Aufgabe 1

$$O(f(n)) = \{ g(n) : \exists c > 0 : \exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : \boxed{g(n) \leq c \cdot f(n)} \}$$

a) 1 $\{n^3\} \subseteq O(n^3 - 2n^2)$

zz. $\exists c > 0 , \exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : n^3 \leq c(n^3 - 2n^2)$

Def 1: $n_0 = 3$, $c = 3$

zz $n^3 \leq 3(n^3 - 2n^2)$

$\overset{n \in \mathbb{N}^+}{\Leftrightarrow}$ $n \leq 3(n-2)$

$\Leftrightarrow$ $3 \leq n$

$\Rightarrow n^3 \subseteq O(n^3 - 2n^2)$

2 $\{n^{\frac{3}{2}}\} \subseteq O(n^2)$

zz: $\exists c > 0 ; \exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : n^{\frac{3}{2}} \leq c n^2$

$\Leftrightarrow 1 \leq c\sqrt{n}$

Let $c = 1$ $\Rightarrow$ $1 \leq \sqrt{n}$ true for all $n \geq n_0 = 1$

since $f(x) = \sqrt{x}$ is strictly increasing

3 $\{n^3\} \not\subseteq O(10^6 n^2)$

zz. $\forall c>0 : \exists n\in\mathbb{N}^+ : g(n) > c\cdot f(n)$

let $g(n) = n^3$ and $f(n) = 10^6 n^2$

zz $\exists n \in \mathbb{N}^+ : n^3 > c\cdot 10^6 n^2$

$n > c\cdot 10^6$ true because $n\in\mathbb{N}^+$

b) 1 $\{2^{n+1}\} \subseteq O(2^n)$

zz. $\underbrace{\exists c>0 : \exists n_0\in\mathbb{N}^+ : \forall n \geq n_0}$: $2^{n+1} \leq c\cdot 2^n$

$\Leftrightarrow 2\cdot 2^n \leq c\cdot 2^n$

$\Leftrightarrow 2 \leq c$ true for all $n_0 \in \mathbb{N}^+$ $c \geq 2$

We will imply this from now on...

2 $\{5\cos(n)+n\} \not\subseteq O(1)$

zz: $\forall c>0\ \exists n\in\mathbb{N}^+ : 5\cos(n)+n > c$

Since $\max(5\cos(n)) = 5$ -> Term will vanish for large n

=> zz: $n > c$, true because $n\in\mathbb{N}^+$

3 $\{(n+1)!\} \not\subseteq O(n!)$

zz: $\forall c>0 : \exists n \in \mathbb{N}^+ : (n+1)! > c \cdot n!$

$\Leftrightarrow (n+1) \cdot n! > c \cdot n!$

$\Leftrightarrow n+1 > c$, true because $n \in \mathbb{N}^+$

c) 1 $\{\log_2 n\} \subseteq O(\log_{10} n) \land \{\log_2 n\} \subseteq \Omega(\log_{10} n)$

$$\boxed{\log_2(x) = \frac{\log(x)}{\log(2)}}$$

zz $\log_2(n) = c \cdot \log_{10}(n)$

$\Leftrightarrow \log_2(n) = c \cdot \frac{\log_2(n)}{\log_2(10)}$

Let $c = \log_2(10) \Rightarrow \log_2(n) = \log_2(n)$, true for all $n \in \mathbb{N}^+$

Since $g(n) \leq c f(n)$ and $g(n) \geq c f(n) \Rightarrow \{\log_2(n)\} \subseteq \Theta(\log_{10} n)$

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c) 2. $\{2^{\log_a(n)}\} \notin \Theta(2^{\log_b(n)})$ for $a \neq b$ (*)

~~case 1: a>b~~

if $g(n) \notin O(f(n)) \Rightarrow g(n) \notin \Theta(f(n))$ (*1)

if $g(n) \notin \Omega(f(n)) \Rightarrow g(n) \notin \Theta(f(n))$ (*2)

To show (*) it suffice to show either (*1) or (*2)

case 1: $a > b$

To show: $\{2^{\log_a(n)}\} \notin O(2^{\log_b(n)})$ (**)

$\cancel{\log_2}$ $2^{\log_a(n)} > c \cdot 2^{\log_b(n)}$

$\Leftrightarrow n^{\log_a(2)} > c \cdot n^{\log_b(n)}$ $\quad | : n^{\log_b(n)}$

$\Leftrightarrow n^\wedge\left(\frac{\log_a(2)}{\log_b(2)}\right) > c$ (*3)

Because $\frac{\log_a(2)}{\log_b(2)} > 0$ for $a > b$ (prop. of log's) and n^d with $d > 0$ is strictly increasing, there is always an "n" for which (*3) for an arbitrary $c > 0$. Therefore (**) is true.

case 2: $b > a$

For symmetry reasons if $b > a$:

$\{2^{\log_a(n)}\} \notin \Omega(2^{\log_b(n)})$ (**2)

In either ~~all~~ cases for ~~both~~ $a > b$ or $b > a$ either (**) and (**2) are true and therefore also (*).

d) 1 $\{f(n)+g(n)\} \not\subseteq O(\min(f(n), g(n)))$

$$\min(f(n), g(n)) = \begin{cases} f(n) : f(n) < g(n) \\ g(n) : f(n) \geq g(n) \end{cases}$$

$\exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : f(n) + g(n) \leq c \cdot \min(f(n), g(n))$

1st case: $f(n) < g(n)$

$\Rightarrow f(n)+g(n) \leq c \cdot f(n)$

$\Leftrightarrow g(n) \leq f(n)(c-1)$

Let $f(n) = n$, $g(n) = n^2$ $n \in \mathbb{N}^+$

$\Rightarrow n^2 \leq n \cdot (c-1)$

$\Leftrightarrow n \leq c-1$ False, because $n \in \mathbb{N}^+$

Proof through contradiction

2 $\{f(n)+g(n)\} \subseteq O(\max[f(n), g(n)])$

$$\max(f(n), g(n)) = \begin{cases} f(n) : f(n) > g(n) \\ g(n) : f(n) \leq g(n) \end{cases}$$

zz $\exists c > 0 : \exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : f(n)+g(n) \leq c \cdot \max(f(n), g(n))$

1st case: Let $f(n) > g(n)$

$\Rightarrow f(n)+g(n) \leq c \cdot f(n)$

$\Leftrightarrow g(n) \leq f(n)(c-1)$, true for $c > 1$

2nd case: Let $f(n) \leq g(n)$

$\Rightarrow f(n)+g(n) \leq c \cdot g(n)$

$\Leftrightarrow f(n) \leq g(n)(c-1)$, true for $c \geq 1$

3 $\{f(n)g(n)\} \subseteq O(f(n)g(n))$

zz $\exists c > 0 : \exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : f(n)g(n) \leq c \cdot f(n)g(n)$

Let $c = 1$: $\Rightarrow f(n)g(n) \leq f(n)g(n)$ true for all $n \in \mathbb{N}^+$

e) $\{f(n)\} \subseteq O(n) \not\Rightarrow \{2^{f(n)}\} \subseteq O(2^n)$

Let $f(n) = 10n$

$\{10n\} \subseteq O(n)$ ✓ but $\{2^{10n}\} \not\subseteq O(2^n)$, because

$2^{10n} = (2^n)^{10} \not\subseteq O(2^n)$

zz: $\forall c > 0 : \exists n \in \mathbb{N}^+ : 2^{10n} > c \cdot 2^n$

$\Leftrightarrow \quad 10n > \log_2(c) + n$, true because $n \in \mathbb{N}^+$

2 $\boxed{\{f(n)\} \subseteq O(n)} \Rightarrow \{f(n)^2\} \subseteq O(n^2)$

$\exists c > 0; \exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : f(n)^2 \leq c \cdot n^2$

$\Leftrightarrow \quad f(n) \leq \sqrt{c}\, n$, true, because $\forall \sqrt{c} : \exists c' = \sqrt{c}$, and $\boxed{f(n) \leq c' \cdot n}$

(assumption)

f)

zz: $\Theta(f) \overset{(*)}{=} \Theta(g) \Leftrightarrow f \in \Theta(g)$

suppose $h(n) \in \Theta(f(n)) \overset{(*)}{\Leftrightarrow} h(n) \in \Theta(g(n))$

$\Leftrightarrow \exists c_1; c_2; c_1'; c_2' > 0 : \exists n_0 \in \mathbb{N}^+ : n_0 < n : h(n) \leq c_1 f(n)$

$\wedge\ h(n) \geq c_2 f(n) \wedge h(n) \leq c_1' g(n) \wedge h(n) \geq c_2' g(n)$

1 case

$\Leftrightarrow c_1 f(n) \geq h(n) \geq c_2' g(n)$

$\Leftrightarrow f(n) \geq \frac{c_2'}{c_1} g(n)$ let $\frac{c_2'}{c_1} = d_1 > 0$

$\Leftrightarrow f(n) \geq d_1 g(n)$

$\Leftrightarrow f(n) \in \Omega(g(n))$ $(*_1)$

2. case

$\Leftrightarrow c_2 f(n) \leq h(n) \leq c_1' g(n)$

$\Leftrightarrow f(n) \leq \frac{c_1'}{c_2} g(n)$ let $\frac{c_1'}{c_2} = d_2 > 0$

$\Leftrightarrow f(n) \leq d_2 g(n)$

$\Leftrightarrow f(n) \in O(g(n))$ $(*_2)$

with $(*_1 \wedge *_2) \Leftrightarrow f(n) \in \Theta(g)$ //