Bootcamp Theorem Proof

1 Lemma Statement

For Z_n, B_n, C_n, D_n finite, non-negative RVs adapted to $\mathcal{F}_n$

$$\mathbb{E}\left[Z_{n+1} \mid \mathcal{F}_n\right] \leq (1+B_n)Z_n + C_n - D_n$$

on the set $\{\sum_n B_n < \infty, \sum_n C_n < \infty\}$ we have

$$\sum_n D_n < \infty \qquad \text{almost surely} \tag{1a}$$

$$Z_n \to Z < \infty \qquad \text{almost surely} \tag{1b}$$

2 Proof of Lemma

2.1 Proof that U_n is supermartingale

Let us define the following stochastic process

$$U_n := \frac{Z_n}{\prod_{k=1}^{n-1}(1+B_k)} - \sum_{m=1}^{n-1} \frac{C_m - D_m}{\prod_{k=1}^{m}(1+B_k)}$$

We want to show that this is a super-martingale. Taking the expectation of

$$U_{n+1} := \frac{Z_{n+1}}{\prod_{k=1}^{n}(1+B_k)} - \sum_{m=1}^{n} \frac{C_m - D_m}{\prod_{k=1}^{m}(1+B_k)}$$

conditional on the sigma algebra $\mathcal{F}_n$ gives us

$$\begin{aligned}
\mathbb{E}\left[U_{n+1} \mid \mathcal{F}_n\right] &= \frac{\mathbb{E}\left[Z_{n+1} \mid \mathcal{F}_n\right]}{\prod_{k=1}^{n}(1+B_k)} - \sum_{m=1}^{n} \frac{C_m - D_m}{\prod_{k=1}^{m}(1+B_k)} \\
&\leq \frac{(1+B_n)Z_n + C_n - D_n}{\prod_{k=1}^{n}(1+B_k)} - \sum_{m=1}^{n} \frac{C_m - D_m}{\prod_{k=1}^{m}(1+B_k)} \\
&= \frac{Z_n}{\prod_{k=1}^{n-1}(1+B_k)} - \sum_{m=1}^{n-1} \frac{C_m - D_m}{\prod_{k=1}^{m}(1+B_k)} \\
&:= U_n
\end{aligned}$$

So that U_n is a super-martingale.

2.2 Proof that $a + U_{n\wedge\nu_a}$ is supermartingale

Next, we define the following random variable

$$\nu_a := \inf\left\{n \;:\; \sum_{m=1}^{n} \frac{C_m}{\prod_{k=1}^{m}(1+B_k)} > a\right\} \qquad a > 0$$

With the convention that $\inf \emptyset = +\infty$. This is a stopping time because $\{\nu_a = n\} \in \mathcal{F}_n$, or equivalently, because it depends only on the information present at time n, and not on any future information. Next we define the stopped process

$$U_{n \wedge \nu_a}$$

where $n \wedge \nu_a$ indicates the minimum between n and $\nu_a(\omega)$, for a given outcome ω. In other words, we can write

$$\begin{aligned} n \wedge \nu_a = \min\{n, \nu_a\} &= \nu_a \mathbb{1}_{\{\nu_a < n\}} + n \mathbb{1}_{\{\nu_a \geq n\}} \\ U_{n \wedge \nu_a} &= U_{\nu_a} \mathbb{1}_{\{\nu_a < n\}} + U_n \mathbb{1}_{\{\nu_a \geq n\}} \end{aligned}$$

Since U_n is a super-martingale, we can show that also the stopped process $U_{\nu_a \wedge n}$ is a super martingale

$$\begin{aligned} \mathbb{E}\left[U_{(n+1)\wedge \nu_a} \mid \mathcal{F}_n\right] &= \mathbb{E}\left[U_{\nu_a} \mathbb{1}_{\{\nu_a < n+1\}} + U_{n+1} \mathbb{1}_{\{\nu_a \geq n+1\}} \mid \mathcal{F}_n\right] \\ &= \mathbb{E}\left[U_{\nu_a} \mathbb{1}_{\{\nu_a < n+1\}} \mid \mathcal{F}_n\right] + \mathbb{E}\left[U_{n+1} \mathbb{1}_{\{\nu_a \geq n+1\}} \mid \mathcal{F}_n\right] \\ &= \mathbb{E}\left[U_{\nu_a} \mathbb{1}_{\{\nu_a \leq n\}} \mid \mathcal{F}_n\right] + \mathbb{E}\left[U_{n+1} \mathbb{1}_{\{\nu_a > n\}} \mid \mathcal{F}_n\right] \\ &= U_{\nu_a} \mathbb{1}_{\{\nu_a \leq n\}} + \mathbb{E}\left[U_{n+1} \mid \mathcal{F}_n\right] \mathbb{1}_{\{\nu_a > n\}} \\ &\leq U_{\nu_a} \mathbb{1}_{\{\nu_a \leq n\}} + U_n \mathbb{1}_{\{\nu_a > n\}} && U_n \text{ is super-martingale} \\ &=: U_{n \wedge \nu_a} \end{aligned}$$

and similarly, we can show that the stochastic process $a + U_{n \wedge \nu_a}$ with $a > 0$ is also a super-martingale because

$$\mathbb{E}\left[U_{(n+1)\wedge \nu_a} + a \mid \mathcal{F}_n\right] \leq U_{n \wedge \nu_a} + a$$

by linearity of the expectation operator.

2.3 Proof that $a + U_{n \wedge \nu_a}$ is bounded in $\mathcal{L}^1$

Notice that $(B_n)_{n \geq 1}$ are non-negative, finite, $\mathcal{F}_n$-measurable random variables. This means that

$$\left(\prod_{k=1}^{n} (1 + B_k)\right)_n$$

are finite, positive, non-decreasing, and $\mathcal{F}_n$-measurable random variables. Similarly, U_n are finite and $\mathcal{F}_n$-measurable random variables. Since

$$a + U_{n \wedge \nu_a} = a + U_{\nu_a} \mathbb{1}_{\{\nu_a < n\}} + U_n \mathbb{1}_{\{\nu_a \geq n\}} \qquad a > 0$$

we have that also $(a + U_{n \wedge \nu_a})_n$ are finite and $\mathcal{F}_n$-measurable random variables, and we can choose a so that they are also positive random variables

$$a + U_{n \wedge \nu_a} > 0 \qquad \forall n$$

Putting together the fact that $a + U_{n \wedge \nu_a}$ is a positive supermartingale, and that $\mathbb{E}[a + U_{n \wedge \nu_a}]$ forms a non-negative, decreasing sequence, thus we can find a value $M > 0$ such that

$$\sup_n \mathbb{E}\left[|a + U_{n \wedge \nu_a}|\right] = \sup_n \mathbb{E}\left[a + U_{n \wedge \nu_a}\right] < M < \infty$$

In other words, $a + U_{n \wedge \nu_a}$ is bounded in $\mathcal{L}^1$.

2.4 Proof of Convergence of $U_{n \wedge \nu_a}$

By Doob's convergence theorem we therefore know that there exists a random variable $U^a \in \mathcal{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$ such that

$$a + U_{n \wedge \nu_a} \overset{\text{a.s.}}{\to} U^a$$

Using the definition of almost sure convergence for random variables we can re-write this as

$$\mathbb{P}\left(\left\{\omega \in \Omega \,:\, \lim_{n \to \infty} a + U_{n \wedge \nu_a} = U^a\right\}\right) = 1$$

which can be equivalently written as

$$\mathbb{P}\left(\left\{\omega \in \Omega \,:\, \lim_{n\to\infty} U_{n\wedge\nu_a} = U^a - a\right\}\right) = 1$$

So we see that

$$U_{n\wedge\nu_a} \overset{\text{a.s.}}{\to} U^{\text{stopped}} \in \mathcal{L}^1$$

2.5 Proof of Convergence of U_n on $\{\nu_a = +\infty\}$

Now consider the following set, where we've used $\inf \emptyset = +\infty$

$$\{\nu_a = +\infty\} := \{\omega \in \Omega \,:\, \nu_a(\omega) = +\infty\} = \left\{\omega \in \Omega \,:\, \sum_{m=1}^{\infty} \frac{C_m}{\prod_{k=1}^{m}(1+B_k)} \leq a\right\} \qquad a > 0$$

Because of how we've defined $U_{n\wedge\nu_a}$ we have that on such as set clearly $n < \nu_a$ and thus

$$U_{n\wedge\nu_a} = U_{n\wedge\infty} = U_n$$

Therefore on the set $\{\nu_a = +\infty\}$ the stopped process and U_n are equivalent, and so

$$U_n \overset{\text{a.s.}}{\to} U^{\text{stopped}} \qquad\qquad \text{on } \{\nu_a = +\infty\}$$

2.6 Proof of Convergence of U_n on Ω_0

Now we want to show that actually U_n converges to a finite random variable not only on such a set, but on the set

$$\Omega_0 := \left\{\omega \in \Omega \,:\, \sum_{n=1}^{\infty} C_n < \infty\right\}$$

Notice that on Ω_0

$$B_n \geq 0 \quad\Longrightarrow\quad \prod_{k=1}^{n}(1+B_k) \geq 1 \quad\Longrightarrow\quad \sum_{n=1}^{\infty} \frac{C_n}{\prod_{k=1}^{n}(1+B_k)} \leq \sum_{n=1}^{\infty} C_n < \infty$$

Now let us define the following family of sets

$$\Omega_a := \Omega_0 \cap \{\nu_a = +\infty\} = \left\{\omega \in \Omega \,:\, \sum_{n=1}^{\infty} C_n < \infty\right\} \cap \left\{\omega \in \Omega \,:\, \sum_{m=1}^{\infty} \frac{C_m}{\prod_{k=1}^{m}(1+B_k)} \leq a\right\}$$

Then we notice that as a goes to $+\infty$, the second set becomes Ω as $\sum_{m=1}^{\infty} \frac{C_m}{\prod_{k=1}^{m}(1+B_k)} \leq +\infty$ is not giving any restriction on the outcomes $\omega \in \Omega$, and therefore

$$\lim_{a\to\infty} \Omega_a = \lim_{a\to\infty} \Omega_0 \cap \{\nu_a = +\infty\} = \Omega_0 \cap \Omega = \Omega_0$$

Since this holds for every $a > 0$, and for every such a we have that U_n converges to U^{stopped} almost surely on $\{\nu_a = +\infty\}$, then we have that

$$U_n \overset{\text{a.s.}}{\to} U^{\text{stopped}}$$

on the set Ω_0 which can be written as

$$\Omega_0 = \bigcup_{a>0} \Omega_a = \bigcup_{a>0} \left(\Omega_0 \cap \{\nu_a = +\infty\}\right)$$

2.7 Proof that $\sum_n \frac{D_n}{\prod_{m=1}^n (1+B_m)}$ converges

In the following we will work on our desired set

$$\Omega_{\text{final}} := \Omega_0 \cap \left\{ \omega \in \Omega \, : \, \sum_{n=1}^{\infty} B_n < \infty \right\} = \left\{ \omega \in \Omega \, : \, \sum_{n=1}^{\infty} B_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} C_n < \infty \right\}$$

where we know that U_n converges to U^{stopped} almost surely, so

$$\begin{aligned}
\lim_{n\to\infty} U_n &= \lim_{n\to\infty} \left(\frac{Z_n}{\prod_{k=1}^{n-1}(1+B_k)} - \sum_{m=1}^{n-1} \frac{C_m}{\prod_{k=1}^{m}(1+B_k)} + \sum_{m=1}^{n-1} \frac{D_m}{\prod_{k=1}^{m}(1+B_k)} \right) \\
&= \lim_{n\to\infty} \frac{Z_n}{\prod_{k=1}^{n-1}(1+B_k)} - \lim_{n\to\infty} \sum_{m=1}^{n-1} \frac{C_m}{\prod_{k=1}^{m}(1+B_k)} + \lim_{n\to\infty} \sum_{m=1}^{n-1} \frac{D_m}{\prod_{k=1}^{m}(1+B_k)} \\
&= U^{\text{stopped}}
\end{aligned}$$

Rearranging the expression above, and keeping in mind that

$$\sum_{n=1}^{\infty} \frac{C_n}{\prod_{k=1}^{n}(1+B_k)} \le \sum_{n=1}^{\infty} C_n < \infty \qquad \text{for } \omega \in \Omega_{\text{final}}$$

we obtain that the following expression converges, and thus is bounded.

$$\frac{Z_n}{\prod_{k=1}^{n-1}(1+B_k)} + \sum_{m=1}^{n-1} \frac{D_m}{\prod_{k=1}^{m}(1+B_k)} \quad \xrightarrow{\text{a.s.}} \quad U^{\text{stopped}} + \sum_{m=1}^{\infty} \frac{C_m}{\prod_{k=1}^{m}(1+B_k)}$$

Notice also that because Z_n, D_n and B_n are non-negative, we have the following inequality

$$\frac{Z_n}{\prod_{k=1}^{n-1}(1+B_k)} + \sum_{m=1}^{n-1} \frac{D_m}{\prod_{k=1}^{m}(1+B_k)} \ge \sum_{m=1}^{n-1} \frac{D_m}{\prod_{k=1}^{m}(1+B_k)}$$

which tells us that also the term on the right hand side is bounded. To show that it converges, we can observe that it is a sum of non-negative terms, thus it is non-decreasing and therefore monotone. A monotone, bounded sequence converges and so it has a finite limit almost surely

$$D_\infty(\omega) := \sum_{n=1}^{\infty} \frac{D_n}{\prod_{k=1}(1+B_k)} < \infty$$

2.8 Proof of Convergence of Z_n

In a similar way, we have that the first term in the definition of U_n converges to some finite limit $Z_\infty(\omega)$:

$$\frac{Z_n}{\prod_{k=1}^{n-1}(1+B_k)} \quad \xrightarrow{\text{a.s.}} \quad Z_\infty(\omega)$$

To prove that Z_n also converges to a finite limit, we can re-write it as follows

$$Z_n = \frac{Z_n}{\prod_{k=1}^{n-1}(1+B_k)} \cdot \prod_{k=1}^{n-1}(1+B_k)$$

Now recall that for a non-negative real number $\alpha \ge 0$ we have the following inequality

$$1 + \alpha \le e^{\alpha}$$

We can use this result together with

$$\sum_{n=1}^{\infty} B_n < \infty$$

as we're working in Ω_{final} to get

$$\prod_{n=1}^{\infty}(1+B_n) \leq \prod_{n=1}^{\infty}\exp(B_n) = \exp\left(\sum_{n=1}^{\infty} B_n\right) < \infty$$

So that on the set

$$\left\{\omega \in \Omega \,:\, \sum_{n=1}^{\infty} B_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} C_n < \infty\right\}$$

we have

$$\lim_{n\to\infty} Z_n = Z_\infty(\omega)\prod_{n=1}^{\infty}(1+B_k) =: Z < \infty$$

2.9 Proof that $\sum_n^\infty D_n < \infty$ almost surely

In a similar fashion we have

$$\begin{aligned}
\sum_{n=1}^{\infty} D_n &= \sum_{n=1}^{\infty}\left(\frac{D_n}{\prod_{m=1}^{n}(1+B_m)}\prod_{m=1}^{n}(1+B_m)\right) \\
&\leq \sum_{n=1}^{\infty}\left(\frac{D_n}{\prod_{m=1}^{n}(1+B_m)}\prod_{m=1}^{\infty}(1+B_m)\right) && \text{as } \prod_{m=1}^{n}(1+B_m) \leq \prod_{m=1}^{\infty}(1+B_m) \\
&= \left(\sum_{n=1}^{\infty}\frac{D_n}{\prod_{m=1}^{n}(1+B_m)}\right)\prod_{m=1}^{\infty}(1+B_m) && \text{independent of } n \\
&= D_\infty(\omega)\prod_{m=1}^{\infty}(1+B_m) \\
&< \infty && \text{almost surely}
\end{aligned}$$

3 Theorem Assumptions

$$\theta_{n+1} = \theta_n + \gamma_{n+1}H(\theta_n, X_{n+1}) \tag{2}$$

$$\mathbb{E}\left[g(\theta_n, X_{n+1}) \mid \mathcal{F}_n\right] = \int g(\theta_n, x)\mu_{\theta_n}(dx) \qquad \text{for } g \text{ positive and Borel-measurable} \tag{3}$$

$$\sigma^2(\theta) = \int |H(\theta, x)|^2\mu_\theta(dx) \leq C(1+|\theta|^2) \qquad C \text{ constant} \tag{4}$$

$$h(\theta) = \int H(\theta, x)\mu_\theta(dx) \tag{5}$$

$$\exists\theta_* \;:\; \sup_{\epsilon \leq |\theta-\theta_*| \leq \frac{1}{\epsilon}} (\theta-\theta_*)^\top h(\theta) < 0 \qquad \forall\epsilon > 0 \tag{6}$$

$$\sum_n \gamma_n = +\infty \tag{7}$$

$$T_n := \theta_n - \theta_* \tag{8}$$

$$Z_n := |T_n|^2 \tag{9}$$

4 Theorem Proof

$$
\begin{aligned}
Z_{n+1} &= |T_{n+1}|^2 \\
&= |\theta_{n+1} - \theta_*|^2 \\
&= \theta_{n+1}^2 + \theta_*^2 - 2\theta_{n+1}\theta_* \\
&= (\theta_n + \gamma_{n+1} H(\theta_n, X_{n+1}))^2 + \theta_*^2 - 2\left(\theta_n + \gamma_{n+1} H(\theta_n, X_{n+1})\right)\theta_* && \text{Def (2)} \\
&= \theta_n^2 + \gamma_{n+1}^2 |H(\theta_n, X_{n+1})|^2 + 2\theta_n \gamma_{n+1} H(\theta_n, X_{n+1}) + \theta_*^2 - 2\theta_n\theta_* - 2\theta_*\gamma_{n+1} H(\theta_n, X_{n+1}) \\
&= |\theta_n - \theta_*|^2 + 2\gamma_{n+1}(\theta_n - \theta_*)^\top H(\theta_n, X_{n+1}) + \gamma_{n+1}^2 |H(\theta_n, X_{n+1})|^2 \\
&= Z_n + 2\gamma_{n+1} T_n^\top H(\theta_n, X_{n+1}) + \gamma_{n+1}^2 |H(\theta_n, X_{n+1})|^2 && \text{Defs (8) and (9)}
\end{aligned}
$$

Taking the conditional expectation with respect to the sub σ-algebra $\mathcal{F}_n$ we have

$$
\begin{aligned}
\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[Z_n \mid \mathcal{F}_n] + 2\gamma_{n+1}\mathbb{E}\left[T_n^\top H(\theta_n, X_{n+1}) \mid \mathcal{F}_n\right] + \gamma_{n+1}^2 \mathbb{E}\left[|H(\theta_n, X_{n+1})|^2 \mid \mathcal{F}_n\right] && \mathbb{E} \text{ linear} \\
&= Z_n + 2\gamma_{n+1} T_n^\top \mathbb{E}\left[H(\theta_n, X_{n+1}) \mid \mathcal{F}_n\right] + \gamma_{n+1}^2 \mathbb{E}\left[|H(\theta_n, X_{n+1})|^2 \mid \mathcal{F}_n\right] && Z_n \text{ and } T_n \text{ known} \\
&= Z_n + 2\gamma_{n+1} T_n^\top h(\theta_n) + \gamma_{n+1}^2 \sigma^2(\theta_n) && \text{Def (5) and (4)} \\
&\leq Z_n + 2\gamma_{n+1} T_n^\top h(\theta_n) + \gamma_{n+1}^2 C(1 + |\theta_n|^2) && \text{By (4)} \\
&= Z_n + 2\gamma_{n+1} T_n^\top h(\theta_n) + \gamma_{n+1}^2 C + \gamma_{n+1}^2 |\theta_n|^2 \\
&= Z_n + 2\gamma_{n+1} T_n^\top h(\theta_n) + \gamma_{n+1}^2 C + \gamma_{n+1}^2 |T_n + \theta_*|^2 C && \text{Def (8)} \\
&= Z_n + 2\gamma_{n+1} T_n^\top h(\theta_n) + \gamma_{n+1}^2 C + \gamma_{n+1}^2 \left(|T_n|^2 + \theta_*^2 + 2T_n\theta_*\right) C \\
&= Z_n + 2\gamma_{n+1} T_n^\top h(\theta_n) + \gamma_{n+1}^2 C + \gamma_{n+1}^2 |T_n|^2 C + \gamma_{n+1}^2 \theta_*^2 C + 2\gamma_{n+1}^2 T_n \theta_* C \\
&= Z_n + 2\gamma_{n+1} T_n^\top h(\theta_n) + \gamma_{n+1}^2 C + \gamma_{n+1}^2 Z_n C + \gamma_{n+1}^2 \theta_*^2 C + 2\gamma_{n+1}^2 T_n \theta_* C && \text{Def (9)} \\
&= Z_n(1 + \gamma_{n+1}^2 C) + \gamma_{n+1}^2 \left(C + \theta_* C + 2T_n\theta_* C\right) + 2\gamma_{n+1} T_n^\top h(\theta_n) \\
&= Z_n(1 + \gamma_{n+1}^2 C) + \gamma_{n+1}^2 \overline{C} + 2\gamma_{n+1} T_n^\top h(\theta_n) && \overline{C} \text{ const.}
\end{aligned}
$$

where we've defined $\overline{C} := C + \theta_* C + 2T_n\theta_* C$, which is a constant with respect to the sub-σ-algebra $\mathcal{F}_n$.

From condition (6) we have that $T_n^\top h(\theta_n) < 0$. This means that we can rewrite $\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]$ as

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = (1 + B_n)Z_n + C_n - D_n$$

where

$$
\begin{aligned}
B_n &:= \gamma_{n+1}^2 C \\
C_n &:= \gamma_{n+1}^2 \overline{C} \\
D_n &:= -2\gamma_{n+1} T_n^\top h(\theta_n)
\end{aligned}
$$

which are all finite, non-negative random variables. Therefore we can apply (1a) and (1b) on the set

$$\left\{ C \sum_n \gamma_{n+1}^2 < \infty \ , \ \overline{C} \sum_n \gamma_{n+1}^2 \right\} = \left\{ \sum_n \gamma_{n+1}^2 \right\}$$

we have

$$
\begin{aligned}
-\sum_n \gamma_{n+1} T_n^\top h(\theta_n) < \infty \quad \text{a.s.} \\
Z_n \to Z < \infty \quad \text{a.s.}
\end{aligned}
$$

To prove that θ_n converges to θ_* almost surely, we want to show that Z is the zero random variable. We prove this by contraddiction. Suppose that Z is not the zero random variable. That is, suppose that there exists an outcome $\omega \in \Omega$ such that

$$Z(\omega) \neq 0$$

Recall that Z_n is a non-negative random variable. We have two cases:

- If $Z_n(\omega) = 0$ for all n then $Z(\omega) = 0$ with probability 1, contraddicting our assumption, and proving the theorem.
- $Z_n(\omega) \neq 0$ for some n. That is $Z_n(\omega) > 0$ for some n.

In this second case, consider those n for which $Z_n(\omega) \neq 0$-. This means that there exists an $N \in \mathbb{N}$ after which we can find some $\epsilon^2 > 0$

$$\epsilon^2 \leq Z_n(\omega) \leq \frac{1}{\epsilon^2} \qquad n > N$$

This is true because

- If $Z_n(\omega) \in [1, \infty)$ then we can simply set $\epsilon^2 \in (0, 1)$ such that $\frac{1}{\epsilon^2} \geq Z_n(\omega)$ (for instance, if $Z_n(\omega) \neq 0$ we can choose $\epsilon^2 := \frac{1}{Z_n(\omega)}$).
- If $Z_n(\omega) \in (0, 1)$ then by the Archimedean property we can always find an $0 < \epsilon^2 \leq Z_n(\omega)$. Clearly then, $\epsilon^2 \in [1, \infty)$.

In other words, there exists $N \in \mathbb{N}$ and $\epsilon^2 \in (0, 1)$ such that

$$\epsilon^2 \leq |T_n(\omega)|^2 \leq \frac{1}{\epsilon^2} \qquad \forall n > N$$

that is,

$$\epsilon \leq |T_n(\omega)| \leq \frac{1}{\epsilon} \qquad \forall n > N$$

Using the fact that for any non-empty set S we have $\sup(S) = -\inf(-S)$ together with condition (6) we have that

$$\sup_{n>N} T_n(\omega)^\top h(\theta_n(\omega)) < 0$$

or equivalently

$$\inf_{n>N} -T_n(\omega)^\top h(\theta_n(\omega)) > 0$$

We can use the well-known fact that for any sequence $x_n \in \mathbb{R}$ the following holds

$$\inf_n x_n \leq \liminf_{n\to\infty} x_n$$

to show

$$0 < \inf_{n>N} -T_n(\omega)^\top h(\theta_n(\omega)) \leq \liminf_{n\to\infty} -T_n(\omega)^\top h(\theta_n(\omega))$$

This means that we can find some constant $\alpha > 0$ such that $0 < \alpha \leq -T_n(\omega)^\top h(\theta_n(\omega))$ for all $n > N$. Using this fact together with assumption (7) and the result

$$-\sum_n \gamma_{n+1} T_n^\top h(\theta_n) < \infty \quad \text{a.s.}$$

we get

$$\sum_n -\gamma_{n+1} T_n^\top(\omega) h(\theta_n(\omega)) \geq \alpha \sum_n \gamma_{n+1} = +\infty$$

But this contradicts our earlier result found thanks to the lemma. Therefore our assumption that $\exists\, \omega \in \Omega$ such that $Z(\omega) \neq 0$ is wrong and Z_n converges almost surely to the zero random variable, that is

$$\theta_n \to \theta_* \qquad \text{almost surely}$$

5 Doob's Theorem

Let X be a supermartingale that is bounded in $\mathcal{L}^1$. That is, there exists a constant $M > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[|X_n|\right] < M$$

then there exists a random variable $X_\infty \in \mathcal{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$ such that

$$X_n \overset{\text{a.s.}}{\to} X_\infty$$

where $\mathcal{F}_\infty := \sigma\left(\bigcup_n \mathcal{F}_n\right)$ is the smallest σ-algebra containing $\bigcup_n \mathcal{F}_n$.