Computational biophysics

Protein's geometry

Centre of mass $\vec{R}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}$

$RMSD(t) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\vec{r}_i(t) - \vec{r}_i(0))^2}$

Radius of gyration $r_g = \sqrt{\frac{\sum_{i=1}^{N} m_i(\vec{r}_i - \vec{R}_{cm})^2}{\sum_{i=1}^{N} m_i}}$

$RMSF_i = \sqrt{\langle \Delta r_i^2 \rangle} = \sqrt{\frac{1}{M}\sum_{f=1}^{M}(\vec{r}_{i,f} - \langle \vec{r}_i \rangle)^2}$

$B_i = \frac{8\pi^2}{3} RMSF_i^2$

Semi-empirical force fields

Bond stretching

Harmonic $U(r_{AB}) = \frac{1}{2}k_{AB}(r_{AB} - r_{AB,eq})^2$

Anarmonic $U(r_{AB}) = \frac{1}{2}\left[k_{AB} + k_{AB}^{(3)}(r_{AB} - r_{AB,eq})\right](r_{AB} - r_{AB,eq})^2$

Quartic correction $U(r_{AB}) = \frac{1}{2}\left[k_{AB} + k_{AB}^{(3)}(r_{AB} - r_{AB,eq}) + k_{AB}^{(4)}(r_{AB} - r_{AB,eq})^2\right] \cdot$
$\cdot (r_{AB} - r_{AB,eq})^2$

Morse $U(r_{AB}) = D_{AB}\left[1 - e^{-\alpha_{AB}(r_{AB} - r_{AB,eq}^2)}\right]$

Valence angle bending

Potential $U(\theta_{ABC}) = \frac{1}{2}[k_{ABC} + k_{ABC}^{(3)}(\theta_{ABC} - \theta_{ABC,eq}) +$
$+ k_{ABC}^{(4)}(\theta_{ABC} - \theta_{ABC,eq})^2 + \cdots](\theta_{ABC} - \theta_{ABC,eq})^2$

Fourier $U(\theta_{ABC}) = \sum_{\{j\}_{ABC}} k_{j,ABC}^{fourier}[1 + \cos(j\theta_{ABC} + \psi_j)]$

$k_{j,ABC}^{fourier} = \frac{2k_{ABC}^{harmonic}}{j^2}$

Torsions

Potential $U(\omega_{ABCD}) = \frac{1}{2}\sum_{\{j\}_{ABCD}} V_{j,ABCD}\left[1 + (-1)^{j+1}\cos(j\omega_{ABCD} + \psi_{j,ABCD})\right]$

Improper $U(\omega_{ABCD}) = \frac{1}{2}\sum_{\{j\}_{ABCD}} V_{j,ABCD}[1 + (-1)^{j+1}\cos(j\omega_{ABCD} + \psi_{j,ABCD})]$

Van der Waals

Lennard-Jones $U(r_{AB}) = 4\epsilon_{AB}\left[\left(\frac{\sigma_{AB}}{r_{AB}}\right)^{12} - \left(\frac{\sigma_{AB}}{r_{AB}}\right)^{6}\right]$

Morse $U(r_{AB}) = D_{AB}\left[1 - e^{-\alpha_{AB}(r_{AB} - r_{AB,eq}^2)}\right]^2$

Hill $U(r_{AB}) = \epsilon\left[\frac{6}{\beta_{AB}-6}e^{\beta_{AB}\frac{1-r_{AB}}{r_{AB}^*}} - \frac{\beta_{AB}}{\beta_{AB}-6}\left(\frac{r_{AB}^*}{r_{AB}}\right)^6\right]$

Electrostatic interactions

Distribution of charges $U_{AB} = \sum_A \sum_{B>A} \vec{M}^{(A)}\vec{V}^{(B)}$

Point like $U_{AB} = \frac{q_A q_B}{\epsilon_{AB} r_{AB}}$

Dipolar interactions $U_{AB/CD} = \frac{\mu_{AB}\mu_{CD}}{\epsilon_{AB/CD} r_{AB/CD}^3}(\cos\chi_{AB/CD} - 3\cos\alpha_{AB}\cos\alpha_{CD})$

Parameterization

Parameters $Z = \sqrt{\sum_i^{observables}\sum_j^{occurrences}\frac{(calc_{i,j} - expt_{i,j})^2}{w_i^2}}$

$\sigma_{AB} = \sigma_A + \sigma_B \qquad \epsilon_{AB} = \sqrt{\epsilon_A \epsilon_B}$

Classical mechanics

Newton's laws

$\vec{F} = m\vec{a} \qquad \vec{F}_{BA} = -\vec{F}_{AB}$

$\vec{v}(t) = \frac{d\vec{r}}{dt} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \qquad m\frac{d^2\vec{r}}{dt^2} = \vec{F}$

Force acting on atom $\vec{F}_i(\vec{r}_1, \ldots, \vec{r}_N, \dot{\vec{r}}_i) = \sum_{j \neq i}\vec{F}_{ij}(\vec{r}_i - \vec{r}_j) + \vec{F}^{(ext)}(\vec{r}_i, \dot{\vec{r}}_i)$

Bond stretching: $U = \frac{k_l}{2}(l - l^0)^2$

Bond bending: $U = \frac{k_\theta}{2}(\theta - \theta^0)^2$

Bond torsion: $U = k_\phi[1 + \cos(n\phi - \phi^0)]$

Van der Waals interactions: $U = \left[\frac{a_{ij}}{r_{ij}^{12}} - \frac{b_{ij}}{r_{ij}^6}\right]$

Electrostatic interactions: $U = \frac{332 q_i q_j}{\epsilon r_{ij}}$

$\vec{p}_i = m_i\vec{v}_i = m\dot{\vec{r}}_i \qquad \vec{F}_i = m_i\ddot{\vec{r}}_i = \dot{\vec{p}}_i$

$\vec{x}(t) = \{\vec{r}_1(t), \ldots, \vec{r}_N(t), \vec{p}_1(t), \ldots, \vec{p}_N(t)\}$

Lagrangian formulation

$\vec{F}_i(\vec{r}_1, \ldots, \vec{r}_N) = -\Delta_i U(\vec{r}_1, \ldots, \vec{r}_N)$

$W_{AB} = \int_A^B \vec{F}_i d\vec{l} = U_A - U_B = -\Delta U_{AB} \qquad \oint \vec{F}_i d\vec{l} = 0$

Kinetic energy $K(\dot{\vec{r}}_1, \ldots, \dot{\vec{r}}_N) = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^2$

$\mathcal{L}(\vec{r}_1, \ldots, \vec{r}_N, \dot{\vec{r}}_1, \ldots, \dot{\vec{r}}_N) = K(\dot{\vec{r}}_1, \dot{\vec{r}}_N) - U(\vec{r}_1, \ldots, \vec{r}_N)$

Euler-Lagrange $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i}\right) - \frac{\partial \mathcal{L}}{\partial \vec{r}_i} = 0$

$E = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^2 + U(\vec{r}_1, \ldots, \vec{r}_N)$

$quad \frac{dE}{dt} = 0$

Generalized coordinates

$q_\alpha = f_\alpha(\vec{r}_1, \ldots, \vec{r}_N) \qquad \alpha = 1, \ldots, 3N \qquad \vec{r}_i = \vec{g}_i(q_1, d \ldots, q_{3N}) \qquad i = 1, \ldots, N$

$\dot{\vec{r}}_i = \sum_{\alpha=1}^{3N}\frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha \qquad \mathcal{L}(q, \dot{q}) = \frac{1}{2}\sum_{\alpha=1}^{3N}\sum_{\beta=1}^{3N} G_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta - U(q_1, \ldots, q_{3N})$

Classical mechanics (contd)

Legendre transforms

$s = f'(x) \equiv g(x) \qquad f'(x) = g(x) = s \Rightarrow x = g^{-1}(s)$

$b(g^{-1}(s)) = f(g^{-1}(s)) - sg^{-1}(s) \equiv \tilde{f}(s) = f(x(s)) - sx(s)$

$\tilde{f}(s_1, \ldots, s_n) = f(x_1(s_1, \ldots, s_n), \ldots x_n(s_1, \ldots, s_n)) - \sum_i s_i x_i(s_1, \ldots, s_n)$

Hamiltonian formulation

$\mathcal{H}(\vec{r}_1, \ldots, \vec{r}_N, \vec{p}_1, \ldots, \vec{p}_N) = -\tilde{\mathcal{L}}(\vec{r}_1, \ldots, \vec{r}_N, \vec{p}_1, \ldots, \vec{p}_N)$

$\mathcal{H}(\vec{r}_1, \ldots, \vec{r}_N, \vec{p}_1, \ldots, \vec{p}_N) = \sum_{i=1}^{N}\frac{\vec{p}_i^2}{2m_i} + U(\vec{r}_1, \ldots, \vec{r}_N)$

$\mathcal{H}(q_1, \ldots, q_{3N}, p_1, \ldots, p_{3N}) = \frac{1}{2}\sum_\alpha\sum_\beta p_\alpha G_{\alpha\beta}^{-1}p_\beta + U(q_1, \ldots, q_{3N})$

Hamilton equations $\dot{q}_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha} \qquad \dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial q_\alpha} \qquad \frac{\mathcal{H}}{dt} = 0 \qquad \mathcal{H} = const$

Some properties

Conservation laws $\frac{da}{dt} = \frac{\partial a}{\partial x_t}\dot{x}(t) = \{a, \mathcal{H}\} = 0$

Incompressibility $\nabla_x \dot{(x)} = 0$

Symplectic structure $M = J^T M J \qquad J_{kl} = \frac{\partial x_k(t)}{\partial x_l(0)}$

Theoretical foundations of statistical mechanics

Thermodynamics

Equilibrium $g(N, P, V, T) = 0$

State function $f(n, P, V, T)$

Reversible work $dW_{rev} = -PdV + \mu dN$

Heat $dQ_{rev} = CdT$

First law $\Delta E = \Delta Q + \Delta W$

Entropy $\Delta S = \int_1^2 \frac{dQ_{rev}}{T}$

The ensemble

Average $A = \frac{1}{Z}\sum_{\lambda=1}^{N} a(x_\lambda) \equiv \langle a \rangle$

Microstate $x_0 = (q_1(0), \ldots, q_{3N}(0), p_1(0), \ldots, p_{3N}(0))$

Phase space volume $dx_t = J(x_t; x_0)dx_0 \qquad \frac{dJ}{dt} = 0 \Rightarrow J(x_t; x_0) = 1 \Rightarrow dx_t = dx_0$

Distribution function
$$f(x_t) : \int f(x)dx = 1 \wedge \frac{df(x_t, t)}{dt} = 0 \Rightarrow$$
$$f(x_t, t)dx_t = f(x_0, 0)dx_0 \Rightarrow$$
$$\frac{\partial f(x, t)}{\partial t} + \{f(x, t), \mathcal{H}(x, t)\} = 0$$

Equilibrium
$$A = \int a(x)f(x, t)dx \Rightarrow \frac{\partial f(x, t)}{\partial t} = 0 \wedge \{f(x, t), \mathcal{H}(x, t) = 0\} \Rightarrow$$
$$f(x) \propto \mathcal{F}(\mathcal{H}(x))$$

$Z = \int dx\mathcal{F}(\mathcal{H}(x)) \Rightarrow f(x) = \frac{1}{Z}\mathcal{F}(\mathcal{H}(x))$

Microcanonical ensemble

State and distribution function

State function $dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$

$\left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{1}{T} \qquad \left(\frac{\partial S}{\partial V}\right)_{N,E} = \frac{P}{T} \qquad \left(\frac{\partial S}{\partial N}\right)_{V,N} = \frac{\mu}{T}$

Boltzmann relation $S(N, V, E) = k\ln\omega(N, V, E)$

Distribution function
$$\Omega(N, V, E) = M_N\int d\vec{p}\int_{D(V)} d\vec{r}\delta(\mathcal{H}(\vec{r}, \vec{p}) - E)$$
$$= M_N\int dx\delta(\mathcal{H}(x) - E)$$
$$M_N = \frac{E_0}{N!h^{3N}}$$

$A = \langle a \rangle = \frac{M_N}{\Omega(N,V,E)}\int dxa(x)\delta(\mathcal{H}(x) - E) = \frac{\int dxa()\delta(\mathcal{H}(x) - E)}{\int dx\delta(\mathcal{H} - E)}$

$\left\langle x_i\frac{\partial \mathcal{H}}{\partial x_j}\right\rangle = \frac{M_N}{\Omega(N,V,E)}\frac{\partial}{\partial E}\int_{\mathcal{H}(x)<E} dxx_i\frac{\partial(\mathcal{H} - E)}{\partial x_j}$

Virial theorem

$\left\langle x_i\frac{\partial \mathcal{H}}{\partial x_j}\right\rangle = \delta_{ij}\frac{\Sigma(E)}{\frac{\partial \Sigma(E)}{\partial E}}$

$\Sigma(N, V, E) = \frac{1}{N!h^{3N}}\int dx\theta(E - \mathcal{H})$

$\Omega(N, V, E) = E_0\frac{\partial \Sigma(N,V,E)}{\partial E} \qquad \left\langle x_i\frac{\partial \mathcal{H}}{\partial x_j}\right\rangle = \delta_{ij}\left(\frac{\ln\Sigma(E)}{\partial E}\right)^{-1}$

$S(N, V, E) = k\ln\Omega(N, V, E) \simeq k\ln\Sigma(N, V, E) = \tilde{S}(N, V, E)$

$\left\langle x_i\frac{\partial \mathcal{H}}{\partial x_j}\right\rangle \simeq \delta_{ij}\left(\frac{S(E)}{\partial E}\right)^{-1} = kT\delta_{ij}$

Thermal contact

$\Omega(N, V, E) = M_N\int dx\delta(\mathcal{H}_1(x_1) + \mathcal{H}_2(x_2) - E)$

$\Omega(N, V, E) = \int dE_1\Omega_1(N_1, V_1, E_1)\Omega_2(N_2, V_2, E - E_1)$

$S(N, V, E) = k\ln\Omega_1(N_1, V_1, \bar{E}_1) + k\ln\Omega_2(N_2, V_2, E - \bar{E}_1)$
$= S_1(N_1, V_1, \bar{E}_1) + S_2(N_2, V_2, E - \bar{E}_1)$

$T_1 = T_2$

Introduction to molecular dynamics

Verlet algorithm

$\vec{r}_i(t + \Delta t) = 2\vec{r}_i(t) - \vec{r}_i(t - \Delta t) + \frac{\Delta t^2}{m_i}\vec{F}_i(t)$

$\vec{v}_i(t + \Delta t) = \vec{v}_i(t) + \frac{\Delta t}{2m_i}\left[\vec{F}_i(t) + \vec{F}_i(t + \Delta t)\right]$

Initial conditions $f(v) = \sqrt{\frac{m}{2\pi kT}}e^{-\frac{mv^2}{2kT}} \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$

Introduction to molecular dynamics (contd)

Action integral

$Q \equiv \{q_1, \ldots, q_{3N}\} \qquad \dot{Q} \equiv \{\dot{q}_1, \ldots, \dot{q}_{3N}\}$

$A[Q] = \int_{t_1}^{t^2}\mathcal{L}(Q(t), \dot{Q}(t))dt$

$\delta Q(t_1) = \delta Q(t_2) = 0 \qquad \delta\dot{Q}(t_1) = \delta\dot{Q}(t_2) = 0$

$\delta A = \int_{\alpha=1}^{3N}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\delta q_\alpha(t)\Big|_{t_1}^{t_2} dt + \int_{t_1}^{t_2}\sum_{\alpha=1}^{3N}\left[\frac{\partial \mathcal{L}}{\partial q_\alpha}\delta q_\alpha(t) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)\delta q_\alpha(t)\right]dt = 0$

Constraints

$\sum_{\alpha=1}^{3N} a_{k\alpha}dq_\alpha + a_{kt}dt = 0, k = 1, \ldots, N_C$

Holonomic $a_{k\alpha} = \frac{\partial \sigma_k}{\partial q_\alpha} \qquad a_{kt} = \frac{\partial \sigma_k}{\partial t}$

Non-holonomic
$$\frac{1}{2}\sum_i m_i\dot{\vec{r}}_i^2 - C = 0 \Rightarrow \frac{1}{2}\sum_i m_i\dot{\vec{r}}_i d\vec{r}_i - Cdt = 0$$
$$\Rightarrow a_{1i} = \frac{1}{2}m_i\dot{\vec{r}}_i \wedge a_{1t} = -C$$

Lagrange multiplier $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = \sum_{k=1}^{N_C}\lambda_k a_{k\alpha}$

$\dot{q}_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha} \qquad \dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial q_\alpha} - \sum_{k=1}^{N_C}\lambda_k a_{k\alpha} \qquad \sum_{\alpha=1}^{3N} a_{k\alpha}\frac{\partial \mathcal{H}}{\partial p_\alpha} = 0$

Simulation $m_i\ddot{\vec{r}}_i = \vec{F}_i + \sum_{k=1}^{N_C}\lambda_k\nabla_i\sigma_k \qquad \dot{\sigma}_k = \sum_{i=1}^{N}\nabla_i\sigma_k \cdot \dot{\vec{r}}_i = 0$

Velocity Verlet $\vec{r}_i(\Delta t) = \vec{r}_i(0) + \Delta t\vec{v}_i(0) + \frac{\Delta t^2}{2m_i}\vec{F}_i(0) + \frac{\Delta t^2}{2m_i}\sum_k\lambda_k\nabla_i\sigma_k(0)$

$\vec{r}_i(\Delta t) = \vec{r}_i + \frac{1}{m_i}\sum_k\tilde{\lambda}_k\nabla_i\sigma_k(0) \qquad \tilde{\lambda}_k = \frac{\Delta t^2}{2}\lambda_k$

$\sigma_l\left(\vec{r}_1^{(1)}, \ldots, \vec{r}_N^{(1)}\right) + \sum_{i=1}^{N}\sum_{k=1}^{N_C}\frac{1}{m_i}\nabla_i\sigma_k\left(\vec{r}_1^{(1)}, \ldots, \vec{r}_N^{(1)}\right) \cdot \nabla_i\sigma_k(\vec{r}_1(0), \cdots, \vec{r}_N(0))\delta\tilde{\lambda}_k \approx 0$

Direct translation

Liouville operator

Computable on $a : \frac{da}{dt} = \{a, \mathcal{H}\}$

$iL = \sum_\alpha\left[\frac{\partial \mathcal{H}}{\partial q_\alpha}\frac{\partial}{\partial q_\alpha} - \frac{\partial \mathcal{H}}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha}\right] \Rightarrow iLa = \{a, \mathcal{H}\} \Rightarrow \frac{da}{dt} = iLa \Rightarrow a(x_t) = e^{iLt}a(x_0)$

Split $iL_1 = \sum_\alpha\frac{\partial \mathcal{H}}{\partial p_\alpha}\frac{\partial}{\partial q_\alpha} \qquad iL_2 = -\sum_\alpha\frac{\partial \mathcal{H}}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha}$

$iL_1 iL_2\phi(x) \neq iL_2 iL_1\phi(x) \Rightarrow iL_1 iL_2 - iL_2 iL_1 \equiv [iL_1, iL_2] \neq 0$

Trotter theorem

$[iL_1, iL_2] \neq 0 \Rightarrow e^{iLt} \neq e^{iL_1 t}e^{iL_2 t}$

$e^{A+B} = \lim_{P\to\infty}\left[e^{\frac{B}{2P}}e^{\frac{A}{P}}e^{\frac{B}{2P}}\right]^P \qquad e^{iLt} = \lim_{P\to\infty}\left[e^{\frac{iL_2 t}{2P}}e^{\frac{iL_1 t}{P}}e^{\frac{iL_2 t}{2P}}\right]^P$

$e^{iLt} \approx e^{\frac{iL_2\Delta t}{2}}e^{iL_1\Delta t}e^{\frac{iL_2\Delta t}{2}}$

Trotter algorithm

Exponential operator $e^{c\frac{\partial}{\partial x}}g(x) = g(x + c)$

$$\begin{pmatrix} x(\Delta t) \\ p(\Delta t) \end{pmatrix} = \begin{pmatrix} x(0) + \frac{\Delta t}{m}\left(p(0) + \frac{\Delta t}{2}F(x(0))\right) \\ p(0) + \frac{\Delta t}{2}F(x(0)) + \frac{\Delta t}{2}F\left(x(0) + \frac{\Delta t}{m}\left(p(0) + \frac{\Delta t}{2}F(x(0))\right)\right) \end{pmatrix}$$

$x(\Delta t) = x(0) + v(0)\Delta t + \frac{\Delta t^2}{2m}F(0) \qquad p(\Delta t) = v(0) + \frac{\Delta t}{2m}[F(0) + F(\Delta t)]$

RESPA

$iL = \frac{p}{m}\frac{\partial}{\partial x} + [F_{fast}(x) + F_{slow}(x)]\frac{\partial}{\partial p} = iL_{fast} + iL_{slow} \qquad \mathcal{H}_{ref} = \frac{p^2}{2m} + U_{fast}(x)$

$e^{iL\Delta t} = e^{iL_{slow}\frac{\Delta t}{2}}e^{iL_{fast}\Delta t}e^{iL_{slow}\frac{\Delta t}{2}}$

$e^{iL_{fast}\Delta t} = \left[e^{\frac{\delta t}{2}F_{fast}\frac{\partial}{\partial p}}e^{\delta t\frac{p}{m}\frac{\partial}{\partial x}}e^{\frac{\delta t}{2}F_{fast}\frac{\partial}{\partial p}}\right]^n \qquad \delta t = \frac{\Delta t}{n}$

Evaluation of energy and forces

Periodic boundary condition

Non bonded interaction $U_{nb}(\vec{r}_1, \ldots, \vec{r}_N) = \sum_{i<j\in nb}\left\{4\epsilon_{ij}\left[\left(\frac{\sigma_{ij}}{r_{ij}}\right)^{12} - \left(\frac{\sigma_{ij}}{r_{ij}}\right)^6\right] + \frac{q_i q_j}{r_{ij}}\right\}$

Error function $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad \lim_{x\to\infty} erf(x) = 1 \qquad erf(0) = 0$

Complement error $erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt \qquad \lim_{x\to\infty} erf(x) = 0 \qquad erf(0) = 1$

$\frac{1}{r} = \overbrace{\frac{erfc(\alpha r)}{r}}^{\text{short-ranged}} + \overbrace{\frac{erf(\alpha r)}{r}}^{\text{long-ranged}} \qquad U_{nb} = U_{short} + U_{long} \qquad \vec{r}_{ij} = |\vec{r}_i - \vec{r}_j + \vec{S}| \qquad \vec{S} = \vec{m}L$

$U_{short}(\vec{r}_1, \ldots, \vec{r}_N) = \sum_{\vec{S}}\sum_{i>j\in nb}\left\{4\epsilon_{ij}\left[\left(\frac{\sigma_{ij}}{r_{ij,\vec{S}}}\right)^{12} - \left(\frac{\sigma_{ij}}{r_{ij,\vec{S}}}\right)^6\right] + \frac{q_i q_j erfc(\alpha r_{ij,\vec{S}})}{r_{ij,\vec{S}}}\right\}$

$U_{long}(\vec{r}_1, \ldots, \vec{r}_N) = \sum_{\vec{S}}\sum_{i>h\in nb}\frac{q_i q_j erf(\alpha r_{ij,\vec{S}})}{r_{ij,\vec{S}}}$

Short range forces

$\tilde{U}_{short} = U_{short}(\vec{r}_{ij})S(\vec{r}_{ij}) \qquad S(r) = \begin{cases} 1 & r < r_C - \lambda \\ 1 + \left(\frac{r - r_C + \lambda}{\lambda}\right)^2\left(2\frac{r - r_C + \lambda}{\lambda} - 3\right) & r_C - \lambda z r \leq r_C \\ 0 & r > r_C \end{cases}$

Evaluation of energy and forces (contd)

Long range forces

$C_{\vec{g}} = \frac{4\pi}{|\vec{g}|^2}e^{-\frac{|\vec{g}|^2}{4\alpha^2}} \qquad \frac{1}{V}\sum_{\vec{g}} C_{\vec{g}}e^{i\vec{g}\cdot\vec{r}} = \frac{1}{V}\sum_{i\neq j} q_i q_j\sum_{\vec{g}\in\mathcal{S}}\frac{4\pi}{|\vec{g}|^2}e^{-\frac{|\vec{g}|^2}{2\alpha^2}}e^{i\vec{g}\cdot(\vec{r}_i - \vec{r}_j)}$

$U_{long} = \frac{1}{V}\sum_{i,j} q_i q_j\sum_{\vec{g}\in\mathcal{S}}\frac{4\pi}{|\vec{g}|^2}e^{-\frac{|\vec{g}|^2}{4\alpha^2}}e^{i\vec{g}\cdot(\vec{r}_i - \vec{r}_j)} - \frac{1}{V}\sum_i q_i^2\sum_{\vec{g}\in\mathcal{S}}\frac{4\pi}{|\vec{g}|^2}e^{-\frac{|\vec{g}|^2}{4\alpha^2}}$

$U_{long} = \frac{1}{V}\sum_{\vec{g}\in\mathcal{S}}\frac{4\pi}{|\vec{g}|^2}e^{-\frac{|\vec{g}|^2}{4\alpha^2}}|S(\vec{g})|^2 - \frac{\alpha}{\sqrt{\pi}}\sum_i q_i^2$

Particle-particle particle-mesh Ewald

$\rho(\vec{r}) = \sum_i q_i\delta(\vec{r} - \vec{r}_i) \qquad \rho(\vec{g}) = \int d\vec{r}\rho(\vec{r})e^{i\vec{g}\cdot\vec{r}} = \sum_i q_i e^{i\vec{g}\cdot\vec{r}_i}$

$\nabla^2\phi(\vec{r}) = -\nabla\cdot\vec{E} = -4\pi\rho(\vec{r}) \qquad g^2\phi(\vec{g}) = -4\pi\rho(\vec{g}) = 4\pi S(\vec{g})$

Canonical ensemble

Thermodynamics

Helmholtz free energy $A(N, V, T) = E(N, V, T) - TS(N, V, T)$

$dA = SdT - PdV + \mu dN \qquad S = -\left(\frac{\partial A}{\partial T}\right)_{N,V} \qquad P = -\left(\frac{\partial A}{\partial V}\right)_{N,T} \qquad \mu = \left(\frac{\partial A}{\partial N}\right)_{V,T}$

Thermal contact

Microcanonical $\Omega(N, V, E) = M_N\int dx_1 dx_2\delta(\mathcal{H}_1(x_1) + \mathcal{H}_2(x_2) - E)$

Distribution function $\ln f(x_1) = \ln\int dx_2\delta(\mathcal{H}_1(x_1) + \mathcal{H}_2(x_2) - E)$

$\ln f(x_1) \approx \ln\int dx_2\delta(\mathcal{H}_2(x_2) - E) - \frac{\partial}{\partial E}\ln\int dx_2\delta(\mathcal{H}_2(x_2) - E)\mathcal{H}_1(x_1)$

$Q(N, V, T) = \frac{1}{N!h^{3N}}\int dxe^{-\beta\mathcal{H}(x)} \qquad \beta = \frac{1}{kT}$

From micro to macro

$A = E - \beta\left(\frac{\partial A}{\partial \beta}\right)_{N,V}$

$E = \langle \mathcal{H} \rangle = \frac{1}{N!h^{3N}}\frac{\int dx\mathcal{H}(x)e^{-\beta\mathcal{H}(x)}}{\int dxe^{-\beta\mathcal{H}(x)}} = -\frac{1}{Q(N,V,\beta)}\frac{\partial Q(N,V,\beta)}{\partial \beta} = -\frac{\partial\ln Q(N,V,\beta)}{\partial \beta}$

$A + \frac{\partial\ln Q}{\partial \beta} + \beta\frac{\partial A}{\partial \beta} = 0 \Rightarrow \ln Q(N, V, \beta) = -\beta A(N, V, \beta)$

$A(N, V, T) = -kT\ln Q(N, V, T) \qquad C_N = \frac{1}{N!h^{3N}}$

Energy $E = \langle \mathcal{H} \rangle = \frac{1}{Q}\frac{\partial Q}{\partial \beta} = 0$

Temperature estimator $\mathcal{T}(x) + \frac{1}{3Nk}\sum_i\frac{\vec{p}_i^2}{m_i} \qquad T = \langle \mathcal{T}(x) \rangle = \frac{C_N\int dx\mathcal{T}(x)e^{-\beta\mathcal{H}(x)}}{C_N\int dxe^{-\beta\mathcal{H}(x)}}$

Energy fluctuation $\Delta E^2 = \frac{\partial^2\ln Q}{\partial \beta^2} = kT^2 C_V \qquad \frac{\Delta E}{E} \sim \frac{1}{\sqrt{N}}$

Pressure estimator $\mathcal{P}(\vec{r}, \vec{p}) = \frac{1}{3V}\sum_i\left[\frac{\vec{p}_i^2}{m_i} + \vec{F}_i\cdot\vec{r}_i\right]$

Thermostats

Velocity rescaling

$\bar{K} = \frac{N_j}{2\beta} \qquad K = \frac{1}{2}\sum_i m_i\vec{v}_i^2 \qquad \vec{v}_i \to \frac{\vec{v}_i}{\alpha} \qquad \alpha = \sqrt{\frac{\bar{K}}{K}}$

$\bar{K} = \frac{1}{2}\sum_i m_i\frac{\vec{v}_i^2}{\alpha^2} = \frac{\bar{K}}{2K}\sum_i m_i\vec{v}_i^2$

Andersen-Heyes Thermostat

$P(p) = \left(\frac{\beta}{2\pi m}\right)^{\frac{3}{2}e^{-\beta\frac{p^2}{2m}}}$

$P(K_t)dK_t \propto K_t^{\frac{N_f}{2}-1}e^{-\beta K_t}dK_t$

$K_t = \frac{1}{2}\sum_i m_i\frac{\vec{v}_i^2}{\alpha^2} = \frac{K_t}{2K}\sum_i m_i\vec{v}_i^2$

Langevin Thermostat

$\dot{\vec{r}}_i(t) = \beta D\vec{F}_i(\vec{r}_1(t), \ldots, \vec{r}_N(t)) + \eta_i(t) \qquad \frac{\partial}{\partial t}P(\vec{x}, t) = D\sum_{i=1}^{N}\nabla_i\cdot[\nabla_i + \beta\nabla\mathcal{H}(\vec{x})]$

$P(\vec{x}) = Ce^{-\beta\mathcal{H}(\vec{x})} \qquad \langle \eta_i(t)\eta_j(t) \rangle = 6D\delta_{ij}\frac{\delta_{kl}}{\Delta t} \qquad \sigma^2 = \frac{6D}{\Delta t}$

$\xi_i(t_k) = \sqrt{\frac{\Delta t}{6D}}\eta_i(t_k) \qquad \langle \xi_i(t_k) \rangle = 0 \qquad \langle \xi_i(t_k)\xi_j(t_l) \rangle = \delta_{ij}\delta_{kl}$

$\vec{r}_i(t + \Delta t) = \vec{r}_i(t) + \Delta t\frac{\vec{F}_i(\vec{r}_1(t), \ldots, \vec{r}_N(t))}{m_i\gamma} + \sqrt{\frac{6kT\Delta t}{m_i\gamma}}\xi_i(t)$

j

Bussi velocity Verlet

$dK = \left(D(K)\frac{\partial\log P(K)}{\partial K} + \frac{\partial D(K)}{\partial K}\right)dt + \sqrt{2D(K)}dW \qquad P(K_t)dK_t \propto K_t^{\frac{N_f}{2}1}e^{\beta K_t}dK$

$dK = \left(\frac{N_f D(K)}{2K\bar{K}}(K - \bar{K}) - \frac{D(K)}{K} + \frac{\partial D(K)}{\partial K}\right)dt + \sqrt{2D(K)}dW \qquad D(K) = \frac{2K\bar{K}}{N_f\tau}$

$dK = (K - \bar{K})\frac{dt}{\tau} + \sqrt{\frac{2K\bar{K}}{N_f}}\frac{dW}{\sqrt{\tau}}$

Berendsen's thermostat $dK = (K - \bar{K})\frac{dt}{\tau}$

$\alpha^2 = e^{-\frac{\Delta t}{\tau}} + \frac{\bar{K}}{N_f K}\left(1 - e^{-\frac{\Delta t}{\tau}}\right)\sum_{i=1}^{N_f}R_i^2 + 2e^{-\frac{\Delta t}{2\tau}}\sqrt{\frac{\bar{K}}{N_f K}\left(1 - e^{-\frac{\Delta t}{\tau}}R_1\right)}$

Nosè Hamiltonian

$\mathcal{H}_N = \sum_i\frac{\vec{p}_i^2}{2m_i s^2} + U(\vec{r}_1, \ldots, \vec{r}_n) + \frac{p_s^2}{2Q} + gkT\log s$

$\Omega = \int d^N\vec{r}d^N\vec{p}dsdp_s s^{dN}\delta\left(\mathcal{H} + \frac{p_s^2}{2Q} + gkT\log s - E\right)$

$\Omega = \frac{1}{dkT}\int d^N\vec{r}d^N\vec{p}dp_s e^{\frac{dN+1}{gkT}\left(E - \mathcal{H} - \frac{p_s^2}{2Q}\right)}$

$g = dN + 1 \Rightarrow \Omega = \frac{e^{\frac{e}{kT}}\sqrt{2\pi QkT}}{(dN+1)kT}\int d^N\vec{r}d^N\vec{p}e^{-\frac{\mathcal{H}}{kT}}$

$\dot{\vec{r}}_i = \frac{\vec{p}_i}{m_i}s^2 \qquad \dot{\vec{p}}_i = \vec{F}_i \qquad \dot{s} = \frac{p_s}{Q} \qquad \dot{p}_s = \frac{1}{s}\left[\sum_i\frac{\vec{p}_i^2}{m_i s^2} - gkT\right]$

Thermostats (contd)

Nosè-Hoover equations

$\vec{p}'_i = \frac{\vec{p}_i}{s}$ $\quad \vec{p}'_s = \frac{p_s}{s}$ $\quad dt' = \frac{dt}{s}$ $\quad \frac{d\vec{r}_i}{dt'} = \frac{\vec{p}'_i}{m_i}$ $\quad \frac{d\vec{p}'_i}{dt'} = \vec{F}_i - \frac{sp'_s}{Q}\vec{p}'_i$

$\frac{ds}{dt'} = \frac{s^2 p'_s}{Q}$ $\quad \frac{dp'_s}{dt'} = \frac{1}{2}\left[\sum_i \frac{(\vec{p}'_i)^2}{m_i} - gkT\right]^2 - \frac{s(p'_s)^2}{Q}$ $\quad \frac{1}{2}\frac{ds}{dt'} = \frac{d\eta}{dt'}$ $\quad p_s = p_\eta = sp'_s$

$\dot{\vec{r}}_i = \frac{\vec{p}_i}{m_i}$ $\quad \dot{\vec{p}}_i = \vec{F}_i - \frac{P_\eta}{Q}\vec{p}_i$ $\quad \dot{\eta} = \frac{p_\eta}{Q}$ $\quad \dot{p}_\eta = \sum_i \frac{\vec{p}_i^2}{m_i} - dNkT$

Non Hamiltonian statistical mechanics

$\dot{x} = \xi(x,t)$ $\quad \nabla \cdot \dot{x} = \nabla \cdot \xi(x,t) = \kappa(x,t) \neq 0$

$J(x_t; x_0) = e^{\int_0^t ds\kappa(x_s,s)}$ $\quad \kappa(x_t,t) = \frac{dw(x_t,t)}{dt} \Rightarrow J(x_t;x_0) = e^{w(x_t,t)-w(x_0,0)}$ $\quad e^{-w(x_t,t)}dx_t = e^{-w(x_0,0)}dx_0$

$J(x_t;x_0) = \frac{\sqrt{g(x_0,0)}}{\sqrt{g(x_t,t)}}$ $\quad \sqrt{g(x_t,t)} = e^{-w(x_t,t)}$

$\frac{\partial}{\partial t}\left[f(x,t)\sqrt{g(x,t)}\right] + \nabla \cdot \left[\dot{x}\sqrt{g(x,t)}f(x,t)\right] = 0$ $\quad f(x_t,t)\sqrt{g(x_t,t)}dx_t = f(x_0)\sqrt{g(x_0)}dx_0$ $\quad \xi(x)\cdot\nabla f(x) = 0$

$\Lambda_k(x_t) - C_k = 0$ $\quad \frac{d\Lambda_k(x_t)}{dt} = 0 \Rightarrow f(x) = \prod_{k=1}^{N_C}\delta(\Lambda_k(x_t)-C_k)$

Microcanonical $\mathcal{E} = \int dx\sqrt{g(x)}f(x) = \int dx\sqrt{g(x)}\prod_{k=1}^{N_C}\delta(\Lambda_k(x_t)-C_k)$

Nosè=Hoover: $\mathcal{H}'(\vec{r},\eta,\vec{p},p_\eta) = \mathcal{H}(\vec{r},\vec{P}) + \frac{p_\eta^2}{2Q} + dNkT\eta$ $\quad \frac{d\mathcal{H}'}{dt} = 0$

$\kappa = -Nd\dot{\eta} \Rightarrow \sqrt{g} = e^{dN\eta}$

Partition function $\mathcal{E}_T(N,V,C_1) = \frac{e^{\beta C_1}\sqrt{2\pi QkT}}{dNkT}\int d^N\vec{p}\int_{\mathcal{D}(V)} d^N\vec{r}e^{-\beta\mathcal{H}(\vec{r},\vec{p})}$

$\vec{P} = \sum_{i=1}^N \vec{p}_1$ $\quad \vec{K} = \vec{P}e^\eta \Rightarrow \frac{d\vec{K}}{dt} = 0$

Nosè-Hoover chains

$\dot{\vec{r}}_i = \frac{\vec{p}_i}{m_i}$ $\quad \dot{\vec{p}}_i = \vec{F}_i - \frac{p_{\eta_1}}{Q_1}\vec{p}_i$ $\quad \dot{\eta}_j = \frac{p_{\eta_j}}{Q_j}$ $\quad j = 1,\ldots,M$

$\dot{p}_{\eta_1} = \left[\sum_i \frac{\vec{p}_i^2}{m_i} - dNkT\right] - \frac{p_{\eta_2}}{Q_2}p_{\eta_1}$ $\quad \dot{p}_{\eta_j} = \left[\sum_i \frac{\vec{p}^2_{\eta_{j-1}}}{Q_{j-1}} - kT\right] - \frac{p_{\eta_{j+1}}}{Q_{j+1}}p_{\eta_j}$ $\quad j = 2,\ldots,M-1$

$\dot{p}_{\eta_M} = \frac{p^2_{\eta_{M-1}}}{Q_{M-1}} - kT$

$\mathcal{H}'(\vec{r},\eta,\vec{p},p_\eta) = \mathcal{H}(\vec{r},\vec{p}) + \sum_{j=1}^M \frac{p^2_{\eta_j}}{2Q_j} + dNkT\eta_1 + kT\sum_{j=2}^M \eta_j$

$\kappa = -dN\dot{\eta}_1 - \dot{\eta}_c$ $\quad \eta_c = \sum_{j=2}^M \eta_j$ $\quad \sqrt{g} = e^{dN\eta_1+\eta_c}$

Partition function $\mathcal{E}_T(N,V,C_1) = \mathcal{M}\int d^N\vec{p}\int_{\mathcal{D}(V)} d^N\vec{r}e^{-\beta\mathcal{H}(\vec{r},\vec{p})}$

Isobaric ensemble

Legendre transforms

Enthalpy $dH = TdS + \mu dN + VdP$ $\quad T = \left(\frac{\partial H}{\partial S}\right)_{N,P}$ $\quad \langle V\rangle = \left(\frac{\partial H}{\partial P}\right)_{N,S}$ $\quad \mu = \left(\frac{\partial H}{\partial N}\right)_{P,S}$

Gibbs $dG = \mu dN + VdP - SdT$ $\quad S = -\left(\frac{\partial G}{\partial T}\right)_{N,P}$ $\quad \langle V\rangle = -\left(\frac{\partial G}{\partial P}\right)_{N,T}$ $\quad \mu = -\left(\frac{\partial G}{\partial N}\right)_{P,T}$

Isoenthalpic-isobaric ensemble

$H = \mathcal{H}(v) + PV$ $\quad f(x) = F(\mathcal{H}(x)) = \mathcal{M}\delta(\mathcal{H}(x) + PV - H)$

$\Gamma(N,P,H) = \mathcal{M}\int_0^\infty dV\int d^N\vec{p}\int_{\mathcal{D}(V)} d^N\vec{r}\delta(\mathcal{H}(\vec{r}m\vec{p}) + PV - H)$

$S(N,P,H) = k\ln\Gamma(N,P,H)$ $\quad \frac{1}{T} = \left(\frac{\partial S}{\partial H}\right)_{N,P}$ $\quad \frac{\langle V\rangle}{T} = -\left(\frac{\partial S}{\partial P}\right)_{N,H}$ $\quad \frac{\mu}{T} = -\left(\frac{\partial S}{\partial N}\right)_{P,H}$

Isothermal-isobaric ensemble

$Q(N,V,T) \propto Q(N_1,V_1,T)Q(N_2,V_2,T)$

$f(x_1) = I_{N_1}e^{\beta\mu N_1}e^{-\beta PV_1}C_{N_1}e^{-\beta\mathcal{H}_1(x_1)}$ $\quad I_{N_1} = \frac{1}{V_0 N_1! h^{3N_1}}$

$\Delta(N,P,T) = \frac{1}{V_0}\int_0^\infty dVe^{-\beta PV}Q(N,V,T)$

$G(N,P,\beta) = -\frac{\partial\ln\Delta(N,P,\beta)}{\partial\beta} - \beta\frac{\partial G}{\partial\beta}$

Virial theorems

Pressure $\langle P^{(int)}\rangle = \frac{P}{\Delta(N,P,T)}\int_0^\infty dVe^{-\beta PV}Q(N,V,T) = P$ Work $\langle P^{(int)}V\rangle + kT = P\langle V\rangle$

Andersen's Hamiltonian

$\mathcal{H}_A = \sum_{i=1}^N \frac{V^{-\frac{2}{3}}\pi_i^2}{2m_i} + U(V^{\frac{1}{3}}\vec{s}_1,\ldots,V^{\frac{1}{3}}\vec{s}_N) + \frac{p_V^2}{2W} + PV$ $\quad W = (3N+1)kT\tau_b^2$

$\dot{\vec{s}}_i = \frac{p_i}{m_i} + \frac{\dot{V}}{3V}r_i$ $\quad \dot{\pi}_i = -\frac{\partial U}{\partial r_i} - \frac{\dot{V}}{3V}p_i$ $\quad \dot{V} = \frac{p_V}{W}$ $\quad \dot{p}_V = \frac{1}{3V}\sum_{i=1}^N\left[\frac{p_i^2}{m_i} - \frac{\partial U}{\partial r_i}r_i\right] - P$

$\dot{\vec{r}}_i = \frac{\vec{p}}{m_i} + \frac{\dot{V}}{3V}\vec{r}_i$ $\quad \dot{\vec{p}}_i = -\frac{\partial U}{\partial\vec{r}_i} - \frac{\dot{V}}{3V}\vec{p}_i$ $\quad \dot{V} = \frac{p_V}{W}$ $\quad \dot{p}_V = \frac{1}{3V}\sum_{i=1}^N\left[\frac{\vec{p}_i^2}{m_i} - \frac{\partial U}{\partial\vec{r}_i}\cdot\vec{r}_i\right] - P$

$\left\langle\frac{p_V^2}{2W}\right\rangle = k\frac{T}{2} \Rightarrow \mathcal{H}(\vec{r},\vec{p}) + PV$ is conserved

MTK algorithm

$\epsilon = \frac{1}{3}\ln\frac{V}{V_0} \Rightarrow \dot{\epsilon} = \frac{\dot{V}}{3V} = \frac{p_\epsilon}{W}$

$\dot{\vec{r}}_i = \frac{\vec{p}_i}{m_i} + \frac{p_\epsilon}{W}\vec{r}_i$ $\quad \dot{\vec{p}}_i = -\frac{\partial U}{\partial\vec{r}_i} - \frac{p_\epsilon}{W}\vec{p}_i$ $\quad \dot{V} = \frac{dVp_\epsilon}{W}$ $\quad \dot{p}_\epsilon = dV(\mathcal{P}^{(int)} - P)$

$\kappa = \frac{\dot{V}}{V}$ $\quad \dot{\vec{p}}_i = -\frac{\partial U}{\partial\vec{r}_i} - \left(1 + \frac{d}{N_F}\right)\frac{p_\epsilon}{W}\vec{p}_i$ $\quad \dot{p}_\epsilon = dV(\mathcal{P}^{(int)} - P) + \frac{d}{N_f}\sum_{i=1}^N\frac{\vec{p}_i^2}{m_i}$

Langevin piston $\dot{\vec{r}}_i = \frac{\vec{p}_i}{m_i} + \frac{\dot{V}}{3V}\vec{r}_i$ $\quad \dot{\vec{p}}_i = -\frac{\partial U}{\partial\vec{r}_i} - \frac{\dot{V}}{3V}\vec{p}_i$ $\quad \dot{V} = \frac{p_V}{W}$

$\dot{p}_V = \frac{1}{3V}\sum_{i=1}^N\left[\frac{\vec{p}_i^2}{m_i} - \frac{\partial U}{\partial\vec{r}_i}\cdot\vec{r}_i\right] - P - \gamma\dot{V} + R(t)$ $\quad \langle R(0)R(t)\rangle = \frac{2\gamma kT}{W}\delta(t)$

Grand canonical ensemble

Euler's theorem

$U(\lambda N,\lambda V,\lambda S) = \lambda U(N,V,S) \Rightarrow U(N,V,S) = \mu N - PV + TS$

$A(\lambda N,\lambda V,T) = \lambda A(N,V,T) \Rightarrow A(N,V,T) = \mu N - PV$

$H(\lambda N,\lambda S,P) = \lambda H(N,S,P) \Rightarrow H(N,S,P) = \mu N + TS$

$G(\lambda N,P,T) = \lambda G(N,P,T) \Rightarrow G(N,P,T) = \mu N$

Thermodynamics

$\tau(A)\left(\frac{\partial A}{\partial N},V,T\right) = -PV$ $\quad d\tilde{A} = -PdVpSdT - Nd\mu$ $\quad \tilde{A}(\mu,\lambda V,T) = \lambda\tilde{A}(\mu,V,T)$

Partition function $\sum_{N=0}^\infty \frac{1}{N!h^{3N}}e^{\beta\mu N}\int dxe^{-\beta\mathcal{H}(x,N)} = e^{\beta PV}$

$\mathcal{E}(\mu,V,T) = \sum_{N=0}^\infty \frac{1}{N!h^{3N}}e^{\beta\mu N}\int dxe^{-\beta\mathcal{H}(x,N)} = \sum_{N=0}^\infty e^{\beta\mu N}Q(N,V,T) = e^{\beta PV}$

$\frac{PV}{kT} = \ln\mathcal{E}(\mu,V,T)$ $\quad z = e^{\beta\mu}$ $\quad \mathcal{E}(z,V,T) = \sum_{N=0}^\infty z^N Q(N,V,T)$

$\langle N\rangle = \frac{1}{\mathcal{E}(\mu,V,T)}\sum_{N=0}^\infty Ne^{\beta\mu N}Q(N,V,T) = kT\left(\frac{\partial}{\partial\mu}\ln\mathcal{E}(\mu,V,T)\right)_{V,T}$

$\langle N\rangle = z\frac{\partial}{\partial z}\ln\mathcal{E}(z,V,T)$

Ideal gas

$Q(N,V,T) = \frac{1}{N!}\left[V\left(\frac{2\pi m}{\beta h^2}\right)^{\frac{3}{2}}\right]^N = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N$ $\quad \mathcal{E}(z,V,T) = e^{zV}\lambda^3$

$\langle N\rangle = V\frac{z}{\lambda^3} \Rightarrow z = \frac{\langle N\rangle\lambda^3}{V}$ $\quad \frac{PV}{PkT} = \ln\mathcal{E}(z,V,T) = \frac{Vz}{\lambda^3} = \langle N\rangle$

Particle number fluctuations

$\Delta N = \sqrt{\langle N^2\rangle - \langle N\rangle^2}$ $\quad z\frac{\partial}{\partial z}z\frac{\partial}{\partial z}\ln\mathcal{E}(z,V,T) = \langle N^2\rangle - \langle N\rangle^2$

$\Delta N^2 = kTV\frac{\partial^2 P}{\partial\mu^2}$ $\quad a(v,T) = \frac{1}{N}A\left(N,\frac{V}{N},T\right) \Rightarrow \mu = a(v,T) - V\frac{\partial a}{\partial v}$

$P = -\frac{\partial a}{\partial v}$ $\quad \frac{\partial P}{\partial\mu} = -\frac{\partial^2 a}{\partial v^2}\frac{\partial v}{\partial\mu}$ $\quad \frac{\partial\mu}{\partial v} = \frac{\partial a}{\partial v} - \frac{\partial a}{\partial v} - v\frac{\partial^2 a}{\partial v^2} \Rightarrow \frac{\partial P}{\partial\mu} = \frac{1}{v}$

$\frac{\partial^2 P}{\partial\mu^2} = -\frac{1}{v^3}\frac{\partial P}{\partial v}$ $\quad \kappa_T = -\frac{1}{v\frac{\partial P}{\partial v}} \Rightarrow \Delta N^2 = \frac{\langle N\rangle kT}{v}\kappa_T$ $\quad \frac{\Delta N}{\langle N\rangle} \sim \frac{1}{\sqrt{\langle N\rangle}}$

Quantifying uncertainties and sampling quality

Autocorrelation analysis

$C_f(t') = \frac{\langle(f(x)-\langle f\rangle)(f(t+t')-\langle f\rangle)\rangle}{\sigma_f^2}$

$C_f(t') = \frac{1}{\sigma_f^2}\frac{1}{N}\sum_{i=1}^{N-\frac{t'}{\Delta t}}(f(j\Delta t) - \langle f\rangle)(f(j\Delta t + t') - \langle f\rangle)$ $\quad \tau_f = \int_0^{+\infty} dt'C_f(t')$

$N_f^{ind} \simeq \frac{t_{sim}}{\tau_f}$ $\quad SE(f) = \frac{\sigma_f}{\sqrt{N_f^{ind}}} \sim \sigma_f\sqrt{\frac{\tau_f}{t_{sim}}}$

Block averaging analysis

$BSE(f,n) = \frac{\sigma_n}{\sqrt{M}}$

Protein motions

Gaussian network models

$U_{ij} = \gamma_{ij}(\Delta\vec{r}_j - \Delta\vec{r}_j)\cdot(\Delta\vec{r}_j - \Delta\vec{r}_j) = \gamma_{ij}\Delta\vec{r}_{ij}^2$ $\quad U_{GNM} = \frac{\gamma}{2}\sum_i\sum_j\Delta\vec{r}_{ij}^2$

$U_{GNM} = \frac{\gamma}{2}\Delta\vec{r}(t)^T\Gamma\Delta\vec{r}(t)$ $\quad \langle\Delta\vec{r}_i\cdot\Delta\vec{r}_j\rangle = \frac{3kT}{\gamma}\left[\Gamma^{-1}\right]_{ij}$

$\langle\Delta\vec{r}_i\cdot\Delta\vec{r}_i\rangle = \sum_k[\Delta r_i^2]_k$ $\quad B_i = \frac{8\pi^2}{3}\langle\Delta r_i^2\rangle$

Normal mode analysis: $U = \frac{1}{2}\sum_i\sum_j H_{ij}(q_i - q_i^0)(q_j - q_j^0)$

$C = \langle\Delta\vec{q}\Delta\vec{q}^T\rangle = \frac{1}{Q}\int\Delta\vec{q}\Delta\vec{q}^T e^{-\frac{\Delta\vec{q}^T H\Delta\vec{q}}{2kT}}d^N\Delta\vec{q} = kTH^{-1}$

Anisotropic network model

$U_{ANM} = \frac{1}{2}\sum_{ij}\gamma(r_{ij} - r_{ij}^0)^2 \Rightarrow \frac{\partial^2 U}{\partial x_i\partial x_j} = -\gamma\frac{(x_i-x_j)(y_i-y_j)}{t_{ij}^2}$ $\quad \langle\Delta\vec{r}_i\cdot\Delta\vec{r}_j\rangle = \frac{3kT}{\gamma}\sum_k\lambda_k^{-1}[\vec{u}_k\vec{u}_k^T]_{ij}$

Correlation cosine $I_k = \frac{\Delta\vec{q}_{AB}\cdot\vec{u}_k}{|\Delta\vec{q}_{AB}|}$ $\quad C_0 = \sqrt{\sum_k I_k^2}$

Degree of connectivity $\kappa_k = N^{-1}e^{-\sum_{i=1}^N\alpha(\Delta r_i)^2\big|_k\log(\alpha(\Delta r_i)^2\big|_k)}$ $\quad \sum_{i=1}^N\alpha(\Delta r_i)^2\big|_k = 1$

Essential dynamics

Covariance matrix $C_{ij} = \overline{(x_i(t) - \overline{x_i(t)})(x_j(t) - \overline{x_j(t)})}$

Correlation matrix $R_{ij} = \frac{\overline{(x_i(t)-\overline{x_i(t)})(x_j(t)-\overline{x_j(t)})}}{\sigma_{x_i}\sigma_{x_j}}$

$PC_k(t) = \sum_{i=1}^N\vec{d}_i(t)\cdot\vec{u}_i^k$ $\quad c_k = \frac{2}{T_{sim}}\left(\int_0^{T_{sim}}\cos\left(\pi\frac{k_BT}{\lambda_k}t\right)PC_k(t)dt\right)^2\left(\int_0^{T_{sim}}PC_k^2(t)dt\right)^{-1}$

$$\Omega_{A,B} = 1 - \left[\frac{\sum_{k=1}^{3N-6}(\lambda_k^A+\lambda_k^B) - 2\sum_{k=1}^{3N-6}\sum_{j=1}^{3N-6}\sqrt{\lambda_k^A\lambda_j^B}(\vec{u}_k^A\cdot\vec{u}_j^B)^2}{\sum_{k=1}^{3N-6}(\lambda_k^A+\lambda_k^B)}\right]^{\frac{1}{2}}$$

Monte Carlo methods

Introduction

$f(x) \geq 0 \wedge \int f(x)dx = 1$ $\quad I = \int dx\phi(x)f(x) \equiv \langle\phi\rangle_f$

$\tilde{I}_M = \frac{1}{M}\int_{i=1}^M\phi(x_i)$ $\quad \lim_{M\to\infty}\tilde{I}_M = I$

$\int dx\phi(x)f(x) = \frac{1}{M}\int_{i=1}^M\phi(x_i) \pm \frac{1}{\sqrt{M}}\left[\langle\phi^2\rangle_f - \langle\phi\rangle_f^2\right]^{\frac{1}{2}}$

$P(X) = \int_a^X f(x)dx$ $\quad f(X) = \frac{dP}{dX}$ $\quad X \geq x \Rightarrow g(X) \geq g(x)$ $\quad \tilde{P}(Y = g(X)) = P(X)$

Monte Carlo methods (contd)

Importance sampling

$I = \int dx\phi(x)f(x) = \int dx\left[\frac{\phi(x)f(x)}{h(x)}\right]h(x) = \int dx\psi(x)h(x) = \frac{1}{M}\sum_{i=1}^M\psi(x_i) \pm \frac{1}{\sqrt{M}}\left[\langle\psi^2\rangle_h - \langle\psi\rangle_h^2\right]^{\frac{1}{2}}$

$\sigma^2[h] = \int dx\frac{\phi^2(x)f^2(x)}{h(x)} - \left[\int dx\phi(x)f(x)\right]^2$

$\delta F[h] = -\frac{\phi^2(x)f^2(x)}{h^2(x)}\delta h(x) - \lambda\delta h(x)$ $\quad \frac{\delta F[h]}{\delta h(x)} = 0 \Rightarrow h(x) = \frac{1}{\sqrt{-\lambda}}\phi(x)f(x)$

Markov chains

Detailed balance $R(x|y)f(y) = R(y|x)f(x)$

Rejection method $R(x|y) = A(x|y)T(x|y)$ $\quad r(x|y) = \frac{T(y|x)f(x)}{T(x|y)f(y)}$

$A(x|y) = \min[1, r(x|y)]$

Metropolis algorithm $r(x_{k+1}|x_k) = \frac{T(x_k|x_{k+1})f(x_{k+1})}{T(x_{k+1}|x_k)f(x_k)}$

Canonical distribution $A(r'|r) = \min\left[1, e^{-\beta(U(r')-U(r))}\right]$

Trial move $\begin{cases} x'_i = x_i + \frac{1}{\sqrt{3}}(\xi_x - 0.5)\Delta \\ y'_i = y_i + \frac{1}{\sqrt{3}}(\xi_y - 0.5)\Delta \\ z'_i = z_i + \frac{1}{\sqrt{3}}(\xi_z - 0.5)\Delta \end{cases}$

Isothermal-isobaric $A(V'|V) = \min\left[1, e^{-\beta P(V'-V)}e^{N\ln\frac{V'}{V}}e^{-\beta(U(r')-I)}\right]$

Gran canonical $\begin{cases} A(N+1|N) = min\left[1, \frac{V}{\lambda^3(N+1)}e^{\beta\mu}e^{-\beta(U(r')-U(r))}\right] \\ A(N-1|N) = \min\left[1, \frac{\lambda^3 N}{V}e^{-\beta\mu}e^{-\beta(U(r')-U(r))}\right] \end{cases}$

Hybrid Monte Carlo

$A(r',p'|r,p) = \min\left[1, e^{-\beta\Delta\mathcal{H}}\right]$ $\quad T(r',p'|r,p) = T(r,-p|r',-p')$

$\int d^Npd^Np'T(r',p'|r,p)A(r',p'|r,p)f(r,p) = \int d^Npd^Np'T(r,p|r',p')A(r,p|r',p')f(r',p')$

Free energy calculations

Free energy perturbation theory

$\Delta A_{AB} = -kT\ln\frac{Z_B}{Z_A}$ $\quad \frac{Z_B}{Z_A} = \left\langle e^{-\beta[U_B(\vec{r}_1,\ldots,\vec{r}_N)-U_A(\vec{r}_1,\ldots,\vec{r}_N)]}\right\rangle_A$

$\Delta A_{AB} = -kT\ln\left\langle e^{-\beta[U_B(\vec{r}_1,\ldots,\vec{r}_N)-U_A(\vec{r}_1,\ldots,\vec{r}_N)]}\right\rangle_A$

Adiabatic switching: $\frac{kT}{Z}\frac{\partial Z}{\partial\lambda} = \frac{kT}{Z}\int d^N\vec{r}\left(-\beta\frac{\partial U}{\partial\lambda}\right)e^{-\beta U(\vec{r}_1,\ldots,\vec{r}_N)} = -\langle\frac{\partial U}{\partial\lambda}\rangle$

Thermodynamics integration: $\Delta A_{AB} = \int_0^1\left(\frac{\partial A}{\partial\lambda}\right)d\lambda = \int_0^1\left\langle\frac{\partial U}{\partial\lambda}\right\rangle_\lambda d\lambda$

Adiabatic free energy dynamics: $\left\langle\frac{p_\lambda^2}{2m_\lambda}\right\rangle = kT_\lambda$ $\quad Z(\lambda,\beta) = \int d^N\vec{r}e^{-\beta U(\vec{r},\lambda)}$

$\mathcal{H}_{eff}(\lambda,p_\lambda) = \frac{p_\lambda^2}{2m_\lambda} - \frac{1}{\beta}\ln Z(\lambda,\beta)$ $\quad \tilde{P}_{adb}(\lambda,\beta,\beta_\lambda) = [Z(\lambda,\beta)]^{\frac{\beta_\lambda}{\beta}}$

$A(\lambda) = -kT\ln Z(\lambda,\beta) + const$

Jarzynski's equality

$W_{AB} = \langle\mathcal{W}_{AB}(x)\rangle_A \geq \Delta A_{AB}$ $\quad \langle e^{-\beta\mathcal{W}_{AB}(x_0)}\rangle_A = e^{-\beta\Delta A_{AB}}$

Pulling experiment: $U(\vec{r}_1,\ldots,\vec{r}_N,t) = U_0(\vec{r}_1,\ldots,\vec{r}_N) + \frac{1}{2}k(|\vec{r}_1 - \vec{r}_N| - r_{eq} - vt)^2$

$\langle e^{-\beta\mathcal{W}_\tau}\rangle = \int\mathcal{W}_\tau P(\mathcal{W}_\tau)e^{-\beta\mathcal{W}_\tau}$

$P(\mathcal{W}_\tau) \sim N \Rightarrow \ln\langle e^{-\beta\mathcal{W}_\tau} \simeq -\beta\langle\mathcal{W}_\tau\rangle + \frac{\beta^2}{2}(\langle\mathcal{W}_\tau^2\rangle - \langle\mathcal{W}_\tau\rangle^2)$

Replica exchange Monte Carlo

$F\left(\vec{r}^{(1)},\ldots,\vec{r}^{(M)}\right) = \prod_{K=1}^M f_K\left(\vec{r}^{(K)}\right)$ $\quad f_K\left(\vec{r}^{(K)}\right) = \frac{e^{-\beta_K U\left(\vec{r}^{(K)}\right)}}{Q(N,V,T_K}$

$T\left(\tilde{\vec{r}}^{(K)},\tilde{\vec{r}}^{(K+1)}|\vec{r}^{(K)},\vec{r}^{(K+1)}\right) = T\left(\vec{r}^{(K)},\vec{r}^{(K+1)}|\tilde{\vec{r}}^{(K)},\tilde{\vec{r}}^{(K+1)}\right)$

$A(\vec{r}^{(K+1)},\vec{r}^{(K)}|\vec{r}^{(K)},\vec{r}^{(K+1)}) = \min\left[1, e^{-\Delta_{K,K+1}}\right]$

$\Delta_{K,K+1} = (\beta_K - \beta_{K+1})\left[U\left(\vec{r}^{(K+1)}\right) - U\left(\vec{r}^{(K)}\right)\right]$

Wang-Landau sampling $Q(\beta) = \int_0^\infty dEe^{-\beta E}\Omega(E)$ $\quad A(E_2|E_1) = \min\left[1, \frac{\Omega(E_2)}{\Omega(E_1)}\right]$