Computational biophysics

Protein's geometry

$$\begin{aligned} \text{Centre of mass } \vec{R}_{cm} &= \frac{\sum\limits_{i=1}^{N} m_i \vec{r}_i}{\sum\limits_{i=1}^{N} m_i} \\ \text{Radius of gyration } r_g &= \sqrt{\frac{\sum\limits_{i=1}^{N} m_i (\vec{r}_i - \vec{R}_{cm})^2}{\sum\limits_{i=1}^{N} m_i}} \\ \text{Radius of gyration } r_g &= \sqrt{\frac{\sum\limits_{i=1}^{N} m_i (\vec{r}_i - \vec{R}_{cm})^2}{\sum\limits_{i=1}^{N} m_i}} \\ B_i &= \frac{8\pi^2}{3} RMSF_i^2 \end{aligned}$$

Semi-empirical force fields

Bond stretching

Harmonic $U(r_{AB}) = \frac{1}{2}k_{AB}(r_{AB} - r_{AB,eq})^2$
Anarmonic $U(r_{AB}) = \frac{1}{2} \left[k_{AB} + k_{AB}^{(3)}(r_{AB} - r_{AB,eq}) \right] (r_{AB} - r_{AB,eq})^2$
Quartic correction $U(r_{AB}) = \frac{1}{2} \left[k_{AB} + k_{AB}^{(3)}(r_{AB} - r_{AB,eq}) + k_{AB}^{(4)}(r_{AB} - r_{AB,eq})^2 \right].$
$\cdot (r_{AB} - r_{AB,eq})^2$
Morse $U(r_{AB}) = D_{AB} \left[1 - e^{-\alpha_{AB}(r_{AB} - r_{AB,eq}^2)} \right]$

-Valence angle bending

Potential
$$U(\theta_{ABC}) = \frac{1}{2} [k_{ABC} + k_{ABC}^{(3)}(\theta_{ABC} - \theta_{ABC,eq}) + k_{ABC}^{(4)}(\theta_{ABC} - \theta_{ABC,eq})^2 + \cdots] (\theta_{ABC} - \theta_{ABC,eq})^2$$

$$U(\theta_{ABC}) = \sum_{\substack{\{j\}_{ABC}\\j,ABC}} k_{j,ABC}^{fourier} [1 + \cos(j\theta_{ABC} + \psi_j)]$$
Fourier
$$k_{j,ABC}^{fourier} = \frac{2k_{ABC}^{harmonic}}{j^2}$$

Potential
$$U(\omega_{ABCD}) = \frac{1}{2} \sum_{\{j\}_{ABCD}} V_{j,ABCD} \left[1 + (-1)^{j+1} \cos(j\omega_{ABCD} + \psi_{j,ABCD}) \right]$$

Improper $U(\omega_{ABCD}) = \frac{1}{2} \sum_{\{j\}_{ABCD}} V_{j,ABCD} \left[1 + (-1)^{j+1} \cos(j\omega_{ABCD} + \psi_{j,ABCD}) \right]$

Van der Waals

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Lennard-Jones U(r_{AB}) = 4\epsilon_{AB} \left[ \left( \frac{\sigma_{AB}}{r_{AB}} \right)^{12} - \left( \frac{\sigma_{AB}}{r_{AB}} \right)^{6} \right]
Morse U(r_{AB}) = D_{AB} \left[ 1 - e^{-\alpha_{AB}(r_{AB} - r_{AB,eq}^2)} \right]^2
\text{Hill } U(r_{AB}) = \epsilon \left[ \frac{6}{\beta_{AB} - 6} e^{\beta_{AB} \frac{1 - r_{AB}}{r_{AB}^*}} - \frac{\beta_{AB}}{\beta_{AB} - 6} \left( \frac{r_{AB}^*}{r_{AB}^*} \right)^6 \right]
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Electrostatic interactions

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Distribution of charges U_{AB} = \sum_{A} \sum_{B \geq A} \vec{M}^{(A)} \vec{V}^{(B)}
Point like U_{AB} = \frac{q_A q_B}{\epsilon_{AB} r_{AB}}
Dipolar interactions U_{AB/CD} = \frac{\mu_{AB} \mu_{CD}}{\epsilon_{AB/CD} r_{AB/CD}^3} (\cos \chi_{AB/CD} - 3\cos \alpha_{AB}\cos \alpha_{CD})
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Parameterization

Parameters
$$Z = \sqrt{\sum_{i}^{observables \ occurrences} \sum_{j}^{(calc_{i,j} - expt_{i,j})^2} \omega_i^2}$$
 $\sigma_{AB} = \sigma_A + \sigma_B$
 $\epsilon_{AB} = \sqrt{\epsilon_A \epsilon_B}$

Classical mechanics

Newton's laws

$$\vec{F} = m\vec{a} \qquad \vec{F}_{BA} = -\vec{F}_{AB}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \qquad m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$
Force acting on atom $\vec{F}_i(\vec{r}_1, \dots, \vec{r}_N, \vec{r}_i) = \sum_{j \neq i} \vec{F}_{ij}(\vec{r}_i - \vec{r}_j) + \vec{F}^{(ext)}(\vec{r}_i, \dot{\vec{r}}_i)$
Bond stretching: $U = \frac{k_t}{2}(l - l^0)^2$
Bond bending: $U = \frac{k_\theta}{2}(\theta - \theta^0)^2$
Bond torsion: $U = k_\phi[1 + \cos(n\phi - \phi^0)]$
Van der Waals interactions: $U = \begin{bmatrix} a_{ij} \\ r_{ij}^{12} - \frac{b_{ij}}{r_{ij}^0} \end{bmatrix}$
Electrostatic interactions: $U = \frac{332q_iq_j}{q_j}$

Lagrangian formulation

 $ec{p}_i = m_i ec{v}_i = m \dot{ec{r}}_i \qquad ec{F}_i = m_i \ddot{ec{r}}_i = \dot{ec{p}}_i$

 $\vec{x}(t) = \{\vec{r}_1(t), \dots, \vec{r}_N(t), \vec{p}_1(t), \dots, \vec{p}_N(t)\}$

$$\begin{split} \vec{F}_i(\vec{r}_1,\ldots,\vec{r}_N) &= -\Delta_i U(\vec{r}_1,\ldots,\vec{r}_N) \\ W_{AB} &= \int_A^B \vec{F}_i d\vec{l} = U_A - U_B = -\Delta U_{AB} \qquad \oint \vec{F}_i d\vec{l} = 0 \\ \text{Kinetic energy } K(\dot{\vec{r}}_1,\ldots,\dot{\vec{r}}_N) &= \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^2 \\ \mathcal{L}(\vec{r}_1,\ldots,\vec{r}_N,\dot{\vec{r}}_1,\ldots,\dot{\vec{r}}_N) &= K(\dot{\vec{r}}_1,\dot{\vec{r}}_N) - U(\vec{r}_1,\ldots,\vec{r}_N) \\ \text{Euler-Lagrange } \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \right) - \frac{\partial \mathcal{L}}{\partial \ddot{\vec{r}}_i} &= 0 \\ E &= \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^2 + U(\vec{r}_1,\ldots,\vec{r}_N) \end{split}$$

 $quad\frac{dE}{dt} = 0$

$q_{\alpha} = f_{\alpha}(\vec{r_1}, \dots, \vec{r_N})$ $\alpha = 1, \dots, 3N$ $\vec{r_i} = \vec{g_i}(q_1, d, \dots, q_{3N})$ $i = 1, \dots, N$

$$q_{\alpha} = f_{\alpha}(\vec{r_1}, \dots, \vec{r_N}) \qquad \alpha = 1, \dots, 3N \qquad \vec{r_i} = \vec{g_i}(q_1, d \dots, q_{3N}) \qquad i = 1, \dots, N$$

$$\dot{\vec{r_i}} = \sum_{\alpha=1}^{3N} \frac{\partial \vec{r_i}}{\partial q_{\alpha}} \dot{q_{\alpha}} \qquad \mathcal{L}(q, \dot{q}) = \frac{1}{2} \sum_{\alpha=1}^{3N} \sum_{\beta=1}^{3N} G_{\alpha\beta} \dot{q_{\alpha}} \dot{q_{\beta}} - U(q_1, \dots, q_{3N})$$

Classical mechanics (contd)

Legendre transforms $s = f'(x) \equiv g(x) \qquad f'(x) = g(x) = s \Rightarrow x = g^{-1}(s)$ $b(g^{-1}(s)) = f(g^{-1}(s)) - sg^{-1}(s) \equiv \tilde{f}(s) = f(x(s)) - sx(s)$ $\tilde{f}(s_1, \dots, s_n) = f(x_1(s_1, \dots, s_n), \dots x_n(s_1, \dots, s_n)) - \sum s_i x_i(s_1, \dots, s_n)$

Hamiltonian formulation

$$\mathcal{H}(\vec{r}_{1},\ldots,\vec{r}_{N},\vec{p}_{1},\ldots,\vec{p}_{N}) = -\tilde{\mathcal{L}}(\vec{r}_{1},\ldots,\vec{r}_{N},\vec{p}_{1},\ldots,\vec{p}_{N})$$

$$\mathcal{H}(\vec{r}_{1},\ldots,\vec{r}_{N},\vec{p}_{1},\ldots,\vec{p}_{N}) = \sum_{i=1}^{N} \frac{\vec{p}_{i}^{2}}{2m_{i}} + U(\vec{r}_{1},\ldots,\vec{r}_{N})$$

$$\mathcal{H}(q_{1},\ldots,q_{3N},p_{1},\ldots,p_{3N}) = \frac{1}{2} \sum_{\alpha} \sum_{\beta} p_{\alpha} G_{\alpha\beta}^{-1} p_{\beta} + U(q_{1},\ldots,q_{3N})$$
Hamilton equations $\dot{q}_{\alpha} = \frac{\partial \mathcal{H}}{\partial p_{\alpha}} \qquad \dot{p}_{\alpha} = -\frac{\partial \mathcal{H}}{\partial q_{\alpha}} \qquad \frac{\mathcal{H}}{dt} = 0 \qquad \mathcal{H} = const$

Some properties

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Conservation laws \frac{da}{dt} = \frac{\partial a}{\partial x_t} \dot{x}(t) = \{a, \mathcal{H}\} = 0
Incompressibility \nabla_x \dot{(}x) = 0
Symplectic structure M = J^T M J J_{kl} = \frac{\partial x_k(t)}{\partial x_l(0)}
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Theoretical foundations of statistical mechanics

Thermodynamics

Equilibrium $g(N, P, V, T) = 0$	First law $\Delta E = \Delta Q + \Delta W$
State function $f(n, P, V, T)$	Entropy $\Delta S = \int_{1}^{2} \frac{dQ_{rev}}{T}$
Reversible work $dW_{rev} = -PdV + \mu dN$ Heat $dQ_{rev} = CdT$	

Average
$$A = \frac{1}{Z} \sum_{\lambda=1}^{N} a(x_{\lambda}) \equiv \langle a \rangle$$

Microstate $x_0 = (q_1(0), \dots, q_{3N}(0), p_1(0), \dots, p_{3N}(0))$
Phase space volume $dx_t = J(x_t; x_0) dx_0$ $\frac{dJ}{dt} = 0 \Rightarrow J(x_t; x_0) = 1 \Rightarrow dx_t = dx_0$
 $f(x_t) : \int f(x) dx = 1 \land \frac{df(x_t, t)}{dt} = 0 \Rightarrow$
Distribution function $f(x_t, t) dx_t = f(x_0, 0) dx_0 \Rightarrow$
 $\frac{\partial f(x, t)}{\partial t} + \{f(x, t), \mathcal{H}(x, t)\} = 0$
Equilibrium $A = \int a(x) f(x, t) dx \Rightarrow \frac{\partial f(x, t)}{\partial t} = 0 \land \{f(x, t), \mathcal{H}(x, t) = 0\} \Rightarrow$
 $f(x) \propto \mathcal{F}(\mathcal{H}(x))$

Microcanonical ensemble

State and distribution function

 $Z = \int dx \mathcal{F}(\mathcal{H}(x)) \Rightarrow f(x) = \frac{1}{Z} \mathcal{F}(\mathcal{H}(x))$

State function
$$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$$

$$\left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{1}{T} \qquad \left(\frac{\partial S}{\partial V}\right)_{N,E} = \frac{P}{T} \qquad \left(\frac{\partial S}{\partial N}\right)_{V,N} = \frac{\mu}{T}$$
Boltzmann relation $S(N,V,E) = k \ln \omega(N,V,E)$

$$\Omega(N,V,E) = M_N \int d\vec{p} \int_{D(V)} d\vec{r} \delta(\mathcal{H}(\vec{r},\vec{p}) - E)$$
Distribution function
$$= M_N \int dx \delta(\mathcal{H}(x) - E)$$

$$M_N = \frac{E_0}{N!h^{3N}}$$

$$A = \langle a \rangle = \frac{M_N}{\Omega(N, V, E)} \int dx a(x) \delta(\mathcal{H}(x) - E) = \frac{\int dx a(x) \delta(\mathcal{H}(x) - E)}{\int dx \delta(\mathcal{H} - E)}$$
$$\left\langle x_i \frac{\partial \mathcal{H}}{\partial x_j} \right\rangle = \frac{M_N}{\Omega(N, V, E)} \frac{\partial}{\partial E} \int_{\mathcal{H}(x) < E} dx x_i \frac{\partial (\mathcal{H} - E)}{\partial x_j}$$

-Virial theorem

$$\left\langle x_{i} \frac{\partial \mathcal{H}}{\partial x_{j}} \right\rangle = \delta_{ij} \frac{\sum (E)}{\frac{\partial \Sigma(E)}{\partial E}}$$

$$\Sigma(N, V, E) = \frac{1}{N!h^{3N}} \int dx \theta(E - \mathcal{H})$$

$$\Omega(N, V, E) = E_{0} \frac{\partial \Sigma(N, V, E)}{\partial E} \left\langle x_{i} \frac{\partial \mathcal{H}}{\partial x_{j}} \right\rangle = \delta_{ij} \left(\frac{\ln \Sigma(E)}{\partial E} \right)^{-1}$$

$$S(N, V, E) = k \ln \Omega(N, V, E) \simeq k \ln \Sigma(N, V, E) = \tilde{S}(N, V, E)$$

$$\left\langle x_{i} \frac{\partial \mathcal{H}}{\partial x_{j}} \right\rangle \simeq \delta_{ij} \left(\frac{S(E)}{\partial E} \right)^{-1} = kT \delta_{ij}$$

$$\frac{\Omega(N, V, E) = M_N \int dx \delta(\mathcal{H}_1(x_1) + \mathcal{H}_2(x_2) - E)}{\Omega(N, V, E) = \int dE_1 \Omega_1(N_1, V_1, E_1) \Omega_2(N_2, V_2, E - E_1)}
S(N, V, E) = k \ln \Omega_1(N_1, V_1, \bar{E}_1) + k \ln \Omega_2(N_2, V_2, E - \bar{E}_1)
= S_1(N_1, V_1, \bar{E}_1) + S_2(N_2, V_2, E - \bar{E}_1)
T_1 = T_2$$

Introduction to molecular dynamics

Verlet algorithm

$$\begin{split} \vec{r}_i(t+\Delta t) &= 2\vec{r}_i(t) - \vec{r}_i(t-\Delta t) + \frac{\Delta t^2}{m_i}\vec{F}_i(t) \\ \vec{v}_i(t+\Delta t) &= \vec{v}_i(t) + \frac{\Delta t}{2m_i}\left[\vec{F}_i(t) + \vec{F}_i(t+\Delta t)\right] \\ \text{Initial conditions } f(v) &= \sqrt{\frac{m}{2\pi kT}}e^{-\frac{mv^2}{2kT}} \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}} \end{split}$$

Introduction to molecular dynamics (contd)

Action integral
$$Q = \{q_1, \dots, q_{3N}\} \qquad \dot{Q} = \{\dot{q}_1, \dots, \dot{q}_{3N}\}$$

$$A[Q] = \int_{t_1}^{t^2} \mathcal{L}(Q(t), \dot{Q}(t)) dt$$

$$\delta Q(t_1) = \delta Q(t_2) = 0 \qquad \delta \dot{Q}(t_1) = \delta \dot{Q}(t_2) = 0$$

$$\delta A = \int_{\alpha=1}^{3N} \frac{\partial \mathcal{L}}{\partial \dot{q}_{\alpha}} \delta q_{\alpha}(t) \Big|_{t_1}^{t_2} dt + \int_{t_1}^{t_2} \sum_{\alpha=1}^{3N} \left[\frac{\partial \mathcal{L}}{\partial q_{\alpha}} \delta q_{\alpha}(t) - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{\alpha}} \right) \delta q_{\alpha}(t) \right] dt = 0$$

$$\begin{split} \sum_{\alpha=1}^{3N} a_{k\alpha} dq_{\alpha} + a_{kt} dt &= 0, k = 1, \dots, N_{C} \\ \text{Holomonic } a_{k\alpha} &= \frac{\partial \sigma_{k}}{\partial q_{\alpha}} \qquad a_{kt} = \frac{\partial \sigma_{k}}{\partial t} \\ \frac{1}{2} \sum_{i} m_{i} \dot{\vec{r}}_{i}^{2} - C &= 0 \Rightarrow \frac{1}{2} \sum_{i} m_{i} \dot{\vec{r}}_{i} d\vec{r}_{i} - C dt = 0 \\ \text{Non-holonomic} \\ &\Rightarrow a_{1i} &= \frac{1}{2} m_{i} \dot{\vec{r}}_{i} \wedge a_{1t} = -C \\ \text{Lagrange multiplier } \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial q_{\alpha}} = \sum_{k=1}^{N_{C}} \lambda_{k} a_{k\alpha} \end{split}$$

$$\dot{q}_{\alpha} = \frac{\partial \mathcal{H}}{\partial p_{\alpha}} \qquad \dot{p}_{\alpha} = -\frac{\partial \mathcal{H}}{\partial q_{\alpha}} - \sum_{k=1}^{N_{C}} \lambda_{k} a_{k\alpha} \qquad \sum_{\alpha=1}^{3N} a_{k\alpha} \frac{\partial \mathcal{H}}{\partial p_{\alpha}} = 0$$
Simulation $m_{i}\ddot{\vec{r}}_{i} = \vec{F}_{i} + \sum_{k=1}^{N_{C}} \lambda_{k} \nabla_{i} \sigma_{k} \qquad \dot{\sigma}_{k} = \sum_{i=1}^{N} \nabla_{i} \sigma_{k} \cdot \dot{\vec{r}}_{i} = 0$

Velocity Verlet
$$\vec{r}_i(\Delta t) = \vec{r}_i(0) + \Delta t \vec{v}_i(0) + \frac{\Delta t^2}{2m_i} \vec{F}_i(0) + \frac{\Delta t^2}{2m_i} \sum_k \lambda_k \nabla_i \sigma_k(0)$$

$$\vec{r}_i(\Delta t) = \vec{r}_i + \frac{1}{m_i} \sum_k \tilde{\lambda}_k \nabla_i \sigma_k(0)$$
 $\tilde{\lambda}_k = \frac{\Delta t^2}{2} \lambda_k$

$$\sigma_{l}\left(\vec{r}_{1}^{(1)}, \dots, \vec{r}_{N}^{(1)}\right) + \sum_{i=1}^{N} \sum_{k=1}^{N_{C}} \frac{1}{m_{i}} \nabla_{i} \sigma_{k}\left(\vec{r}_{1}^{(1)}, \dots, \vec{r}_{N}^{(1)}\right) \cdot \nabla_{i} \sigma_{k}\left(\vec{r}_{1}(0), \dots, \vec{r}_{N}(0)\right) \delta \tilde{\lambda}_{k} \approx 0$$

Direct translation

Liouville operator

Computable on
$$a: \frac{da}{dt} = \{a, \mathcal{H}\}\$$

$$iL = \sum_{\alpha} \left[\frac{\partial \mathcal{H}}{\partial q_{\alpha}} \frac{\partial}{\partial q_{\alpha}} - \frac{\partial \mathcal{H}}{\partial q_{\alpha}} \frac{\partial}{\partial p_{\alpha}} \right] \Rightarrow iLa = \{a, \mathcal{H}\} \Rightarrow \frac{da}{dt} = iLa \Rightarrow a(x_{t}) = e^{iLt}a(x_{0})$$
Split $iL_{1} = \sum_{\alpha} \frac{\partial \mathcal{H}}{\partial p_{\alpha}} \frac{\partial}{\partial q_{\alpha}} \qquad iL_{2} = -\sum_{\alpha} \frac{\partial \mathcal{H}}{\partial q_{\alpha}} \frac{\partial}{\partial p_{\alpha}}$

$$iL_{1}iL_{2}\phi(x) \neq iL_{2}iL_{1}\phi(x) \Rightarrow iL_{1}iL_{2} - iL_{2}iL_{1} \equiv [iL_{1}, iL_{2}] \neq 0$$

$$[iL_1, iL_2] \neq 0 \Rightarrow e^{iLt} \neq e^{iL_1t}e^{iL_2t}$$

$$e^{A+B} = \lim_{P \to \infty} \left[e^{\frac{B}{2P}} e^{\frac{A}{P}} e^{\frac{B}{2P}} \right]^P \qquad e^{iLt} = \lim_{P \to \infty} \left[e^{\frac{iL_2t}{2P}} e^{\frac{iL_1t}{P}} e^{\frac{iL_2t}{2P}} \right]^P$$

$$e^{iLt} \approx e^{\frac{iL_2\Delta t}{2}} e^{iL_1\Delta t} e^{\frac{iL_2\Delta t}{2}}$$

Trotter algorithm

Exponential operator
$$e^{c\frac{\partial}{\partial x}}g(x) = g(x+c)$$

$$\begin{pmatrix} x(\Delta t) \\ p(\Delta t) \end{pmatrix} = \begin{pmatrix} x(0) + \frac{\Delta t}{m} \left(p(0) + \frac{\Delta t}{2} F(x(0)) \right) \\ p(0) + \frac{\Delta t}{2} F(x(0)) + \frac{\Delta t}{2} F\left(x(0) + \frac{\Delta t}{m} \left(p(0) + \frac{\Delta t}{2} F(x(0)) \right) \right) \end{pmatrix}$$

$$x(\Delta t) = x(0) + v(0)\Delta t + \frac{\Delta t^2}{2m} F(0) \qquad p(\Delta t) = v(0) + \frac{\Delta t}{2m} \left[F(0) + F(\Delta t) \right]$$

-RESPA-

$$iL = \frac{p}{m} \frac{\partial}{\partial x} + [F_{fast}(x) + F_{slow}(x)] \frac{\partial}{\partial p} = iL_{fast} + iL_{slow} \qquad \mathcal{H}_{ref} = \frac{p^2}{2m} + U_{fast}(x)$$

$$e^{iL\Delta t} = e^{iL_{slow}} \frac{\Delta t}{2} e^{iL_{fast}\Delta t} e^{iL_{slow}} \frac{\Delta t}{2}$$

$$e^{iL_{fast}\Delta t} = \left[e^{\frac{\delta t}{2}F_{fast}} \frac{\partial}{\partial p} e^{\delta t} \frac{p}{m} \frac{\partial}{\partial x} e^{\frac{\delta t}{2}F_{fast}} \frac{\partial}{\partial p} \right]^n \qquad \delta t = \frac{\Delta t}{n}$$

Evaluation of energy and forces

Periodic boundary condition

Error function
$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$
 $\lim_{x \to \infty} erf(x) = 1$ $erf(0) = 0$

Complement error $erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$ $\lim_{x \to \infty} erf(x) = 0$

$$erf(0) = 1$$

$$\lim_{\text{short-ranged}} \lim_{\text{long-ranged}} \lim_{\text{long-ranged}} \frac{1}{r} = \underbrace{\frac{erfc(\alpha r)}{r}}_{r} + \underbrace{\frac{erf(\alpha r)}{r}}_{r} \qquad U_{nb} = U_{short} + U_{long} \qquad \vec{r}_{ij} = |\vec{r}_i - \vec{r}_j + \vec{S}| \qquad \vec{S} = \vec{m}L$$

$$U_{short}(\vec{r}_1, \dots, \vec{r}_N) = \sum_{\vec{S}} \sum_{i > j \in nb} \left\{ 4\epsilon_{ij} \left[\left(\frac{\sigma_{ij}}{r_{ij,\vec{S}}} \right)^{12} - \left(\frac{\sigma_{ij}}{r_{ij,\vec{S}}} \right)^6 \right] + \underbrace{\frac{q_i q_j erfc(\alpha r_{ij,\vec{S}})}{r_{ij,\vec{S}}}} \right\}$$

$$U_{long}(\vec{r}_1, \dots, \vec{r}_N) = \sum_{\vec{S}} \sum_{i > h \in nb} \frac{q_i q_j erf(\alpha r_{ij,\vec{S}})}{r_{ij,\vec{S}}}$$

Non bonded interaction $U_{nb}(\vec{r}_1, \dots, \vec{r}_N) = \sum_{i < j \in nb} \left\{ 4\epsilon_{ij} \left[\left(\frac{\sigma_{ij}}{r_{ij}} \right)^{12} - \left(\frac{\sigma_{ij}}{r_{ij}} \right)^6 \right] + \frac{q_i q_j}{r_{ij}} \right\}$

Short range forces

$$\tilde{U}_{short} = U_{short}(\vec{r}_{ij})S(\vec{r}_{ij}) \qquad S(r) = \begin{cases} 1 & r < r_C - \lambda \\ 1 + \left(\frac{r - r_C + \lambda}{\lambda}\right)^2 \left(2\frac{r - r_C + \lambda}{\lambda} - 3\right) & r_C - \lambda zr \le r_C \\ 0 & r > r_C \end{cases}$$

Evaluation of energy and forces (contd)

$$C_{\vec{g}} = \frac{4\pi}{|\vec{g}|^2} e^{-\frac{|\vec{g}|^2}{4\alpha^2}} \qquad \frac{1}{V} \sum_{\vec{g}} C_{\vec{g}} e^{i\vec{g}\cdot\vec{r}} = \frac{1}{V} \sum_{i\neq j} q_i q_j \sum_{\vec{g}\in\mathcal{S}} \frac{4\pi}{|\vec{g}|^2} e^{-\frac{|\vec{g}|^2}{2\alpha^2}} e^{i\vec{g}\cdot(\vec{r}_i - \vec{r}_j)}$$

$$U_{long} = \frac{1}{V} \sum_{i,j} q_i q_j \sum_{\vec{g}\in\mathcal{S}} \frac{4\pi}{|\vec{g}|^2} e^{-\frac{|\vec{g}|^2}{4\alpha^2}} e^{i\vec{g}\cdot(\vec{r}_i - \vec{r}_j)} - \frac{1}{V} \sum_i q_i^2 \sum_{\vec{g}\in\mathcal{S}} \frac{4\pi}{|\vec{g}|^2} e^{-\frac{|\vec{g}|^2}{4\alpha^2}}$$

$$U_{long} = \frac{1}{V} \sum_{\vec{z}\in\mathcal{S}} \frac{4\pi}{|\vec{g}|^2} e^{-\frac{|\vec{g}|^2}{4\alpha^2}} |S(\vec{g})|^2 - \frac{\alpha}{\sqrt{\pi}} \sum_i q_i^2$$

Particle-particle particle-mesh Ewald

$$\rho(\vec{r}) = \sum_{i} q_{i} \delta(\vec{r} - \vec{r}_{i}) \qquad \rho(\vec{g}) = \int d\vec{r} \rho(\vec{r}) e^{i\vec{g} \cdot \vec{r}} = \sum_{i} q_{i} e^{i\vec{g} \cdot \vec{r}_{i}}$$

$$\nabla^{2} \phi(\vec{r}) = -\nabla \cdot \vec{E} = -4\pi \rho(\vec{r}) \qquad g^{2} \phi(\vec{g}) = -4\pi \rho(\vec{g}) = 4\pi S(\vec{g})$$

Canonical ensemble

Helmholtz free energy A(N,V,T) = E(N,V,T) - TS(N,V,T) $dA = SdT - PdV + \mu dN$ $S = -\left(\frac{\partial A}{\partial T}\right)_{N,V}$ $P = -\left(\frac{\partial A}{\partial V}\right)_{N,T}$ $\mu = \left(\frac{\partial A}{\partial N}\right)_{V,T}$

Thermal contact

Microcanonical $\Omega(N, V, E) = M_N \int dx_1 dx_2 \delta(\mathcal{H}_1(x_1) + \mathcal{H}_2(x_2) - E)$ Distribution function $\ln f(x_1) = \ln \int dx_2 \delta(\mathcal{H}_1(x_1) + \mathcal{H}_2(x_2) - E)$ $\ln f(x_1) \approx \ln \int dx_2 \delta(\mathcal{H}_2(x_2) - E) - \frac{\partial}{\partial E} \ln \int dx_2 \delta(\mathcal{H}_2(x_2) - E) \mathcal{H}_1(x_1)$ $Q(N, V, T) = \frac{1}{N \ln^{3N}} \int dx e^{-\beta \mathcal{H}(x)}$ $\beta = \frac{1}{1 \cdot T}$

From micro to macro

$$A = E - \beta \left(\frac{\partial A}{\partial \beta}\right)_{N,V}$$

$$E = \langle \mathcal{H} \rangle = \frac{1}{N!h^{3N}} \frac{\int dx \mathcal{H}(x)e^{-\beta \mathcal{H}(x)}}{\int dx e^{-\beta \mathcal{H}(x)}} = -\frac{1}{Q(N,V,\beta)} \frac{\partial Q(N,V,\beta)}{\partial \beta} = -\frac{\partial \ln Q(N,V,\beta)}{\partial \beta}$$

$$A + \frac{\partial \ln Q}{\partial \beta} + \beta \frac{\partial A}{\partial \beta} = 0 \Rightarrow \ln Q(N,V,\beta) = -\beta A(N,V,\beta)$$

$$A(N,V,T) = -kT \ln Q(N,V,T) \qquad C_N = \frac{1}{N!h^{3N}}$$
Energy $E = \langle \mathcal{H} \rangle = \frac{1}{Q} \frac{\partial Q}{\partial \beta} = 0$

Temperature estimator
$$\mathcal{T}(x) + \frac{1}{3Nk} \sum_{i} \frac{\vec{p}_{i}^{2}}{m_{i}}$$
 $T = \langle \mathcal{T}(x) \rangle = \frac{C_{N} \int dx \mathcal{T}(x) e^{-\beta \mathcal{H}(x)}}{C_{N} \int dx e^{-\beta \mathcal{H}(x)}}$
Energy fluctuation $\Delta E^{2} = \frac{\partial^{2} \ln Q}{\partial \beta^{2}} = kT^{2}C_{V}$ $\frac{\Delta E}{E} \sim \frac{1}{\sqrt{N}}$

Pressure estimator
$$\mathcal{D}(\vec{r}, \vec{p}) = \frac{1}{3V} \sum_{i} \left[\frac{\vec{p}_{i}^{2}}{m_{i}} + \vec{F}_{i} \cdot \vec{r}_{i} \right]$$

-Velocity rescaling

$$\begin{split} \bar{K} &= \frac{N_j}{2\beta} \qquad K = \frac{1}{2} \sum_i m_i \vec{v}_i^2 \qquad \vec{v}_i \to \frac{\vec{v}_i}{\alpha} \qquad \alpha = \sqrt{\frac{\bar{K}}{K}} \\ \bar{K} &= \frac{1}{2} \sum_i m_i \frac{\vec{v}_i^2}{\alpha^2} = \frac{\bar{K}}{2K} \sum_i m_i \vec{v}_i^2 \end{split}$$

Andersen-Heyes Thermostat

$$P(p) = \left(\frac{\beta}{2\pi m}\right)^{\frac{3}{2}e^{-\beta\frac{p^2}{2m}}} \\ P(K_t)dK_t \propto K_t^{\frac{N_f}{2}-1}e^{-\beta K_t}dK_t \\ K_t = \frac{1}{2}\sum_i m_i \frac{\vec{v}_i^2}{\alpha^2} = \frac{K_t}{2K}\sum_i m_i \vec{v}_i^2$$

Langevin Thermostat

Bussi velocity Verlet

$$dK = \left(D(K) \frac{\partial \log P(K)}{\partial K} + \frac{\partial D(K)}{\partial K}\right) dt + \sqrt{2D(K)} dW \qquad P(K_t) dK_t \propto K_t^{\frac{N_f}{2} - 1} e^{\beta K_t} dK$$

$$dK = \left(\frac{N_f D(K)}{2K\bar{K}} (K - \bar{K}) - \frac{D(K)}{K} + \frac{\partial D(K)}{\partial K}\right) dt + \sqrt{2D(K)} dW \qquad D(K) = \frac{2K\bar{K}}{N_f \tau}$$

$$dK = (K - \bar{K}) \frac{dt}{\tau} + \sqrt{\frac{2K\bar{K}}{N_f}} \frac{dW}{\sqrt{\tau}}$$
Berendsen's thermostat $dK = (K - \bar{K}) \frac{dt}{\tau}$

$$\alpha^2 = e^{-\frac{\Delta t}{\tau}} + \frac{\bar{K}}{N_f K} \left(1 - e^{-\frac{\Delta t}{\tau}}\right) \sum_{i=1}^{N_f} R_i^2 + 2e^{-\frac{\Delta t}{2\tau}} \sqrt{\frac{\bar{K}}{N_f K}} \left(1 - e^{-\frac{\Delta t}{\tau}} R_1\right)$$

$$\mathcal{H}_{N} = \sum_{i} \frac{\vec{p}_{i}^{2}}{2m_{i}s^{2}} + U(\vec{r}_{1}, \dots, \vec{r}_{n}) + \frac{p_{s}^{2}}{2Q} + gkT \log s$$

$$\Omega = \int d^{N}\vec{r}d^{N}\vec{p}dsdp_{s}s^{dN}\delta \left(\mathcal{H} + \frac{p_{s}^{2}}{2Q} + gkT \log s - E\right)$$

$$\Omega = \frac{1}{dkT} \int d^{N}\vec{r}d^{N}\vec{p}dp_{s}e^{\frac{dN+1}{gkT}\left(E-\mathcal{H}-\frac{p_{s}^{2}}{2Q}\right)}$$

$$g = dN + 1 \Rightarrow \Omega = \frac{e^{\frac{\kappa}{kT}}\sqrt{2\pi QkT}}{(dN+1)kT} \int d^{N}\vec{r}d^{N}\vec{p}e^{-\frac{\mathcal{H}}{kT}}$$

$$\dot{\vec{r}}_{i} = \frac{\vec{p}_{i}}{m_{i}}s^{2} \qquad \dot{\vec{p}}_{i} = \vec{F}_{i} \qquad \dot{s} = \frac{p_{s}}{Q} \qquad \dot{p}_{s} = \frac{1}{s} \left[\sum_{i} \frac{\vec{p}_{i}^{2}}{m_{i}s^{2}} - gkT\right]$$

Thermostats (contd)

$ec{p}_i' = rac{ec{p}_i}{s}$	$\vec{p}_s' = \frac{p_s}{s} \qquad dt' = \frac{dt}{s}$	$\frac{d\vec{r}_i}{dt'} = \frac{\vec{p}_i'}{m_i}$	$\frac{d\vec{p}_i'}{dt'} = \vec{F}_i - \frac{sp_s'}{Q}\vec{p}_i'$	
$\frac{ds}{dt'} = \frac{s^2 p_s'}{Q}$	$\frac{dp_s'}{dt'} = \frac{1}{2} \left[\sum_i \frac{(\vec{p}_i')^2}{m_i} \right]$	$-gkT\bigg]^2 - \frac{s(p_s')^2}{Q}$	$\frac{1}{2}\frac{ds}{dt'} = \frac{d\eta}{dt'}$	$p_s = p_{\eta} = sp_s'$
	$\dot{ec{p}}_i = ec{F}_i - rac{P_\eta}{Q}ec{p}_i$	_	_	

-Non Hamiltonian statistical mechanics

$$\dot{x} = \xi(x,t) \qquad \nabla \cdot \dot{x} = \nabla \cdot \xi(x,t) = \kappa(x,t) \neq 0
J(x_t; x_0) = e^{\int_0^t ds \kappa(x_s,s)} \qquad \kappa(x_t,t) = \frac{dw(x_t,t)}{dt} \implies J(x_t; x_0) = e^{w(x_t,t)-w(x_0,0)} e^{-w(x_t,t)} dx_t = e^{-w(x_0,0)} dx_0$$

$$J(x_t; x_0) = \frac{\sqrt{g(x_0, 0)}}{\sqrt{g(x_t, t)}}$$
 $\sqrt{g(x_t, t)} = e^{-w(x_t, t)}$

$$\frac{\partial}{\partial t} \left[f(x,t) \sqrt{g(x,t)} \right] + \nabla \cdot \left[\dot{x} \sqrt{g(x,t)} f(x,t) \right] = 0 \qquad f(x_t,t) \sqrt{g(x_t,t)} dx_t = f(x_0) \sqrt{g(x_0)} dx_0 \qquad \xi(x) \cdot \nabla f(x) = 0$$

$$\Lambda_k(x_t) - C_k = 0$$
 $\frac{d\Lambda_k(x_t)}{dt} = 0 \Rightarrow f(x) = \prod_{k=1}^{N_C} \delta(\Lambda_k(x_t) - C_k)$

Microcanonical
$$\mathcal{E} = \int dx \sqrt{g(x)} f(x) = \int dx \sqrt{g(x)} \prod_{k=1}^{N_C} \delta(\Lambda_k(x_t) - C_k)$$

Nosè=Hoover:
$$\mathcal{H}'(\vec{r}, \eta, \vec{p}, p_{\eta}) = \mathcal{H}(\vec{r}, \vec{P}) + \frac{p_{\eta}^2}{2Q} + dNkT\eta$$
 $\frac{d\mathcal{H}'}{dt} = 0$
 $\kappa = -Nd\dot{\eta} \Rightarrow \sqrt{g} = e^{dN\eta}$

Partition function
$$\mathcal{E}_T(N, V, C_1) = \frac{e^{\beta C_1} \sqrt{2\pi QkT}}{dNkT} \int d^N \vec{p} \int_{\mathcal{D}(V)} d^N \vec{r} e^{-\beta \mathcal{H}(\vec{r}, \vec{p})}$$

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_1 \qquad \vec{K} = \vec{P}e^{\eta} \Rightarrow \frac{d\vec{K}}{dt} = 0$$

Nosè-Hoover chains

$$\dot{\vec{r}}_{i} = \frac{\vec{p}_{i}}{m_{i}} \qquad \dot{\vec{p}}_{i} = \vec{F}_{i} - \frac{p_{\eta_{1}}}{Q_{1}} \vec{p}_{i} \qquad \dot{\eta}_{j} = \frac{p_{\eta_{j}}}{Q_{j}} \quad j = 1, \dots, M$$

$$\dot{p}_{\eta_{1}} = \left[\sum_{i} \frac{\vec{p}_{i}^{2}}{m_{i}} - dNkT \right] - \frac{p_{\eta_{2}}}{Q_{2}} p_{\eta_{1}} \qquad \dot{p}_{\eta_{j}} = \left[\sum_{i} \frac{\vec{p}_{\eta_{j-1}}^{2}}{Q_{j-1}} - kT \right] - \frac{p_{\eta_{j+1}}}{Q_{j+1}} p_{\eta_{j}} \quad j = 2, \dots, M-1$$

$$\dot{p}_{\eta_{j}} = \frac{p_{\eta_{M-1}}}{q_{j}} \quad kT$$

$$\mathcal{H}'(\vec{r}, \eta, \vec{p}, p_{\eta}) = \mathcal{H}(\vec{r}, \vec{p}) + \sum_{j=1}^{M} \frac{p_{\eta_{j}}^{2}}{2Q_{j}} + dNkT\eta_{1} + kT\sum_{j=2}^{M} \eta_{j}$$

$$\kappa = -dN\dot{\eta}_1 - \dot{\eta}_c$$
 $\eta_c = \sum_{j=2}^{M} \eta_j$ $\sqrt{g} = e^{dN\eta_1 + \eta_c}$

Partition function
$$\mathcal{E}_T(N, \vec{V}, \vec{C}_1) = \mathcal{M} \int d^N \vec{p} \int_{\mathcal{D}(V)} d^N \vec{r} e^{-\beta \mathcal{H}(\vec{r}, \vec{p})}$$

Isobaric ensemble

Enthalpy
$$dH = TdS + \mu dN + VdP$$
 $T = \left(\frac{\partial H}{\partial S}\right)_{N,P} \langle V \rangle = \left(\frac{\partial H}{\partial P}\right)_{N,S} \mu = \left(\frac{\partial H}{\partial N}\right)_{P,S}$
Gibbs $dG = \mu dN + VdP - SdT$ $S = -\left(\frac{\partial G}{\partial T}\right)_{N,P} \langle V \rangle = -\left(\frac{\partial G}{\partial P}\right)_{N,T} \mu = -\left(\frac{\partial G}{\partial N}\right)_{P,T}$

Isoenthalpic-isobaric ensemble

 $H = \mathcal{H}(v) + PV \qquad f(x) = F(\mathcal{H}(x)) = \mathcal{M}\delta(\mathcal{H}(x) + PV - H)$ $\Gamma(N, P, H) = \mathcal{M} \int_0^\infty dV \int d^N \vec{p} \int_{\mathcal{D}(V)} d^N \vec{r} \delta(\mathcal{H}(\vec{r}m\vec{p}) + PV - H)$

$$S(N,P,H) = k \ln \Gamma(N,P,H) \quad \frac{1}{T} = \left(\frac{\partial S}{\partial H}\right)_{N,P} \quad \frac{\langle V \rangle}{T} = -\left(\frac{\partial S}{\partial P}\right)_{N,H} \quad \frac{\mu}{T} = -\left(\frac{\partial S}{\partial N}\right)_{P,H}$$

Isothermal-isobaric ensemble

$$Q(N, V, T) \propto Q(N_1, V_1, T)Q(N_2, V_2, T)$$

$$f(x_1) = I_{N_1} e^{\beta \mu N_1} e^{-\beta P V_1} C_{N_1} e^{-\beta \mathcal{H}_1(x_1)} \qquad I_{N_1} = \frac{1}{V_0 N_1! h^{3N_1}}$$

$$\Delta(N, P, T) = \frac{1}{V_0} \int_0^\infty dV e^{-\beta P V} Q(N, V, T)$$

$G(N, P, \beta) = -\frac{\partial \ln \Delta(N, P, \beta)}{\partial \beta} - \beta \frac{\partial G}{\partial \beta}$

Pressure
$$\langle P^{(int)} \rangle = \frac{P}{\Delta(N,P,T)} \int_0^\infty dV e^{-\beta PV} Q(N,V,T) = P \text{ Work } \langle P^{(int)}V \rangle + kT = P \langle V \rangle$$

Andersen's Hamiltonian

$$\mathcal{H}_{A} = \sum_{i=1}^{N} \frac{V^{-\frac{2}{3}} \pi_{i}^{2}}{2m_{i}} + U(V^{\frac{1}{3}\vec{s}_{1}, \dots, V^{\frac{1}{3}}\vec{s}_{N}}) + \frac{p_{V}^{2}}{2W} + PV \qquad W = (3N+1)kT\tau_{b}^{2}$$

$$\dot{\vec{s}}_{i} = \frac{p_{i}}{m_{i}} + \frac{\dot{V}}{3V}r_{i} \quad \dot{\pi}_{i} = -\frac{\partial U}{\partial r_{i}} - \frac{\dot{V}}{3V}p_{i} \quad \dot{V} = \frac{p_{V}}{W} \quad \dot{p}_{V} = \frac{1}{3V}\sum_{i=1}^{N} \left[\frac{p_{i}^{2}}{m_{i}} - \frac{\partial U}{\partial r_{i}}r_{i}\right] - P$$

$$\dot{\vec{r}}_{i} = \frac{\vec{p}}{m_{i}} + \frac{\dot{V}}{3V}\vec{r}_{i} \quad \dot{\vec{p}}_{i} = -\frac{\partial U}{\partial \vec{r}_{i}} - \frac{\dot{V}}{3V}\vec{p}_{i} \quad \dot{V} = \frac{p_{V}}{W} \quad \dot{p}_{V} = \frac{1}{3V}\sum_{i=1}^{N} \left[\frac{\vec{p}_{i}^{2}}{m_{i}} - \frac{\partial U}{\partial \vec{r}_{i}} \cdot \vec{r}_{i}\right] - P$$

$$\left\langle \frac{p_{V}^{2}}{2W} \right\rangle = k\frac{T}{2} \Rightarrow \mathcal{H}(\vec{r}, \vec{p}) + PV \text{ is conserved}$$

MTK algorithm

$$\begin{split} &\epsilon = \frac{1}{3} \ln \frac{\vec{V}_{0}}{\vec{V}_{0}} \Rightarrow \dot{\epsilon} = \frac{\vec{V}_{W}}{3V} = \frac{p_{\epsilon}}{W} \\ &\dot{\vec{r}}_{i} = \frac{\vec{p}_{i}}{m_{i}} + \frac{p_{\epsilon}}{W} \vec{r}_{i} \quad \dot{\vec{p}}_{i} = -\frac{\partial U}{\partial \vec{r}_{i}} - \frac{p_{\epsilon}}{W} \vec{p}_{i} \quad \dot{V} = \frac{dVp_{\epsilon}}{W} \quad \dot{p}_{\epsilon} = dV(\mathcal{P}^{(int)} - P) \\ &\kappa = \frac{\dot{V}}{V} \qquad \dot{\vec{p}}_{i} = -\frac{\partial U}{\partial \vec{r}_{i}} - \left(1 + \frac{d}{N_{F}}\right) \frac{p_{\epsilon}}{W} \vec{p}_{i} \quad \dot{p}_{\epsilon} = dV(\mathcal{P}^{(int)} - P) + \frac{d}{N_{f}} \sum_{i=1}^{N} \frac{\vec{p}_{i}^{2}}{m_{i}} \\ &\text{Langevin piston } \dot{\vec{r}}_{i} = \frac{\vec{p}_{i}}{m_{i}} + \frac{\dot{V}}{3V} \vec{r}_{i} \quad \dot{\vec{p}}_{i} = -\frac{\partial U}{\partial \vec{r}_{i}} - \frac{\dot{V}}{3V} \vec{p}_{i} \quad \dot{V} = \frac{p_{V}}{W} \\ &\dot{p}_{V} = \frac{1}{3V} \sum_{i=1}^{N} \left[\frac{\vec{p}_{i}^{2}}{m_{i}} - \frac{\partial U}{\partial \vec{r}_{i}} \cdot \vec{r}_{i} \right] - P - \gamma \dot{V} + R(t) \quad \langle R(0)R(t) \rangle = \frac{2\gamma kT}{W} \delta(t) \end{split}$$

-Grand canonical ensemble

Thermodynamics $\tau(A)\left(\frac{\partial A}{\partial N},V,T\right) = -PV \qquad d\tilde{A} = -PdVpSdT - Nd\mu \qquad \tilde{A}(\mu,\lambda V,T) = \lambda \tilde{A}(\mu,V,T)$

Partition function
$$\sum_{N=0}^{\infty} \frac{1}{N!h^{3N}} e^{\beta\mu N} \int dx e^{-\beta\mathcal{H}(x,N)} = e^{\beta PV}$$

$$\mathcal{E}(\mu,V,T) = \sum_{N=0}^{\infty} \frac{1}{N!h^{3N}} e^{\beta\mu N} \int dx e^{-\beta\mathcal{H}(x,N)} = \sum_{N=0}^{\infty} e^{\beta\mu N} Q(N,V,T) = e^{\beta PV}$$

$$\frac{PV}{kT} = \ln \mathcal{E}(\mu,V,T) \qquad z = e^{\beta\mu} \qquad \mathcal{E}(z,V,T) = \sum_{N=0}^{\infty} z^N Q(N,V,T)$$

$$\langle N \rangle = \frac{1}{\mathcal{E}(\mu, V, T)} \sum_{N=0}^{\infty} N e^{\beta \mu N} Q(N, V, T) = kT \left(\frac{\partial}{\partial \mu} \ln \mathcal{E}(\mu, V, T) \right)_{V, T}$$
$$\langle N \rangle = z \frac{\partial}{\partial z} \ln \mathcal{E}(z, V, T)$$

 $U(\lambda N, \lambda V, \lambda S) = \lambda U(N, V, S) \Rightarrow U(N, V, S) = \mu N - PV + TS$

$$\begin{split} A(\lambda N, \lambda V, T) &= \lambda A(N, V, T) \Rightarrow A(N, V, T) = \mu N - PV \\ H(\lambda N, \lambda S, P) &= \lambda H(N, S, P) \Rightarrow H(N, S, P) = \mu N + TS \end{split}$$

 $G(\lambda N, P, T) = \lambda G(N, P, T) \Rightarrow G(N, P, T) = \mu N$

Ideal g

$$Q(N, V, T) = \frac{1}{N!} \left[V \left(\frac{2\pi m}{\beta h^2} \right)^{\frac{3}{2}} \right]^N = \frac{1}{N!} \left(\frac{V}{\lambda^3} \right)^N \qquad \mathcal{E}(z, V, T) = e^{zV} \lambda^3$$
$$\langle N \rangle = V \frac{z}{\lambda^3} \Rightarrow z = \frac{\langle N \rangle \lambda^3}{V} \qquad \frac{PV}{PkT} = \ln \mathcal{E}(z, V, T) = \frac{Vz}{\lambda^3} = \langle N \rangle$$

Particle number fluctuations

$$\begin{split} \Delta N &= \sqrt{\langle N^2 \rangle - \langle N \rangle^2} \qquad z \frac{\partial}{\partial z} z \frac{\partial}{\partial z} \ln \mathcal{E}(z, V, T) = \langle N^2 \rangle - \langle N \rangle^2 \\ \Delta N^2 &= k T V \frac{\partial^2 P}{\partial \mu^2} \qquad a(v, T) = \frac{1}{N} A \left(N, \frac{V}{N}, T \right) \Rightarrow \mu = a(v, T) - V \frac{\partial a}{\partial v} \\ P &= -\frac{\partial a}{\partial v} \qquad \frac{\partial P}{\partial \mu} = -\frac{\partial^2 a}{\partial v^2} \frac{\partial v}{\partial \mu} \qquad \frac{\partial \mu}{\partial v} = \frac{\partial a}{\partial v} - \frac{\partial a}{\partial v} - v \frac{\partial^2 a}{\partial v^2} \Rightarrow \frac{\partial P}{\partial \mu} = \frac{1}{v} \\ \frac{\partial^2 P}{\partial \mu^2} &= -\frac{1}{v^3} \frac{\partial P}{\partial v} \qquad \kappa_T = -\frac{1}{v \frac{\partial P}{\partial v}} \Rightarrow \Delta N^2 = \frac{\langle N \rangle k T}{v} \kappa_T \qquad \frac{\Delta N}{\langle N \rangle} \sim \frac{1}{\sqrt{\langle N \rangle}} \end{split}$$

Quantifying uncertainties and sampling quality

Autocorrelation analysis—

$$C_f(t') = \frac{\langle (f(x) - \langle f \rangle)(f(t+t') - \langle f \rangle) \rangle}{\sigma_f^2}$$

$$C_f(t') = \frac{1}{\sigma_f^2} \frac{1}{N} \sum_{i=1}^{N - \frac{t'}{\Delta t}} (f(j\Delta t) - \langle f \rangle)(f(j\Delta t + t') - \langle f \rangle) \qquad \tau_f = \int_0^{+\infty} dt' C_f(t')$$

$$N_f^{ind} \simeq \frac{t_{sim}}{\tau_f} \qquad SE(f) = \frac{\sigma_f}{\sqrt{N_f^{ind}}} \sim \sigma_f \sqrt{\frac{\tau_f}{t_{sim}}}$$

Block averaging analysis

$$BSE(f,n) = \frac{\sigma_n}{\sqrt{M}}$$

Protein motions

$$U_{ij} = \gamma_{ij} (\Delta \vec{r}_j - \Delta \vec{r}_j) \cdot (\Delta \vec{r}_j - \Delta \vec{r}_j) = \gamma_{ij} \Delta \vec{r}_{ij}^2 \qquad U_{GNM} = \frac{\gamma}{2} \sum_i \sum_j \Delta \vec{r}_{ij}^2$$

$$U_{GNM} = \frac{\gamma}{2} \Delta \vec{r}(t)^T \Gamma \Delta \vec{r}(t) \qquad \langle \Delta \vec{r}_i \cdot \Delta \vec{r}_j \rangle = \frac{3kT}{\gamma} \left[\Gamma^{-1} \right]_{ij}$$

$$\langle \Delta \vec{r}_i \cdot \Delta \vec{r}_i \rangle = \sum_k [\Delta r_i^2]_k \qquad B_i = \frac{8\pi^2}{3} \langle \Delta r_i^2 \rangle$$
Name of mode analysis, $U_i = \frac{1}{2} \sum_i \sum_j U_i (\pi_i - \pi^0) (\pi_i - \pi^0)$

Normal mode analysis:
$$U = \frac{1}{2} \sum_{i} \sum_{j} H_{ij} (q_i - q_i^0) (q_j - q_j^0)$$

 $C = \langle \Delta \vec{q} \Delta \vec{q}^T \rangle = \frac{1}{Q} \int \Delta \vec{q} \Delta \vec{q}^T e^{-\frac{\Delta \vec{q}^T H \Delta \vec{q}}{2kT}} d^N \Delta \vec{q} = kTH^{-1}$

Anisotropic network mod

$$U_{ANM} = \frac{1}{2} \sum_{ij} \gamma (r_{ij} - r_{ij}^0)^2 \Rightarrow \frac{\partial^2 U}{\partial x_i \partial x_j} = -\gamma \frac{(x_i - x_j)(y_i - y_j)}{t_{ij}^2} \qquad \langle \Delta \vec{r}_i \cdot \Delta \vec{r}_j \rangle = \frac{3kT}{\gamma} \sum_k \lambda_k^{-1} [\vec{u}_k \vec{u}_k^T]_{ij}$$

Correlation cosine
$$I_k = \frac{\Delta \vec{q}_{AB} \cdot \vec{u}_k}{|\Delta \vec{q}_{AB}|}$$
 $C_0 = \sqrt{\sum_k I_k^2}$

Degree of connectivity
$$\kappa_k = N^{-1} e^{-\sum_{i=1}^N \alpha(\Delta r_i)^2 \big|_k \log(\alpha(\Delta r_i)^2 \big|_k)}$$

$$\sum_{i=1}^N \alpha(\Delta r_i)^2 \big|_k = 1$$

Essential dynamics

Covariance matrix
$$C_{ij} = \overline{(x_i(t) - \overline{x_i(t)})(x_j(t) - \overline{x_j(t)})}$$

Correlation matrix $R_{ij} = \frac{\overline{(x_i(t) - \overline{x_i(t)})(x_j(t) - \overline{x_j(t)})}}{\sigma_{x_i}\sigma_{x_j}}$

$$PC_k(t) = \sum_{i=1}^{N} \vec{d_i}(t) \cdot \vec{u_i}^k \quad c_k = \frac{2}{T_{sim}} \left(\int_0^{T_{sim}} \cos\left(\pi \frac{k_B T}{\lambda_k} t\right) PC_k(t) dt \right)^2 \left(\int_0^{T_{sim}} PC_k^2(t) dt \right)^{-1}$$

$$\Omega_{A,B} = 1 - \begin{bmatrix} \frac{3N-6}{k+1} (\lambda_k^A + \lambda_k^B) - 2\sum_{k=1}^{N-6} \sum_{j=1}^{N-6} \sqrt{\lambda_k^A \lambda_j^B} (\vec{u_k}^A \cdot \vec{u_j}^B)^2 \\ \sum_{k=1}^{N-6} (\lambda_k^A + \lambda_k^B) \end{bmatrix}^{\frac{1}{2}}$$