Shape Functions

S. Salon

Polynomial Approximation - 1D

$$f(x) = a_0 + a_1 x + a_2 x^2 \ \cdots \ + a_n x^n$$

$$f = \sum_{i=0}^{n} a_i x^i$$

In 2 Dimensions

$$f = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 \cdots + a_m y^n$$

n is the order, and there are *m+1* coefficients, where

$$m = \frac{(n+1)(n+2)}{2}$$

In summation form

$$f = \sum_{i=0}^{m} a_i x^j y^k, \quad j + k < n$$

j and *k* are exponents, related to *i* by

$$i = \frac{(k+1)(j+k+2) + j(j+k) - 2}{2}$$

Consider a 1st order 2D Polynomial

$$f = a_{i1}x^0y^0 + a_{i2}x^1y^0 + a_{i3}x^0y^1$$

$$i1 = \frac{(0+1)(0+0+2)+0(0+0)}{2} - 1 = 0$$

$$i2 = \frac{(0+1)(1+0+2)+1(1+0)}{2} - 1 = 1$$

$$i3 = \frac{(1+1)(0+1+2)+0(0+1)}{2} - 1 = 2$$

For this case

$$N = \frac{(1+1)(1+2)}{2} = 3$$

Pascal's Triangle

$$a_0$$

$$a_1 x \qquad a_2 y$$

$$a_3 x^2 \qquad a_4 xy \qquad a_5 y^2$$

$$a_6 x^3 \qquad a_7 x^2 y \qquad a_8 xy^2 \qquad a_9 y^3$$

$$a_{10} x^4 \qquad a_{11} x^3 y \qquad a_{12} x^2 y^2 \qquad a_{13} xy^3 \qquad a_{14} y^4$$

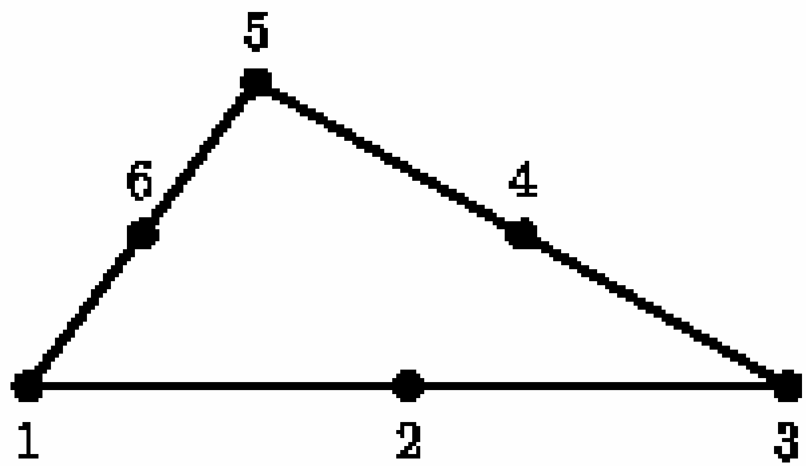
$$a_{15} x^5 \qquad a_{16} x^4 y \qquad a_{17} x^3 y^2 \qquad a_{18} x^2 y^3 \qquad a_{19} xy^4 \qquad a_{20} y^5$$

Complete Polynomials, 2D

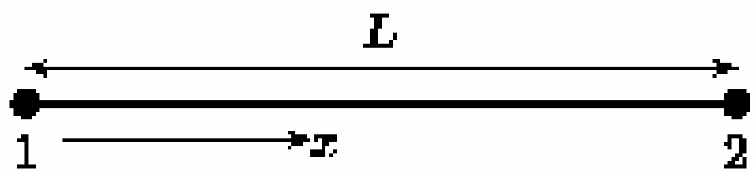
$$f(x, y) = \sum_{i=0}^{m} a_i x^j y^k,$$

$$\text{for} \quad j + k \leq n, \; m = \frac{(n+1)(n+2)}{2}$$

Second Order Triangle



1-D Linear Element



$$\phi(x) = a_0 + a_1 x$$

$$\phi_1 = a_0 + a_1 x_1$$
$$\phi_2 = a_0 + a_1 x_2$$

$$\{\phi\} = (C)\{a\}$$

Where

$$\{\phi\} = \left\{ \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right\}$$

$$(C) = \left(\begin{array}{cc} 1 & x_1 \\ 1 & x_2 \end{array} \right)$$

$$\{a\} = \left\{ \begin{array}{c} a_0 \\ a_1 \end{array} \right\}$$

We now solve

$$\{a\} = (C)^{-1}\{\phi\}$$

$$\phi = (D)\{a\}$$

where

$$(D) = [1, x]$$

$$\phi = (D)(C)^{-1}\{\phi\}$$

$$\phi = (\xi)\{\phi\}$$

We can also find a standard inverse

$$(C)^{-1} = \frac{1}{x_2 - x_1} \begin{pmatrix} x_2 & -1 \\ -x_1 & 1 \end{pmatrix}^T = \frac{1}{x_2 - x_1} \begin{pmatrix} x_2 & -x_1 \\ -1 & 1 \end{pmatrix}$$

$$(\xi) = (D)(C)^{-1} = \frac{[1, x]}{x_2 - x_1} \begin{pmatrix} x_2 & -x_1 \\ -1 & 1 \end{pmatrix}$$

$$(\xi) = \left(\frac{x_2 - x}{x_2 - x_1}, \ \frac{-x_1 + x}{x_2 - x_1} \right)$$

This gives

$$\phi = \frac{x_2 - x}{x_2 - x_1}\phi_1 + \frac{x - x_1}{x_2 - x_1}\phi_2 = \xi_1\phi_1 + \xi_2\phi_2$$

where

$$\begin{aligned} \xi_1 &= \frac{x_2 - x}{x_2 - x_1} \\ \xi_2 &= \frac{x - x_1}{x_2 - x_1} \end{aligned}$$

Shape functions

a b

$$\xi_1 = \frac{(L-x)}{L} = \left(1 - \frac{x}{L}\right)$$

$$\xi_2 = \frac{(x-0)}{L} = \frac{x}{L}$$

The potential in terms of the natural coordinates

$$\phi = \left(1 - \frac{x}{L}\right)\phi_1 + \frac{x}{L}\phi_2 = (1-\xi)\phi_1 + \xi\phi_2$$

$$\xi_1 = \frac{\xi - 1}{2} \text{ and } \xi_2 = \frac{\xi + 1}{2}$$

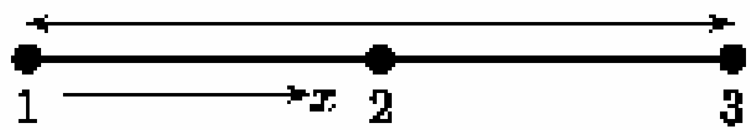
ξ ranges between -1 and $+1$

Lagrange Interpolation

$$\xi_i = \frac{\prod_{j=0,\ j=i}^{n}(x - x_j)}{\prod_{j=0,\ j=i}^{n}(x_i - x_j)}$$

$$\xi_1 = \frac{x - x_2}{x_1 - x_2} \text{ and } \xi_2 = \frac{x - x_1}{x_2 - x_1}$$

1-D Second Order Element



$$\xi_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$\xi_2 = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$\xi_3 = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

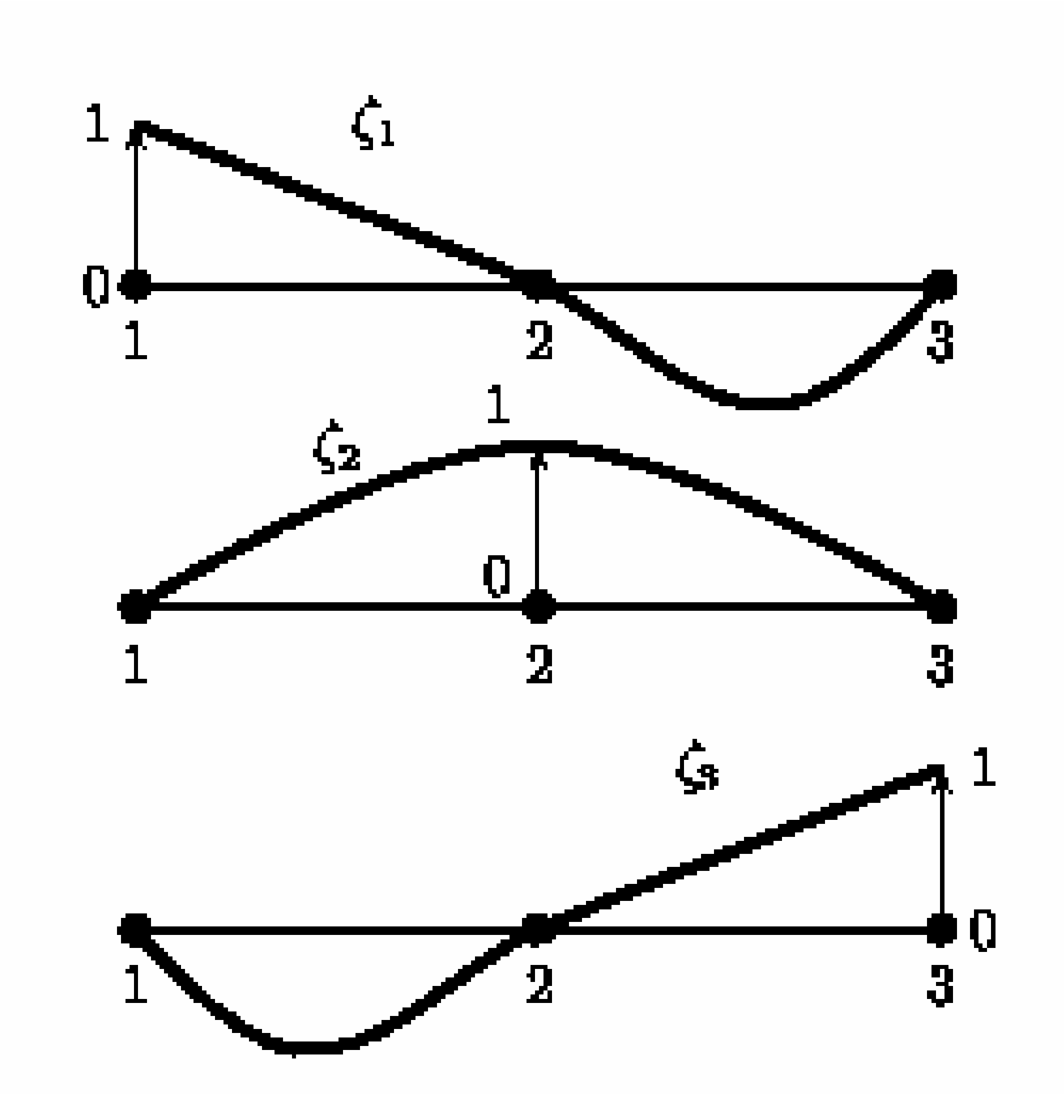
Substituting

$$\xi_1 = \frac{x(x-a)}{2a^2}, \quad \xi_2 = \frac{-(x+a)(x-a)}{a^2}, \quad \xi_3 = \frac{x(x+a)}{2a^2}$$

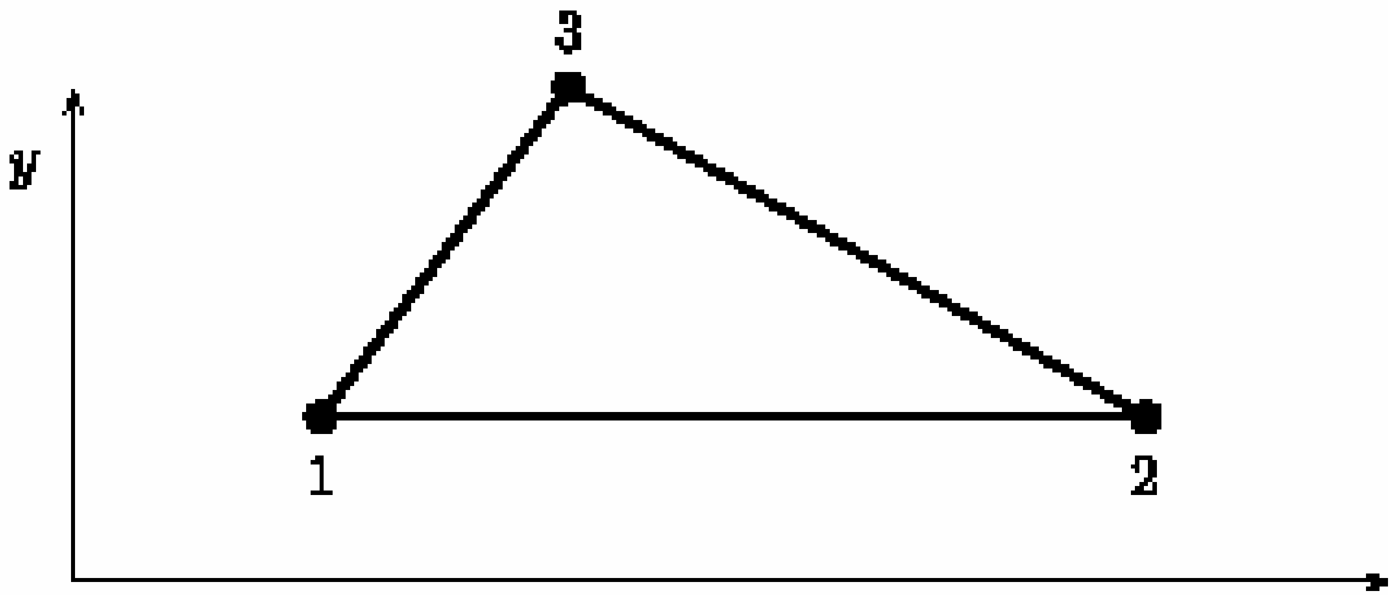
defining $\xi = \frac{(x-x_0)}{a}$

$$\xi_1 = -\frac{\xi}{2}(1-\xi), \quad \xi_2 = 1-\xi^2, \quad \xi_3 = \frac{\xi}{2}(1+\xi)$$

2^{nd} order 1D shape functions



2^D 1^{st} Order Elements



$$\phi = a_1 + a_2 x + a_3 y$$

$$\phi = [1, x, y]\{a\}$$

As before

$$\phi = \sum_{i=1}^{3} \xi_i \phi_i$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$[\phi_i] = [D]\{a\}$$

Generalized Coordinates are found

$$[a] = [D]^{-1}\{\phi_i\}$$

$$\phi = [1, x, y](D)^{-1}(\phi_i)$$

$$(D)^{-1} = \frac{1}{|D|}\begin{pmatrix} (x_2y_3 - x_3y_2) & -(y_3 - y_2) & (x_3 - x_2) \\ -(x_1y_3 - x_3y_1) & (y_3 - y_1) & -(x_3 - x_1) \\ (x_1y_2 - x_2y_1) & -(y_2 - y_1) & (x_2 - x_1) \end{pmatrix}^T$$

where the determinate is (2x area)

$$|D| = (x_2y_3 - x_3y_2) + x_1(y_2 - y_3) + y_1(x_3 - x_2)$$

In terms of new variables

$$
\begin{aligned}
p_1 &= (x_2 y_3 - x_3 y_2) \\
q_1 &= (y_2 - y_3) \\
r_1 &= (x_3 - x_2)
\end{aligned}
$$

and so on in cyclic form

we can show

$$
(D)^{-1} = \frac{1}{2\Delta} \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}
$$

Substituting

$$\phi = \frac{[1, x, y]}{2\Delta} \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

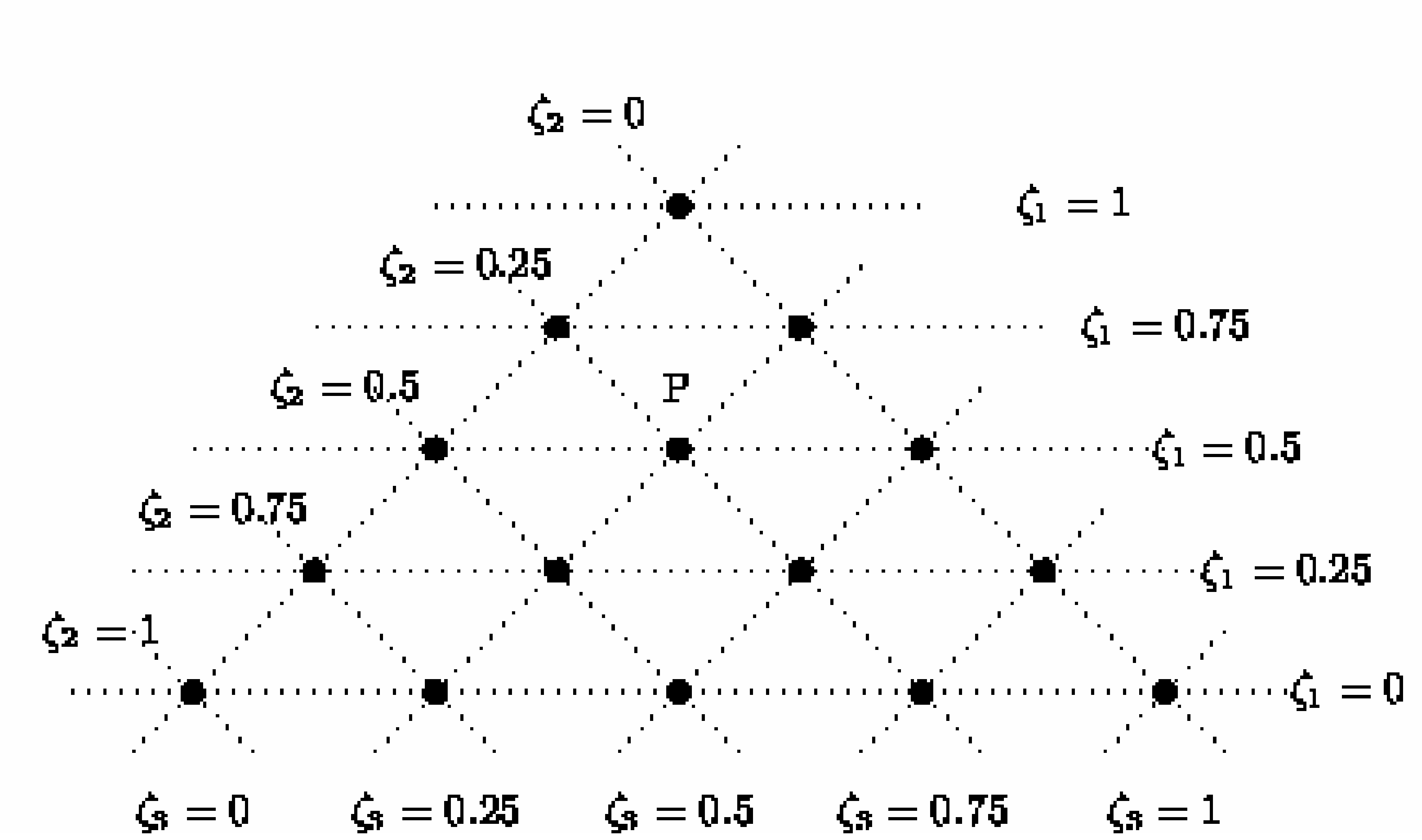
rearranging

$$\phi = \frac{(p_1 + q_1 x + r_1 y)\phi_1}{2\Delta} + \frac{(p_2 + q_2 x + r_2 y)\phi_2}{2\Delta} + \frac{(p_3 + q_3 x + r_3 y)\phi_3}{2\Delta}$$

We see that the shape function is

$$\zeta_i = \frac{p_i + q_i x + r_i y}{2\Delta}, \quad i = 1, 2, 3$$

These are interpolation functions



At point P

$$\zeta_1 = \frac{1}{2}, \quad \zeta_2 = \frac{1}{4}, \quad \zeta_3 = \frac{1}{4}$$

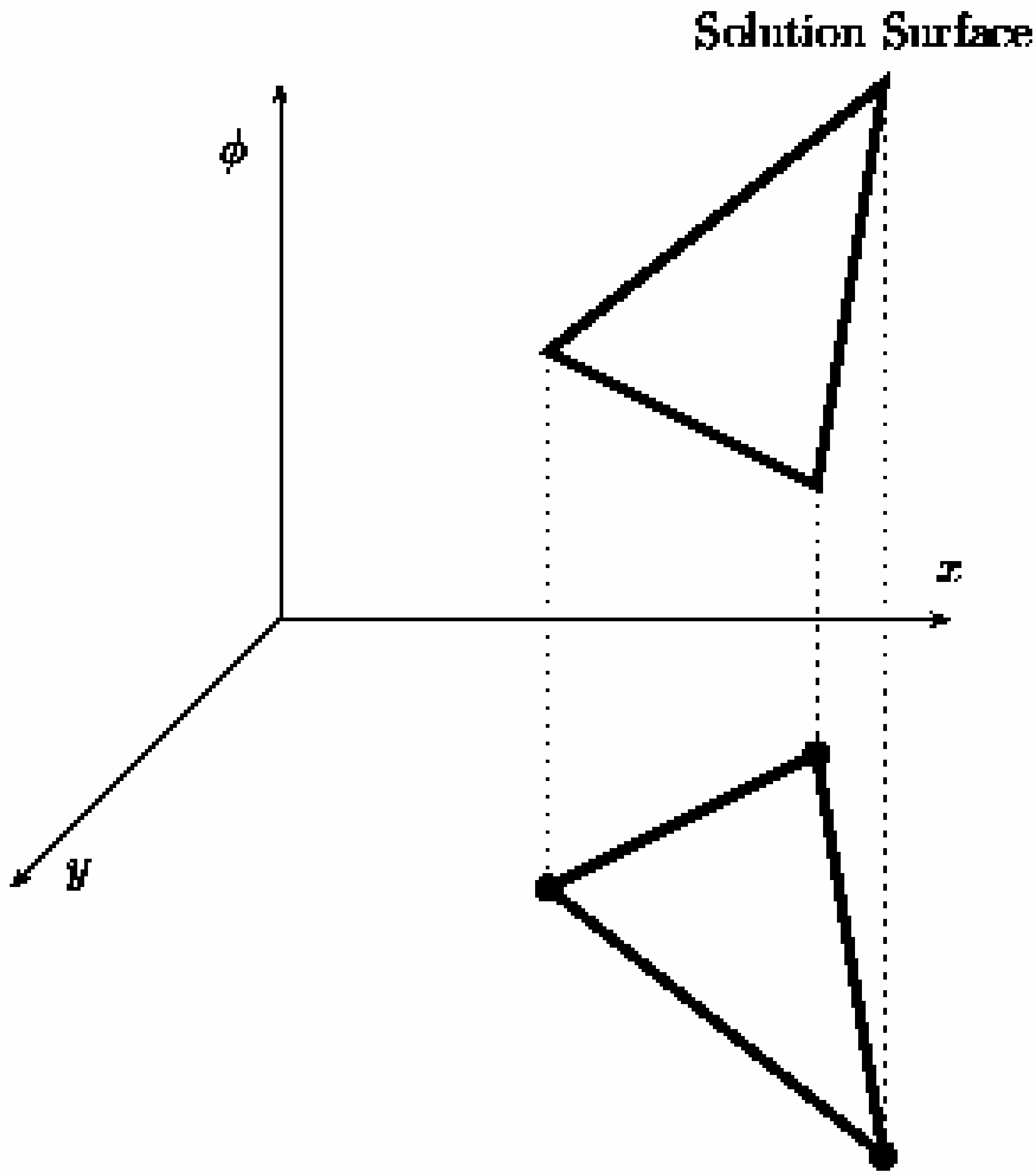
$$\sum \zeta_i = 1.$$

Natural Coordinates in 2D

$$x = \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_3$$

$$y = \zeta_1 y_1 + \zeta_2 y_2 + \zeta_3 y_3$$

$$\zeta_1 + \zeta_2 + \zeta_3 = 1$$


Solution Surface
ϕ
x
y

The equations are now

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}$$

using the same process

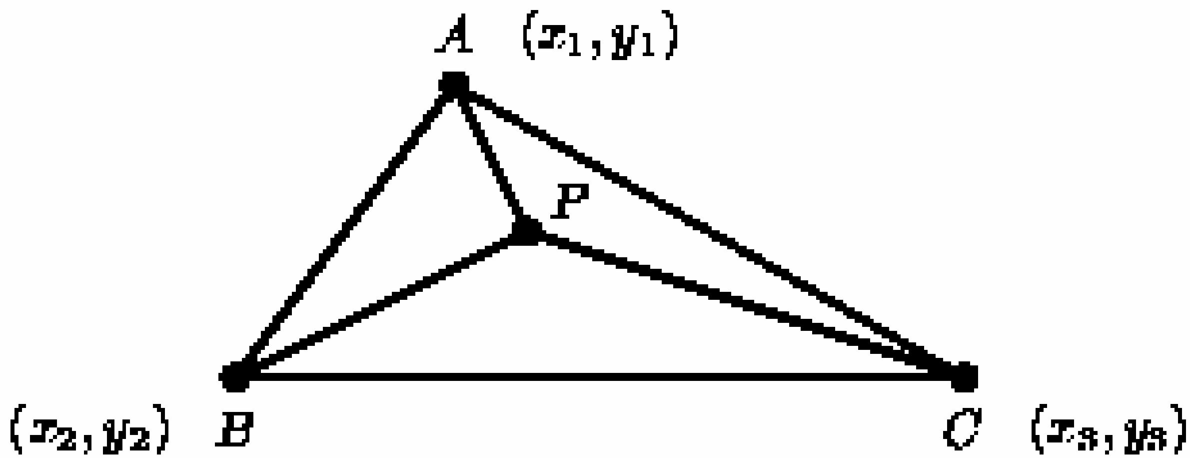
$$\begin{aligned} \zeta_1 &= \frac{1}{2\Delta}(a_1 + b_1 x + c_1 y) \\ \zeta_2 &= \frac{1}{2\Delta}(a_2 + b_2 x + c_2 y) \\ \zeta_3 &= \frac{1}{2\Delta}(a_3 + b_3 x + c_3 y) \end{aligned}$$

Natural Coordinates are the same as the shape function

$$\det = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} = 2\Delta$$

$a = (x_2 y_3 - x_3 y_2),\ b = (y_2 - y_3),\ c = (x_3 - x_2)$, and so on

Coordinate in terms of area



$$\frac{\text{area of } \Delta\ PBC}{\text{area of } \Delta\ ABC} = \frac{a_1 + b_1 x + c_1 y}{2\Delta}$$

$$\frac{\text{area of } \Delta\ PAC}{\text{area of } \Delta\ ABC} = \frac{a_2 + b_2 x + c_2 y}{2\Delta}$$

$$\frac{\text{area of } \Delta\ PBA}{\text{area of } \Delta\ ABC} = \frac{a_3 + b_3 x + c_3 y}{2\Delta}$$

Using the chain rule

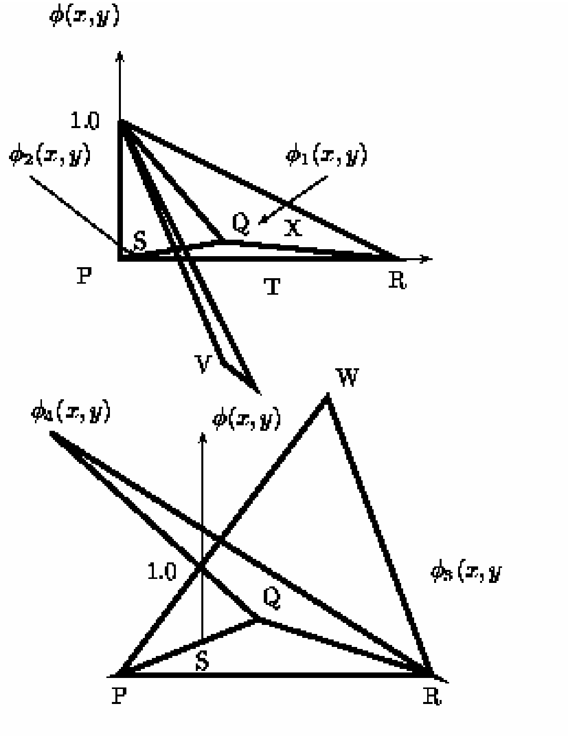
$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{\partial \phi}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial x} + \frac{\partial \phi}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial x} + \frac{\partial \phi}{\partial \zeta_3}\frac{\partial \zeta_3}{\partial x} \\
\frac{\partial \phi}{\partial y} &= \frac{\partial \phi}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial y} + \frac{\partial \phi}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial y} + \frac{\partial \phi}{\partial \zeta_3}\frac{\partial \zeta_3}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \zeta_i}{\partial x} &= b_i \\
\frac{\partial \zeta_i}{\partial y} &= c_i \quad \text{for} \;\; i = 1, 2, 3
\end{aligned}
$$

High Order Functions
Product of 2 linear terms

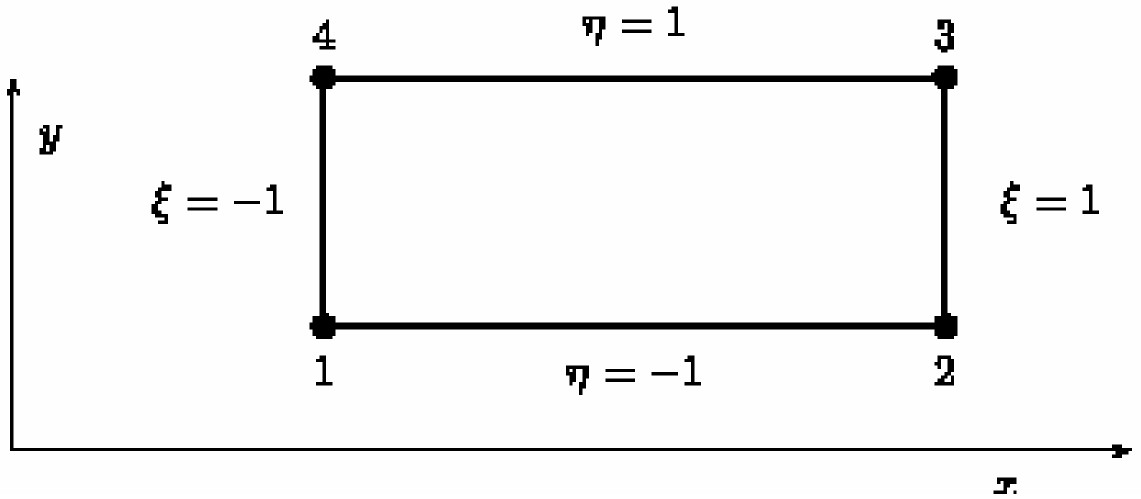
$$\phi(x, y) = \phi_1(x, y)\phi_2(x, y)$$

Second Order Shape Function



Rectangular Elements

$$\psi(\xi,\eta) = \zeta_1(\xi,\eta)\psi_1 + \zeta_2(\xi,\eta)\psi_2 + \zeta_3(\xi,\eta)\psi_3 + \zeta_4(\xi,\eta)\psi_4$$



$$\zeta_i(\xi,\eta) = L_i(\xi)L_i(\eta), \quad i = 1,2,3,4$$

Lagrange Interpolation gives

$$L_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2}, \quad L_2(\xi) = \frac{\xi - \xi_1}{\xi_2 - \xi_1}$$
$$L_1(\eta) = \frac{\eta - \eta_4}{\eta_1 - \eta_4}, \quad L_2(\eta) = \frac{\eta - \eta_3}{\eta_2 - \eta_3}$$

So $$\zeta_1(\xi, \eta) = L_1(\xi) L_1(\eta)$$

or $$\zeta_1 = \frac{(\xi - \xi_2)(\eta - \eta_4)}{(\xi_1 - \xi_2)(\eta_1 - \eta_4)}$$

For our rectangle

$$
\begin{aligned}
\xi_1 = -1, \quad & \eta_1 = -1 \\
\xi_2 = 1, \quad & \eta_2 = -1 \\
\xi_3 = 1, \quad & \eta_3 = 1 \\
\xi_4 = -1, \quad & \eta_4 = 1
\end{aligned}
$$

So

$$\zeta_1 = \frac{(\xi - 1)(\eta - 1)}{(-1 - 1)(-1 - 1)} = \frac{(\xi - 1)(\eta - 1)}{4}$$

Similarly

$$\zeta_2 = -\frac{(\xi + 1)(\eta - 1)}{4}$$

$$\zeta_3 = \frac{(\xi + 1)(\eta + 1)}{4}$$

$$\zeta_4 = -\frac{(\xi - 1)(\eta + 1)}{4}$$

Note that

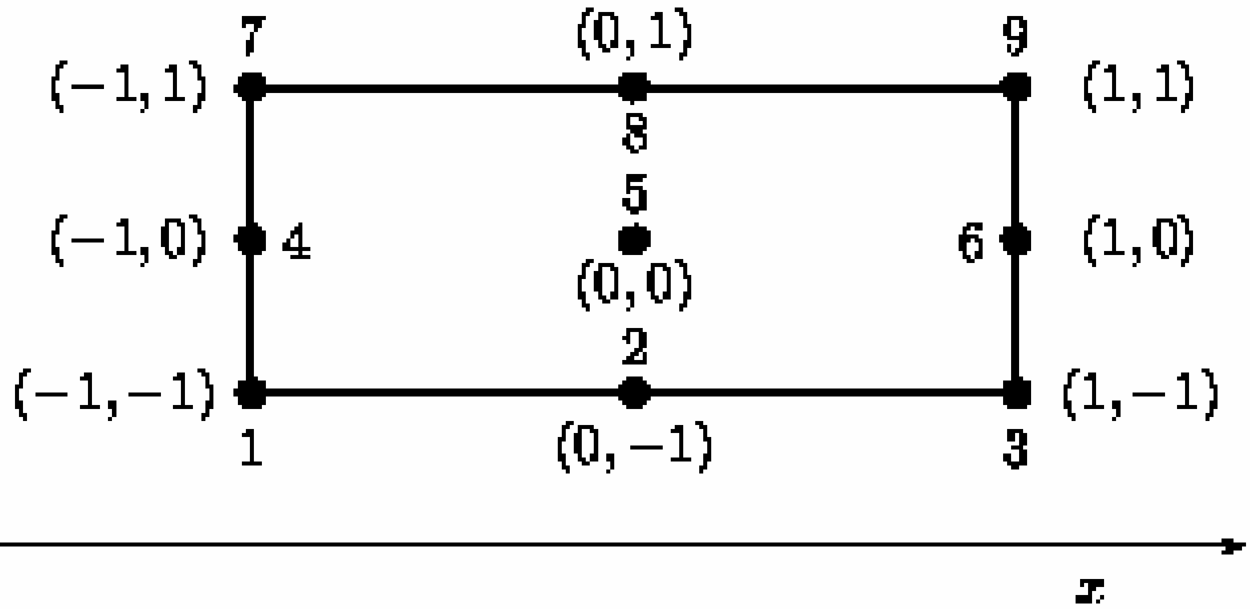
$$\zeta_1(\xi_1, \eta_1) = \frac{(-1-1)(-1-1)}{4} = 1$$

at node 2, $\xi_2 = 1,\ \eta_2 = -1,\ L_1(\xi_2) = 0.$

$$\zeta_1(\xi_2, \eta_2) = \frac{(1-1)(-1-1)}{4} = 0$$

and so forth at the other nodes

Second Order Rectangle



Bi-quadratic

$$\psi(\xi,\eta) = \sum_{j=1}^{N} \zeta_j(\xi_i,\eta_i)\psi_i, \quad -1 < \xi_i < 1, \ \ -1 < \eta_i < 1$$

as in the first order case

$$\zeta_j(\xi_i,\eta_i) = L_j(\xi_i)L_j(\eta_i)$$

Expanding

$$L_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} = \frac{\xi(\xi - 1)}{2}$$

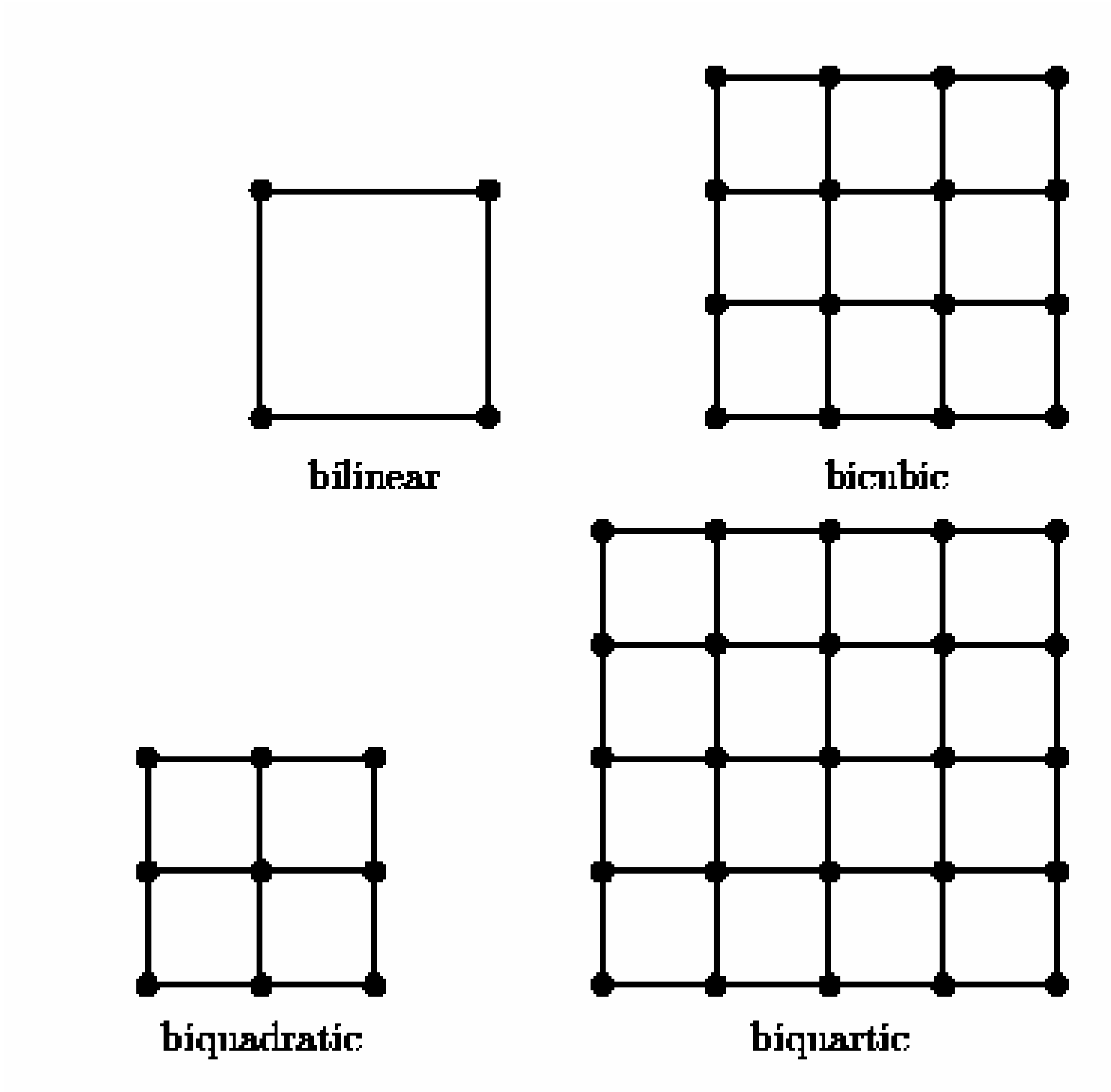
$$L_1(\eta) = \frac{(\eta - \eta_4)(\eta - \eta_7)}{(\eta_1 - \eta_4)(\eta_1 - \eta_7)} = \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)} = \frac{\eta(\eta - 1)}{2}$$

Therefore

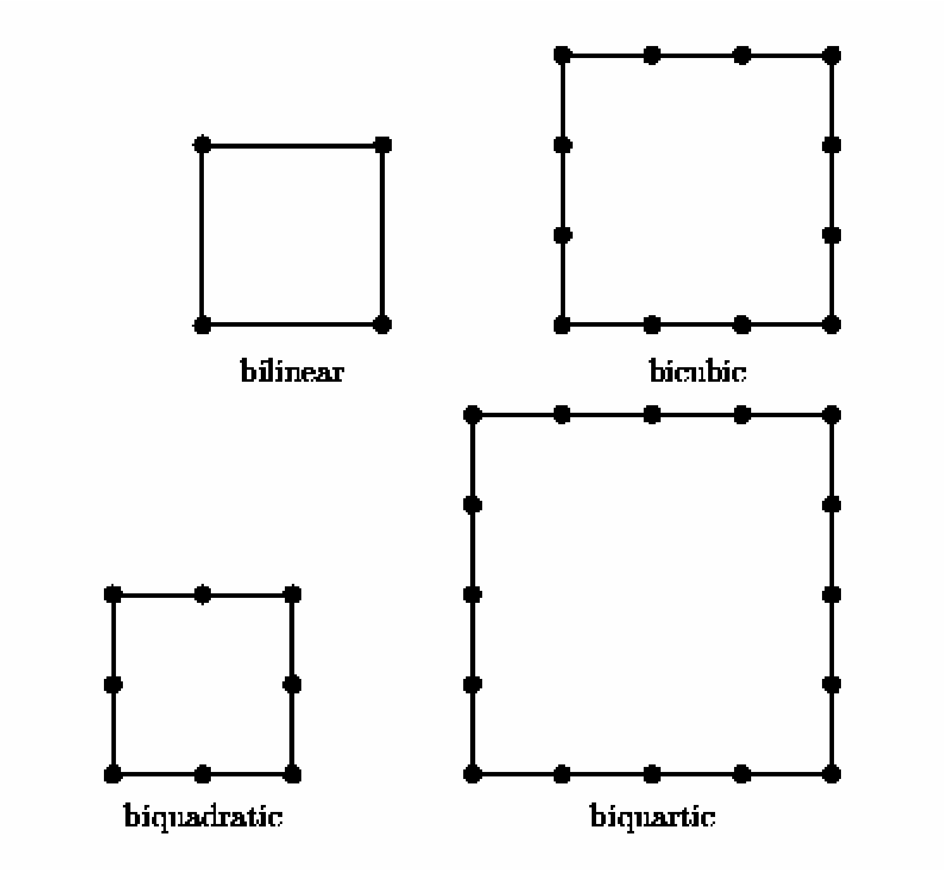
$$\zeta_1(\xi, \eta) = L_1(\xi)L_1(\eta) = \frac{\xi\eta(\xi - 1)(\eta - 1)}{4}$$

same for node 2

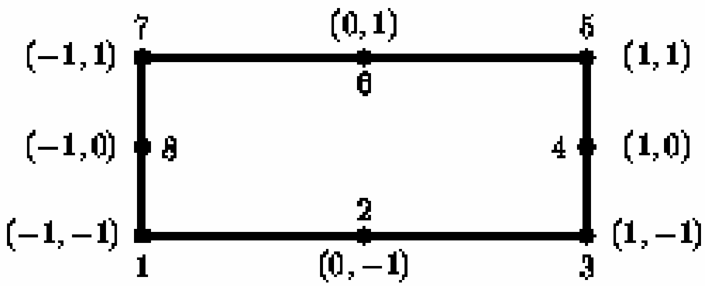
Rectangular Lagrange Elements



Serendipity Elements



Shape Function



4 corner nodes of rectangle

$$\zeta_k = \frac{(1+\xi_p\xi)(1+\eta_p\eta)}{4}$$

$\xi_p = 1$ for nodes 3 and 5,

$\xi_p = -1$ for nodes 1 and 7,

$\eta_p = 1$ for nodes 5

$\eta_p = -1$ for nodes 1 and 3.

for nodes 2 and 8

$$\zeta_2 = L_2(x)L_2(y)$$

$$L_2(y) = \frac{\eta - \eta_6}{\eta_2 - \eta_6}$$

$$L_2(y) = -\frac{(\eta - 1)}{2}$$

Similarly

$$L_2(x) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{(\xi + 1)(\xi - 1)}{(0 + 1)(0 - 1)} = -(\xi^2 - 1)$$

$$\zeta_2 = \frac{(\eta - 1)(\xi^2 - 1)}{2}$$

The shape functions at nodes 1 and 3 do not vanish at node 2

$$\zeta_1 = \frac{(1-\xi)(1-\eta)}{4} = \frac{1}{2}$$

$$\zeta_3 = \frac{(1+\xi)(1-\eta)}{4} = \frac{1}{2}$$

The shape functions at 5 and 7 do vanish at 2.
We therefore correct those at 1 and 3.

Therefore

$$\zeta_1 = \frac{(1-\xi)(1-\eta)}{4} - \frac{(\xi^2-1)(\eta-1)}{4}$$

After some algebra this becomes

$$\zeta_1 = -\frac{\xi(\xi-1)(\eta-1)}{4}$$

Also

$$\zeta_3 = -\frac{\xi(\xi+1)(\eta-1)}{4}$$

Now look at the shape function of node 8 (mid-side)

$$\zeta_8 = \frac{(\xi - \xi_4)(\eta - \eta_1)(\eta - \eta_7)}{(\xi_8 - \xi_4)(\eta_8 - \eta_1)(\eta_8 - \eta_7)} = \frac{(\xi - 1)(\eta^2 - 1)}{2}$$

The modified shape function at node 1 does not vanish at 8 so we define

$$\zeta_1(new) = -\frac{\xi(\xi - 1)(\eta - 1)}{4} - \frac{(\xi - 1)(\eta^2 - 1)}{4}$$

which becomes $$\zeta_1 = \frac{(\eta - 1)(\xi - 1)(-\xi - \eta - 1)}{4}$$

A similar process give us for node 3, 5 and 7

$$\zeta_3 = -\frac{(\xi + 1)(\eta - 1)(\xi - \eta - 1)}{4}$$

$$\zeta_5 = \frac{(\xi + 1)(\eta + 1)(\xi + \eta - 1)}{4}$$

$$\zeta_7 = \frac{(1 - \xi)(1 + \eta)(-\xi + \eta - 1)}{4}$$

3D elements (same)

tetrahedron

$$\phi = a_0 + a_1 x + a_2 y + a_3 z$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\phi = [1, x, y, z](D)^{-1}(\phi_i) = (\zeta_i)(\phi_i)$$

$$(\zeta_i) = [1, x, y, z](D)^{-1}$$

Derivatives and Integrals

$$\frac{\partial \phi}{\partial x, y, \text{ or } z} = \sum_{j=1}^{4} \frac{\partial \phi}{\partial \xi_j} \frac{\partial \xi_j}{\partial x, y, \text{ or } z} = \sum_{j=1}^{4} \frac{\partial \phi}{\partial \xi_j} \frac{b_i, \ c_i, \ \text{ or } d_i}{6V}$$

$$I = \frac{1}{V} \int_{vol} \xi_1^{\alpha} \xi_2^{\beta} \xi_3^{\gamma} \xi_4^{\theta} \, dV = \frac{\alpha! \beta! \gamma! \theta!}{(\alpha + \beta + \gamma + \theta + 3)!} 6V$$

Isoparametric Elements

$$
\begin{aligned}
\zeta_1 &= -\frac{\xi(1-\xi)}{2} \\
\zeta_2 &= 1-\xi^2 \\
\zeta_3 &= \frac{\xi(1+\xi)}{2}
\end{aligned}
$$

$$x = \sum \zeta_i x_i$$

$$\phi = \sum \zeta_i \phi_i$$

x can be described on an arc as

$$x = \frac{\xi(\xi-1)}{2}x_1 + (1-\xi^2)x_2 + \frac{\xi(\xi+1)}{2}x_3$$

to evaluate x at $\xi = -\frac{1}{2}$

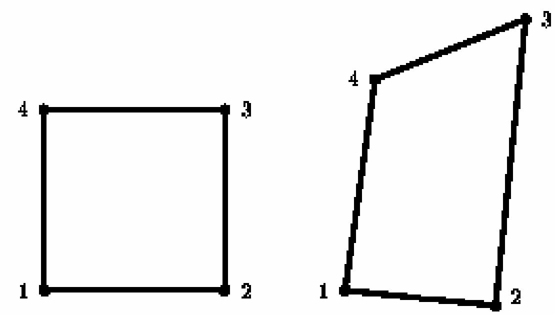
$$x = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-l)/2 - \left(\frac{1}{4}-1\right)(0) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\frac{l}{2} = -\frac{l}{2}$$

4 node rectangular element

$$\zeta_i = \frac{(1+\xi_i\xi)(1+\eta_i\eta)}{4}$$

$$x = \sum_{i=1}^{4} \zeta_i x_i$$

$$y = \sum_{i=1}^{4} \zeta_i y_i$$



(a) Parent element (b) Mapped element

This gives

$$\zeta_1 = \frac{1}{4}(1-0)(1-1) = 0$$

$$\zeta_2 = \frac{1}{4}(1+0)(1-1) = 0$$

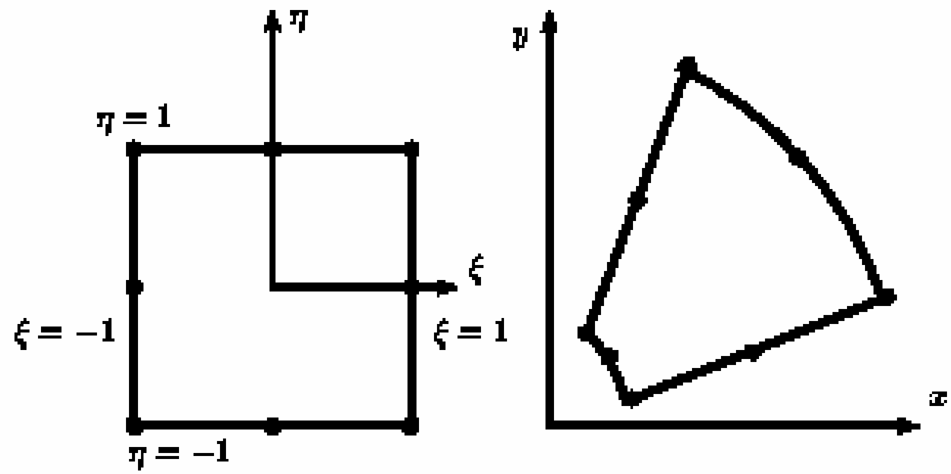
$$\zeta_3 = \frac{1}{4}(1+0)(1+1) = \frac{1}{2}$$

$$\zeta_4 = \frac{1}{4}(1-0)(1+1) = \frac{1}{2}$$

therefore

$$x = \frac{x_3 + x_4}{2}$$

$$y = \frac{y_3 + y_4}{2}$$



(a) Parent element (b) Mapped curved element

Two irregular or curved elements adjacently placed will preserve continuity if the shape functions of the parent elements from which they are generated satisfy interelement continuity.

If the interpolations functions expressed in local coordinates preserve continuity of the field variables in the parent element, the field variables in the global coordinate system in the irregular or curved elements will also be continuous.

If the completeness criterion is satisfied in the parent element, it will also be satisfied in the irregular or curved element.

2D Quadrilateral Element

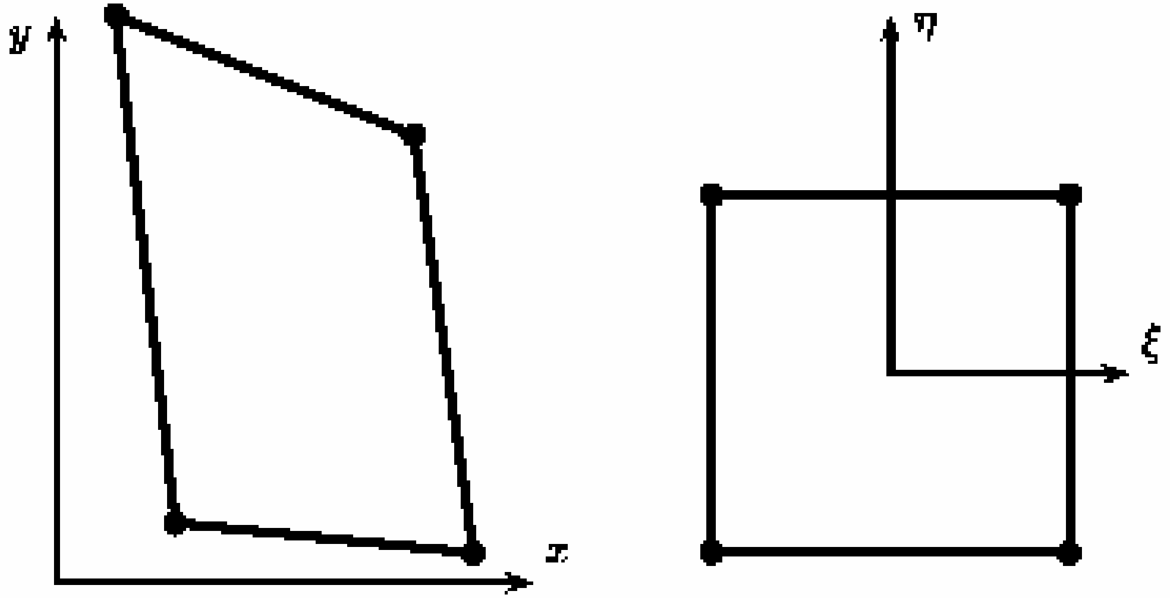
$$I = \int \left(\frac{\partial \phi}{\partial x} i \cdot \frac{\partial \phi}{\partial x} j + \frac{\partial \phi}{\partial y} i \cdot \frac{\partial \phi}{\partial y} j \right) \; dx \; dy$$

We can not evaluate the derivatives of the shape functions directly

$$\frac{\partial \phi}{\partial x} = \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial x} \phi_i$$

$$\frac{\partial \phi}{\partial y} = \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial y} \phi_i$$

Irregular Elements



(a) 2D element (b) Parent element

We find the derivatives with respect to local coordinated by the chain rule

$$\frac{\partial \zeta_i}{\partial \xi} = \frac{\partial \zeta_i}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial \zeta_i}{\partial y} \cdot \frac{\partial y}{\partial \xi}$$

$$\frac{\partial \zeta_i}{\partial \eta} = \frac{\partial \zeta_i}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial \zeta_i}{\partial y} \cdot \frac{\partial y}{\partial \eta}$$

or

$$\begin{pmatrix} \frac{\partial \zeta_i}{\partial \xi} \\ \frac{\partial \zeta_i}{\partial \eta} \end{pmatrix} = (J) \begin{pmatrix} \frac{\partial \zeta_i}{\partial x} \\ \frac{\partial \zeta_i}{\partial y} \end{pmatrix}$$

Using the Jacobian

$$(J) = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

then

$$\begin{pmatrix} \frac{\partial \zeta_i}{\partial x} \\ \frac{\partial \zeta_i}{\partial y} \end{pmatrix} = (J)^{-1} \begin{pmatrix} \frac{\partial \zeta_i}{\partial \xi} \\ \frac{\partial \zeta_i}{\partial \eta} \end{pmatrix}$$

and

$$dx\ dy = |J| d\xi\ d\eta$$

The integral then becomes

$$
\begin{aligned}
I &= \iint \left(\left(\frac{\partial \phi}{\partial x}\right)_i \left(\frac{\partial \phi}{\partial x}\right)_j + \left(\frac{\partial \phi}{\partial y}\right)_i \left(\frac{\partial \phi}{\partial y}\right)_j \right) \, dx \, dy \\
&= \iint \left(\sum \sum \left(\frac{\partial \zeta_i}{\partial x} \frac{\partial \zeta_j}{\partial x} + \frac{\partial \zeta_i}{\partial y} \frac{\partial \zeta_j}{\partial y} \right) \phi_j \right) \, dx \, dy
\end{aligned}
$$

The derivatives in global coordinates can be found from the parent element in local coordinates

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi, \eta)|J| \, d\xi \, d\eta$$

$$I = \sum\sum F(\xi, \eta) W_i W_j$$

Summary

- There is a general method to generate shape functions
- Same process in 1, 2 or 3D
- Can be used on curved elements
- May have internal nodes