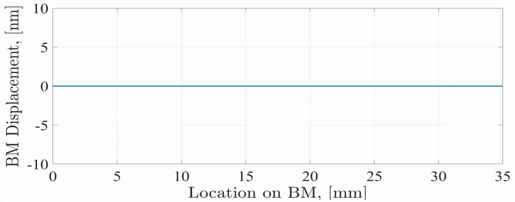
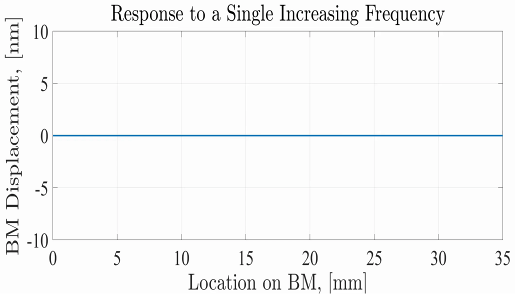
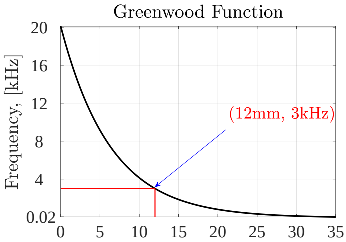
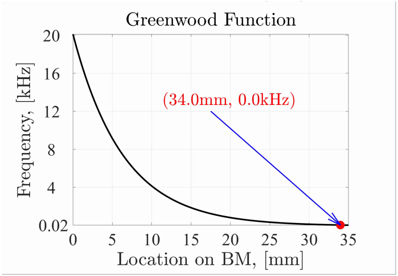


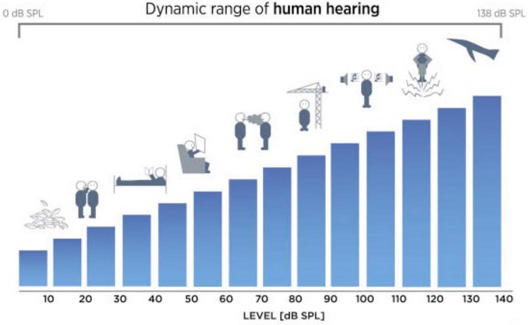






Cochlear Response, Wide Dynamic Range

Wide Dynamic Range: More than 120 dB in Sound Pressure Level (SPL)



20×10^{7} Pa

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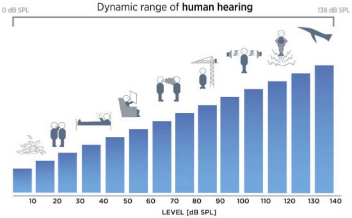

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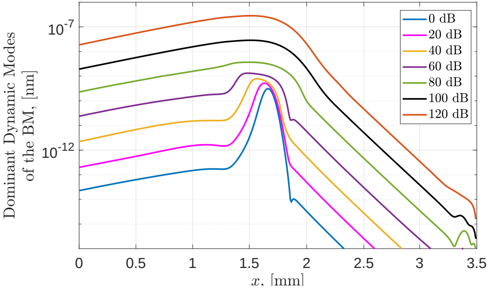
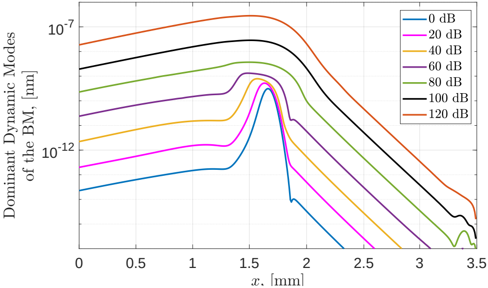


Cochlear Response, Wide Dynamic Range

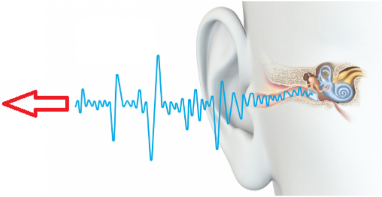
A feedback mechanism that amplifies small inputs
$\rightarrow$ but pushes the dynamics to the edge of stability!



Spontaneous Response: Cochlear Instabilities

The ear is an active device that can produce sound!

- **Spontaneous Otoacoustic Emissions (SOAE)**
  (Not necessarily perceived)



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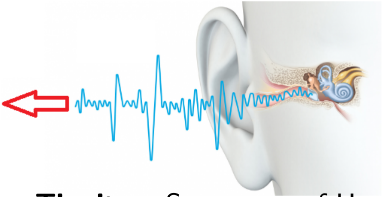

- **Tinnitus**: Symptoms of Hearing Loss Diseases
  (Perceived as harsh and consistent ringing)

Spontaneous Response: Cochlear Instabilities

$\longrightarrow$ Can be modeled as instabilities in stochastic cochlear dynamics...

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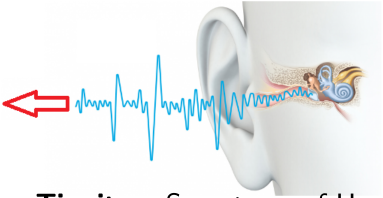

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Cochlear Response, Distortion Products

Two tones simultaneously: $\rightarrow f_{Low}$: fixed $\rightarrow f_{High}$: time-varying

https://maurice-filo.github.io/Cochlear%20Modeling.html

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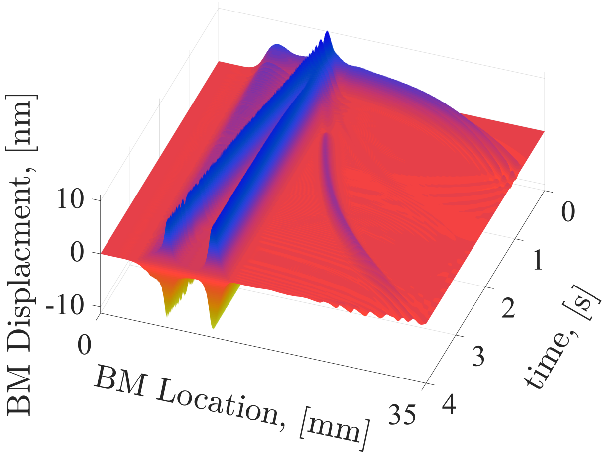
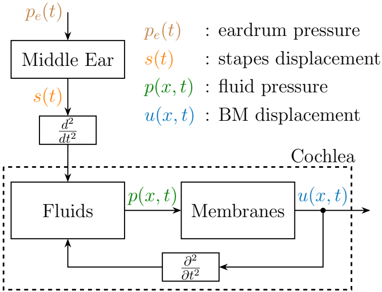


A Deterministic Biomechanical Model

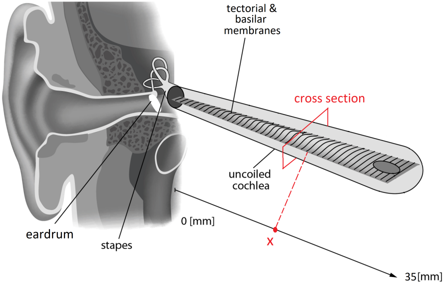


$p_e(t)$: eardrum pressure
$s(t)$: stapes displacement
$p(x,t)$: fluid pressure
$u(x,t)$: BM displacement

$$p(x,t) = -[\mathcal{M}_f \ddot{u}](x,t) - [\mathcal{M}_s \ddot{s}](x,t)$$

where:

$\mathcal{M}_f$ and $\mathcal{M}_s$ are linear spatial operators



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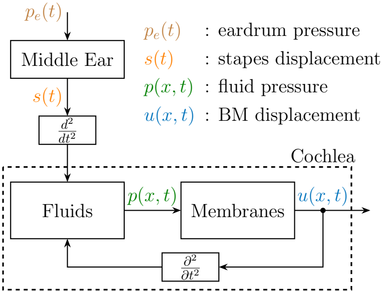
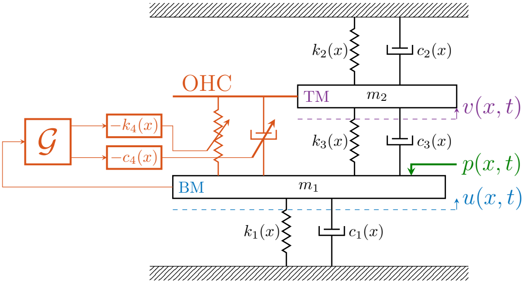
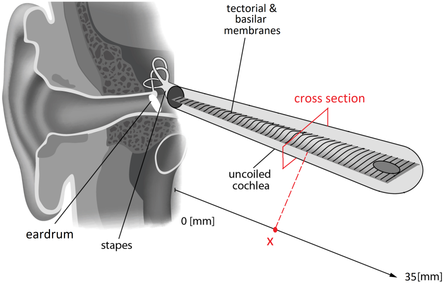




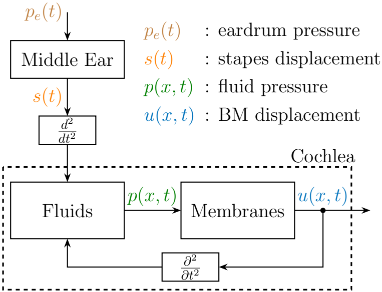


$v(x,t)$: TM displacement
$\gamma(x)$: gain coefficient

$$[\mathcal{G}(u)](x,t) = \frac{\gamma(x)}{1+\theta[\Phi_\eta(u^2)](x,t)}$$

$\mathcal{G}$: Active Gain (small $u \to$ large gain) gives wide dynamic range

A Stochastic Biomechanical Model



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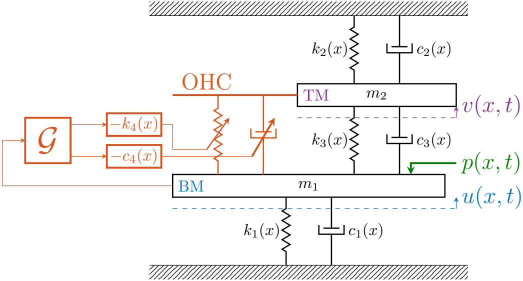

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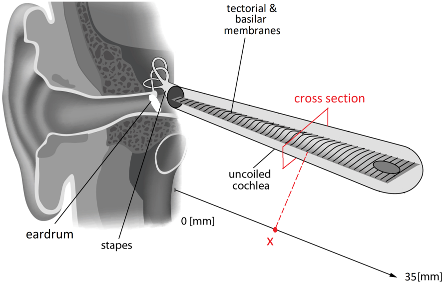


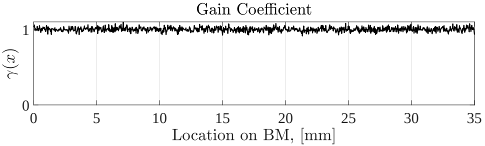
$$[\mathcal{G}(u)](x,t) = \frac{\gamma(x,t)}{1+\theta[\Phi_\eta(u^2)](x,t)}$$

$\gamma(x,t)$: Random Field

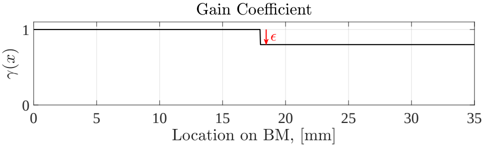
Cochlear Instabilities, Perturbations of the Active Gain

$$[\mathcal{G}(u)](x,t) = \frac{\gamma(x)}{1+\theta[\Phi_\eta(u^2)](x,t)}$$

- $\gamma(x) = 1 + \tilde{\gamma}(x) \rightarrow$ Eigenvalue analysis via Monte Carlo methods [1]



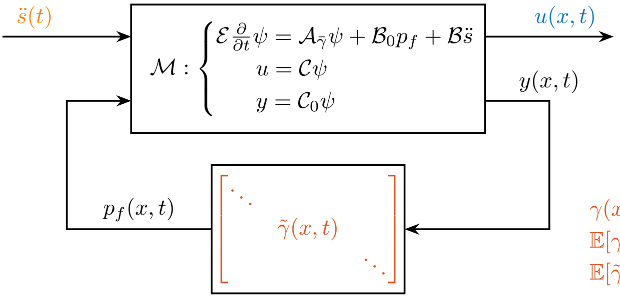
- $\gamma(x) = 1 - \epsilon \text{step}(x - x_0) \rightarrow$ Eigenvalue perturbation analysis [2]



- $\gamma(x,t)$: Random Field $\rightarrow$ Structured Stochastic Uncertainty

[1] Ku, E. M., Elliott, S. J., & Lineton, B. (2008).
[2] Filo, M. G. (2017), Master's Thesis, UCSB.

Structured Stochastic Uncertainty Setting



$$\psi(x,t) := \begin{bmatrix} u(x,t) \\ v(x,t) \\ \dot{u}(x,t) \\ \dot{v}(x,t) \end{bmatrix}$$

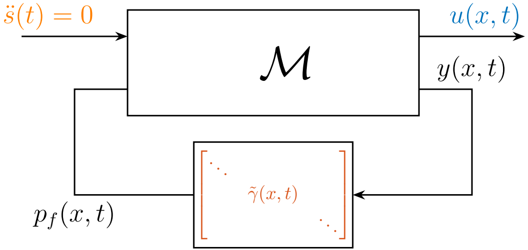
$$\gamma(x,t) = \bar{\gamma}(x) + \tilde{\gamma}(x,t)$$
$$\mathbb{E}[\gamma(x,t)] = \bar{\gamma}(x)$$
$$\mathbb{E}[\tilde{\gamma}(x,t)\tilde{\gamma}(\xi,\tau)] = \mathbf{\Gamma}(x,\xi)\delta(t-\tau)$$

$$\mathcal{E} := \begin{bmatrix} \mathcal{I} & 0 & 0 & 0 \\ 0 & \mathcal{I} & 0 & 0 \\ 0 & 0 & m_1\mathcal{I} + \mathcal{M}_f & 0 \\ 0 & 0 & 0 & m_2\mathcal{I} \end{bmatrix}; \quad \mathcal{B} := \begin{bmatrix} 0 \\ 0 \\ -\mathcal{M}_s \\ 0 \end{bmatrix}; \quad \mathcal{B}_0 := \begin{bmatrix} 0 \\ 0 \\ \mathcal{I} \\ 0 \end{bmatrix}$$

$$\mathcal{A}_{\bar{\gamma}} := \begin{bmatrix} 0 & \mathcal{I} \\ \mathcal{K}_{\bar{\gamma}} & \mathcal{C}_{\bar{\gamma}} \end{bmatrix}; \quad \mathcal{C} := \begin{bmatrix} \mathcal{I} & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{K}_{\bar{\gamma}}, \mathcal{C}_{\bar{\gamma}}$ and $\mathcal{C}_0$ are spatially decoupled operators and are functions of k_i, c_i and $\bar{\gamma}$.

Mean-Square Stability Conditions & Performance



- **Forward block** ($\mathcal{M}$): causal & stable LTI system
- **Feedback block**: Diagonal, temporally independent but possibly spatially correlated: $\mathbb{E}[\tilde{\gamma}(x,t)\tilde{\gamma}(\xi,\tau)] = \mathbf{\Gamma}(x,\xi)\delta(t-\tau)$

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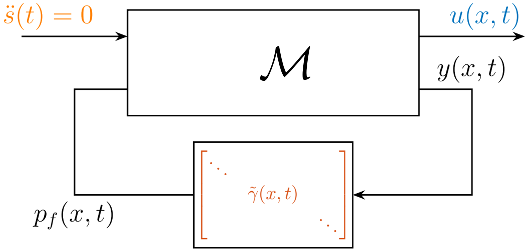

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Definition: **MSS** $\Longleftrightarrow$ covariances of all signals remain bounded for all time

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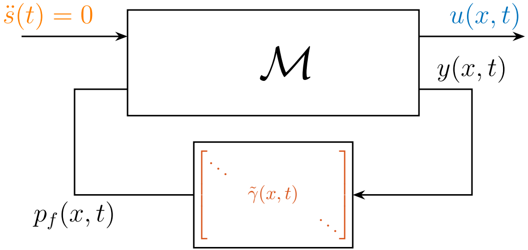

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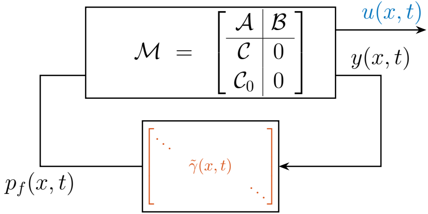
Questions:

1. Conditions on $\mathbf{\Gamma}(x,\xi)$ that guarantee MSS?
2. If MSS is violated, how do covariances grow?

Covariance Evolution & Loop Gain Operator

Original Block Diagram (**Stochastic**)



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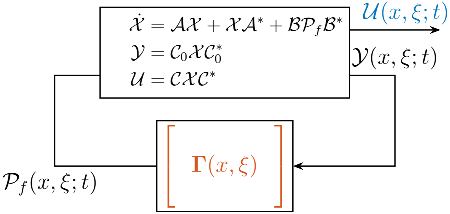
Covariance Block Diagram (**Deterministic**)



$$\mathcal{P}_f = \mathbf{\Gamma} \circ \mathcal{Y}$$

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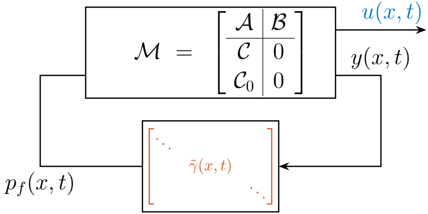

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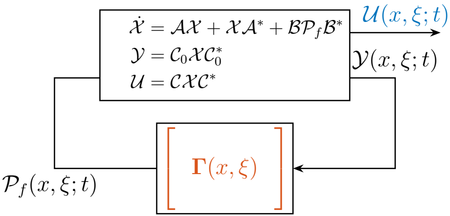

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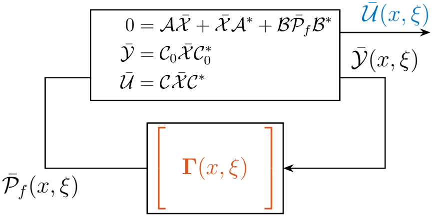
Steady State Covariances



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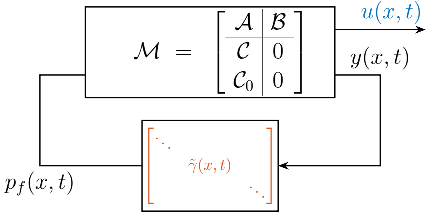

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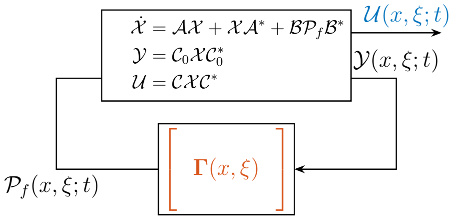

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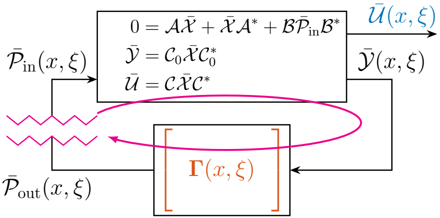

$$\bar{\mathcal{P}}_{\text{out}} = \mathbf{\Gamma} \circ \bar{\mathcal{Y}}$$

Loop Gain Operator:

$$\mathbb{L} : \bar{\mathcal{P}}_{\text{in}} \to \bar{\mathcal{P}}_{\text{out}}$$

- MSS condition:
$$\rho(\mathbb{L}) < 1$$
- Worst-case covariance:
$$\mathbb{L}(\mathbf{P}) = \rho(\mathbb{L})\mathbf{P}$$

MSS Analysis of the Cochlea

$\gamma(x,t) = \bar{\gamma}(x) + \tilde{\gamma}(x,t)$

Expectation: $\mathbb{E}\left[\gamma(x,t)\right] = \bar{\gamma}(x)$

Covariance: $\mathbb{E}\left[\tilde{\gamma}(x,t)\tilde{\gamma}(\xi,\tau)\right] = \frac{\epsilon^2}{\lambda\sqrt{2\pi}} e^{\frac{(x-\xi)^2}{2\lambda^2}} \delta(t-\tau)$

$\mathbf{U}(x,\xi)$: worst case covariance of the basilar membrane displacement $u(x,t)$

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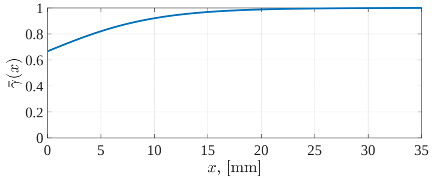

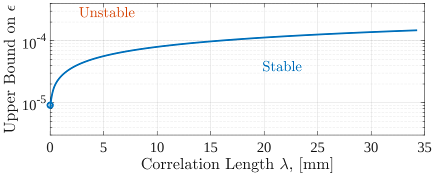

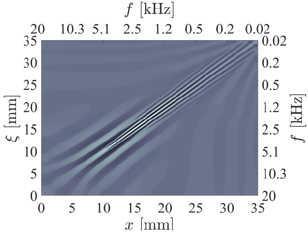

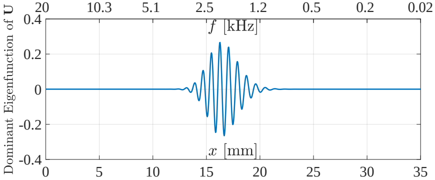

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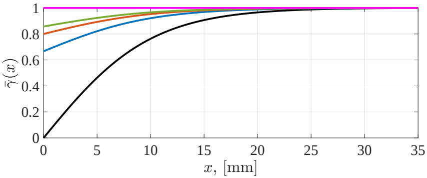
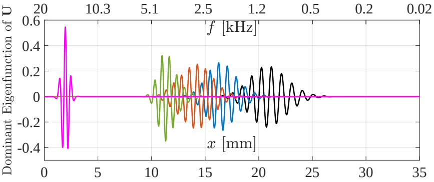
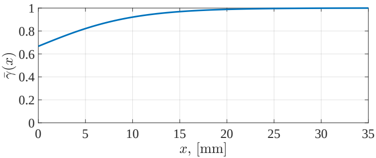
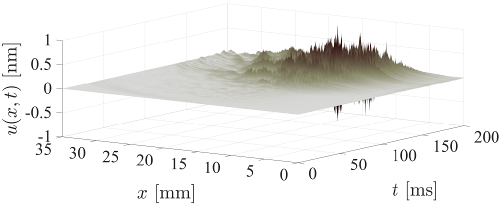


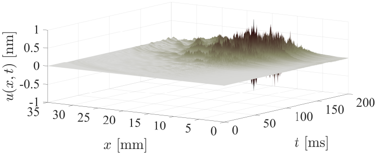


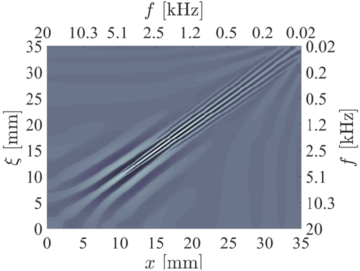
Stochastic Simulation of the Nonlinear Cochlear Dynamics





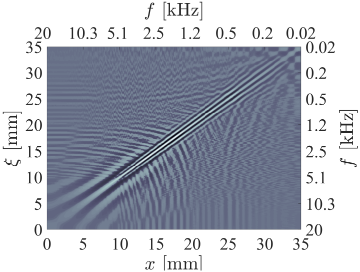
Stochastic Simulation of the Nonlinear Cochlear Dynamics





⟵ Predicted Worst-Case Covariance

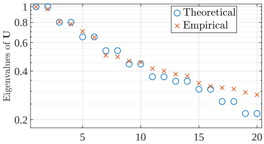
(MSS analysis for linearized dynamics)

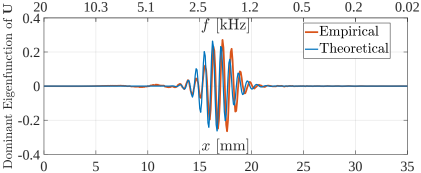


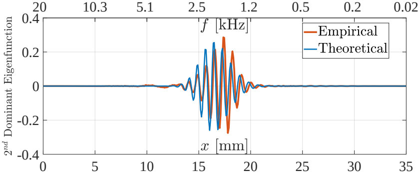
⟵ Empirical Covariance

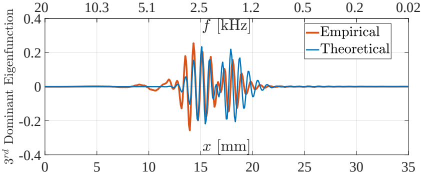
$$\mathbf{U}_{\text{Emp}} := \int_0^{t_f} u(x,t)u(\xi,t)dt$$

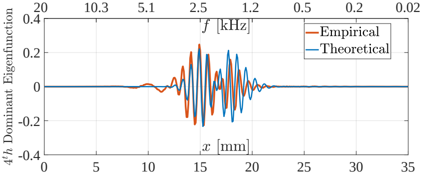
Stochastic Simulation of the Nonlinear Cochlear Dynamics











No significant difference: Nonlinearity only saturates the unstable response!

Conclusion & Future Work

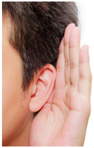
Key Messages:

- Cochlear models are extremely sensitive to stochastic perturbations
- Structured stochastic uncertainty is a suitable framework for MSS analysis of the cochlea
- Various stochastic uncertainties in the cochlea are possible sources for cochlear instabilities such as SOAEs and tinnitus.

Future Directions:

- Uncertainties in different cochlear parameters (such as fluid density)
- Suppressing undesired instabilities such as tinnitus

Questions?



Localized Stochastic Perturbation

$\gamma(x,t) = \bar{\gamma}(x) + \tilde{\gamma}(x,t);$ Expectation: $\mathbb{E}\left[\gamma(x,t)\right] = \bar{\gamma}(x) = 1$

Covariance: $\mathbb{E}\left[\tilde{\gamma}(x,t)\tilde{\gamma}(\xi,\tau)\right] = \epsilon^2 \phi_{\sigma}(x-\mu)\delta(t-\tau, x-\xi)$

$$\phi_\sigma(x-\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}}$$

$\mathbf{U}(x,\xi)$: worst case covariance of the basilar membrane displacement $u(x,t)$

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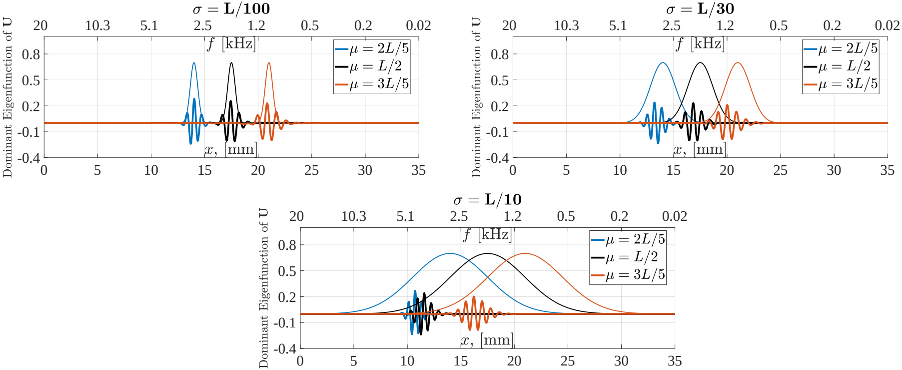
U(x,ξ): worst case covariance of the basilar membrane displacement $u(x,t)$



Thin lines $\longrightarrow \phi_{\sigma}(x-\mu)$; Thick lines $\longrightarrow$ Dominant Eigenfunction of **U**