# Mixed / Multilevel models

# Example dataset: trees

- Data on 1000 trees from 10 plots.

```r
head(trees)
```

```
  plot   dbh height    sex dead  dbh.c
1    2 38.85   37.8 female    0  13.85
2    4 26.05   38.1 female    0   1.05
3    5 42.66   50.2 female    0  17.66
4    2 20.72   30.1 female    0  -4.28
5    4 21.83   34.0 female    0  -3.17
6    4  8.23   21.9   male    0 -16.77
```

# Example dataset: trees

- Data on 1000 trees from 10 plots.
- Trees per plot: 4 - 392.

```
head(trees)
```

```
  plot   dbh height    sex dead  dbh.c
1    2 38.85   37.8 female    0  13.85
2    4 26.05   38.1 female    0   1.05
3    5 42.66   50.2 female    0  17.66
4    2 20.72   30.1 female    0  -4.28
5    4 21.83   34.0 female    0  -3.17
6    4  8.23   21.9   male    0 -16.77
```

Q: What's the relationship between tree diameter and height?

# A simple linear model

```r
lm.simple <- lm(height ~ dbh, data = trees)
```

```

Call:
lm(formula = height ~ dbh, data = trees)

Residuals:
     Min       1Q   Median       3Q      Max 
-13.7384  -4.7652   0.4759   4.2931  13.5282 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 13.18767    0.41476   31.80   <2e-16 ***
dbh          0.60967    0.01351   45.14   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.549 on 998 degrees of freedom
Multiple R-squared:  0.6712,	Adjusted R-squared:  0.6709 
F-statistic:  2038 on 1 and 998 DF,  p-value: < 2.2e-16
```

# Remember our model structure

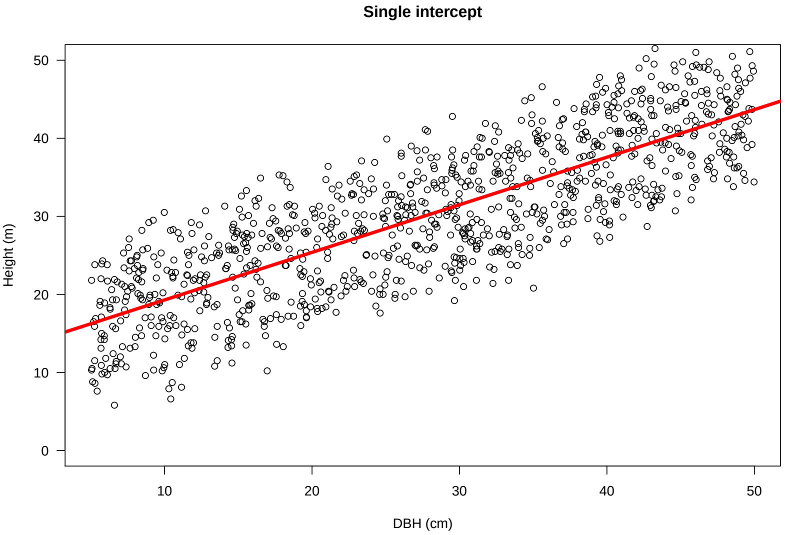
$$y_i \sim N(\mu_i, \sigma^2)$$
$$\mu_i = \alpha + \beta x_i$$

In this case:

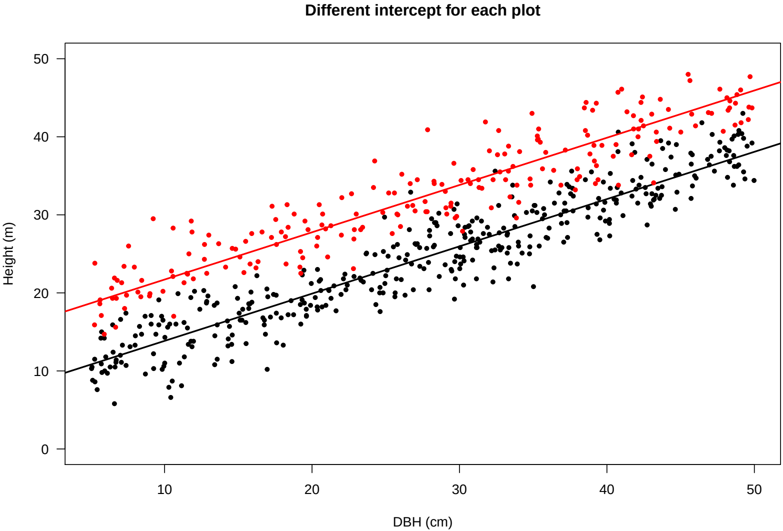
$$Height_i \sim N(\mu_i, \sigma^2)$$
$$\mu_i = \alpha + \beta DBH_i$$

$\alpha$: expected height when DBH = 0
$\beta$: how much height increases with every unit increase of DBH

# There is only one intercept



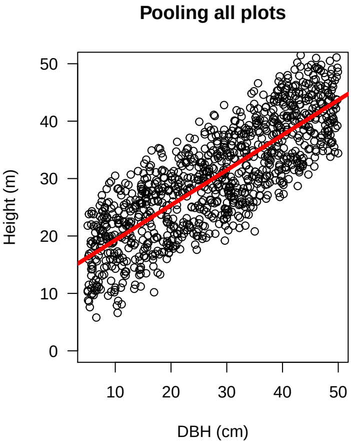
# What if allometry varies among plots?

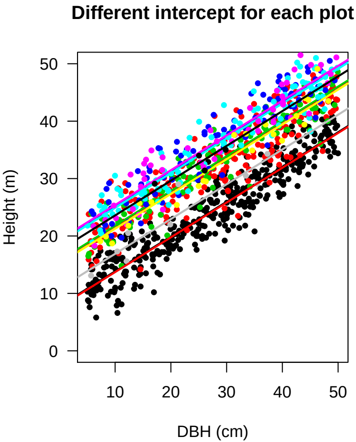


# Fitting a varying intercepts model with `lm`

```
lm(formula = height ~ factor(plot) + dbh, data = trees)
                coef.est coef.se
(Intercept)      7.79     0.24  
factor(plot)2    7.86     0.24  
factor(plot)3    7.95     0.32  
factor(plot)4   11.48     0.33  
factor(plot)5   11.05     0.32  
factor(plot)6   11.55     0.43  
factor(plot)7    7.41     0.63  
factor(plot)8    3.05     0.97  
factor(plot)9    9.73     1.45  
factor(plot)10 -0.14     0.92  
dbh              0.61     0.01  
---
n = 1000, k = 11
residual sd = 2.89, R-Squared = 0.91
```

# Single vs varying intercept





# Mixed models enable us to account for variability

- Varying intercepts

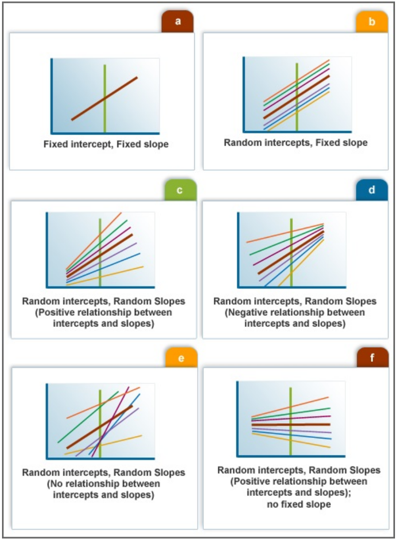
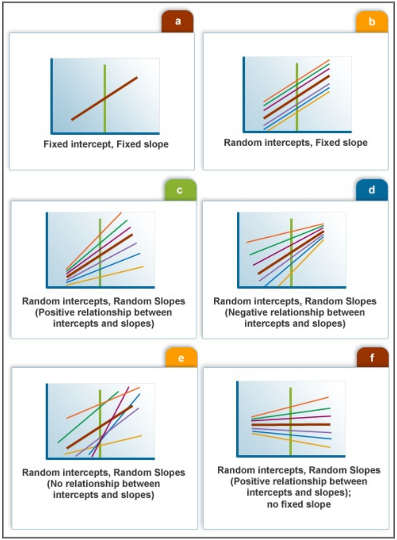

a
Fixed intercept, Fixed slope
b
Random intercepts, Fixed slope
c
Random intercepts, Random Slopes (Positive relationship between intercepts and slopes)
d
Random intercepts, Random Slopes (Negative relationship between intercepts and slopes)
e
Random intercepts, Random Slopes (No relationship between intercepts and slopes)
f
Random intercepts, Random Slopes (Positive relationship between intercepts and slopes); no fixed slope

Figure 1:

# Mixed models enable us to account for variability

- Varying intercepts
- Varying slopes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Random intercepts, Random Slopes (Negative relationship between intercepts and slopes)
e
Random intercepts, Random Slopes (No relationship between intercepts and slopes)
f
Random intercepts, Random Slopes (Positive relationship between intercepts and slopes); no fixed slope

Figure 1:

# Mixed model with varying intercepts

$$
\begin{aligned}
y_i &= a_j + bx_i + \varepsilon_i \\
a_j &\sim N\left(0, \tau^2\right) \\
\varepsilon_i &\sim N\left(0, \sigma^2\right)
\end{aligned}
$$

En nuestro ejemplo:

$$
\begin{aligned}
Height_i &= plot_j + bDBH_i + \varepsilon_i \\
plot_j &\sim N\left(0, \tau^2\right) \\
\varepsilon_i &\sim N\left(0, \sigma^2\right)
\end{aligned}
$$

Mixed models estimate varying parameters (intercepts and/or slopes) with pooling among levels (rather than considering them fully independent)

# Hence there’s gradient between

- **complete pooling**: Single overall intercept.

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  - `lm (height ~ dbh)`

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- **no pooling**: One *independent* intercept for each plot.

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  - `lm (height ~ dbh + factor(plot))`

# Hence there's gradient between

- **complete pooling**: Single overall intercept.
  - `lm (height ~ dbh)`
- **no pooling**: One *independent* intercept for each plot.
  - `lm (height ~ dbh + factor(plot))`
- **partial pooling**: Inter-related intercepts.

# Hence there's gradient between

- **complete pooling**: Single overall intercept.
  - `lm (height ~ dbh)`
- **no pooling**: One *independent* intercept for each plot.
  - `lm (height ~ dbh + factor(plot))`
- **partial pooling**: Inter-related intercepts.
  - `lmer(height ~ dbh + (1 | plot))`

# Random vs Fixed effects?

1. Fixed effects constant across individuals, random effects vary.

http://andrewgelman.com/2005/01/25/why_i_dont_use/

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2. Effects are fixed if they are interesting in themselves; random if interest in the underlying population.

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1. Fixed effects constant across individuals, random effects vary.
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3. Fixed when sample exhausts the population; random when the sample is small part of the population.

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4. Random effect if it's assumed to be a realized value of random variable.

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# Random vs Fixed effects?

1. Fixed effects constant across individuals, random effects vary.
2. Effects are fixed if they are interesting in themselves; random if interest in the underlying population.
3. Fixed when sample exhausts the population; random when the sample is small part of the population.
4. Random effect if it's assumed to be a realized value of random variable.
5. Fixed effects estimated using least squares or maximum likelihood; random effects estimated with shrinkage.

http://andrewgelman.com/2005/01/25/why_i_dont_use/

# What is a random effect, really?

1. Varies by group

Random effects are estimated with partial pooling, while fixed effects are not (infinite variance).

# What is a random effect, really?

1. Varies by group
2. Variation estimated with probability model

Random effects are estimated with partial pooling, while fixed effects are not (infinite variance).

# Shrinkage improves parameter estimation

Especially for groups with low sample size

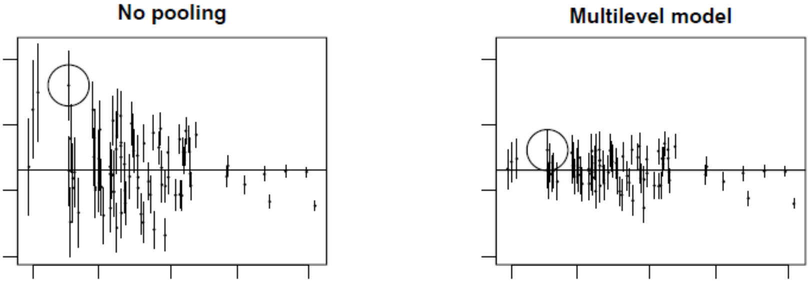


Figure 2:

*From Gelman & Hill p. 253*

# Fitting mixed/multilevel models

```r
library(lme4)
mixed <- lmer(height ~ dbh + (1|plot), data = trees)
```

```
lmer(formula = height ~ dbh + (1 | plot), data = trees)
            coef.est coef.se
(Intercept) 14.80     1.44
dbh          0.61     0.01

Error terms:
 Groups   Name        Std.Dev.
 plot     (Intercept) 4.45
 Residual             2.89
---
number of obs: 1000, groups: plot, 10
AIC = 5015.6, DIC = 4996.4
deviance = 5002.0
```

# Retrieve model coefficients

```r
coef(mixed)
```

```
$plot
   (Intercept)       dbh
1     7.798373 0.6056549
2    15.647613 0.6056549
3    15.735397 0.6056549
4    19.253661 0.6056549
5    18.819467 0.6056549
6    19.306574 0.6056549
7    15.197908 0.6056549
8    11.016485 0.6056549
9    17.265447 0.6056549
10    7.940715 0.6056549

attr(,"class")
[1] "coef.mer"
```

# Broom: model estimates in tidy form

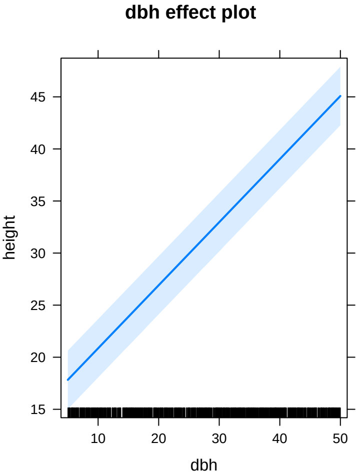
```r
library(broom)
tidy(mixed)
```

```
                     term   estimate   std.error statistic    group
1             (Intercept) 14.7981641 1.437421280  10.29494    fixed
2                     dbh  0.6056549 0.007040079  86.02956    fixed
3     sd_(Intercept).plot  4.4535702          NA        NA     plot
4 sd_Observation.Residual  2.8852942          NA        NA Residual
```

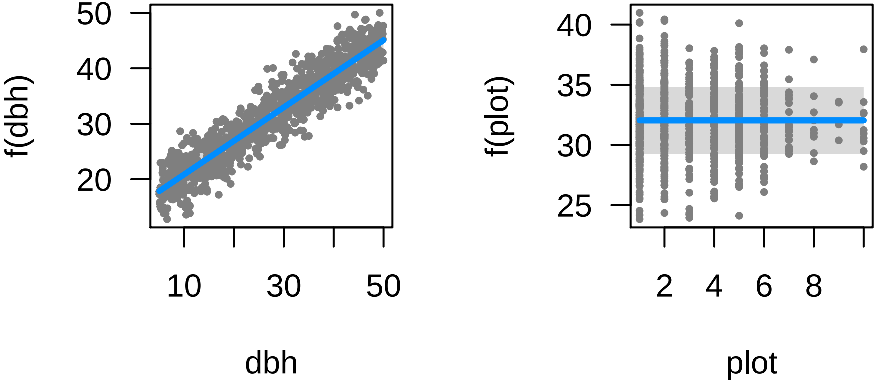
# Visualising model: `allEffects`

```
 model: height ~ dbh

 dbh effect
dbh
       5       20       30       40       50
17.82644 26.91126 32.96781 39.02436 45.08091
```

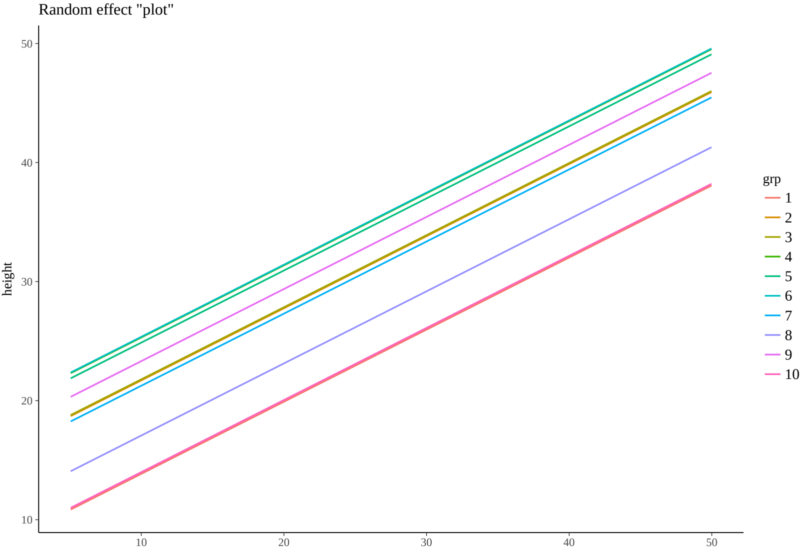


# Visualising model: `visreg`



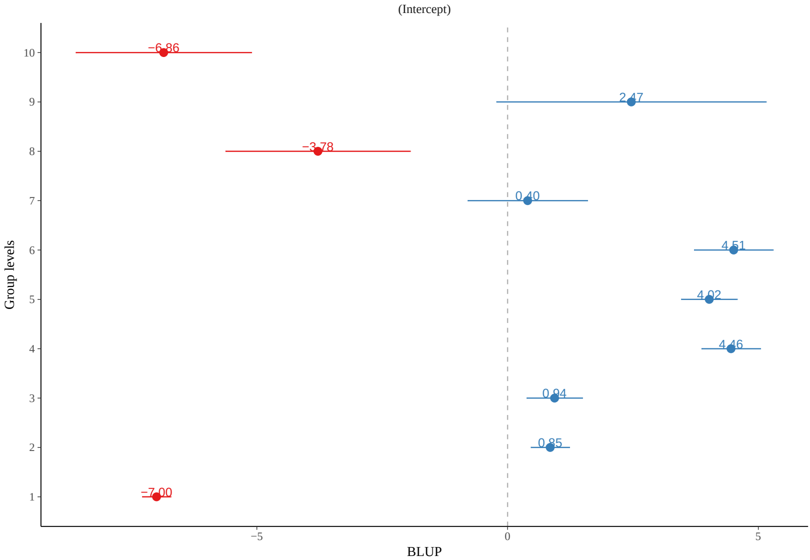
# Visualising model: `sjPlot`

```r
library(sjPlot)
sjp.lmer(mixed, type = "ri.slope")
```



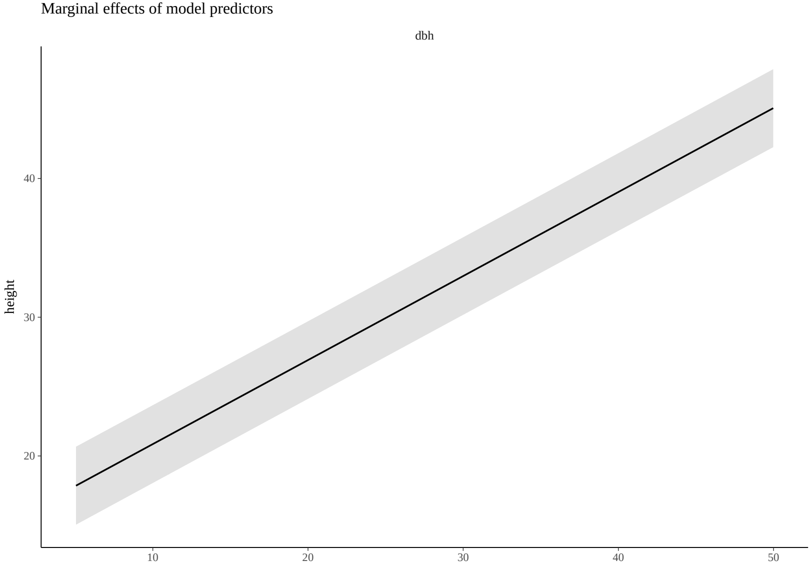
# Visualising model: `sjPlot`

```
sjp.lmer(mixed)
```



# Visualising model: `sjPlot`

```
sjp.lmer(mixed, type = "eff", show.ci = TRUE)
```

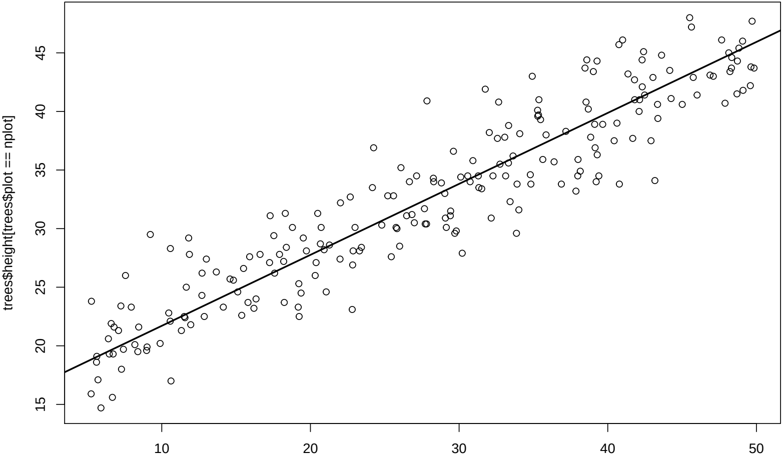


# Using merTools to understand fitted model

```r
library(merTools)
shinyMer(mixed)
```

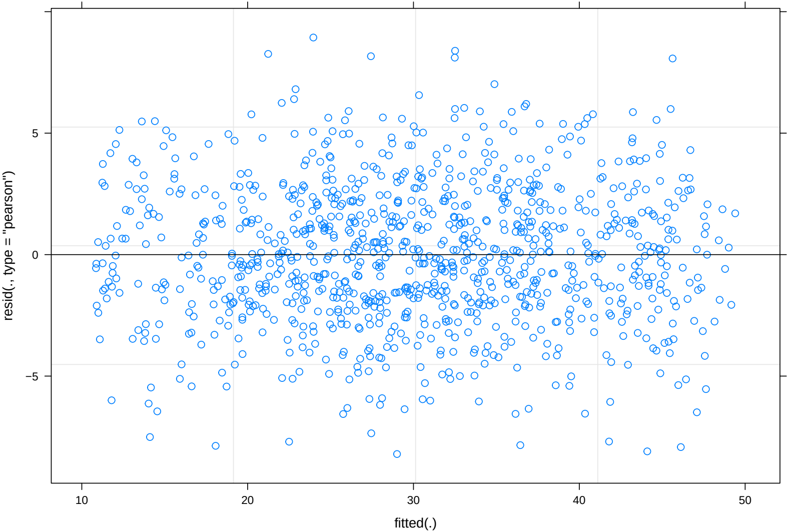
# Plotting regression for individual forest plots

```
nplot <- 2
plot(trees$dbh[trees$plot==nplot], trees$height[trees$plot==nplo
abline(a=coef(mixed)$plot[nplot, 1], b=coef(mixed)$plot[nplot, 2
```



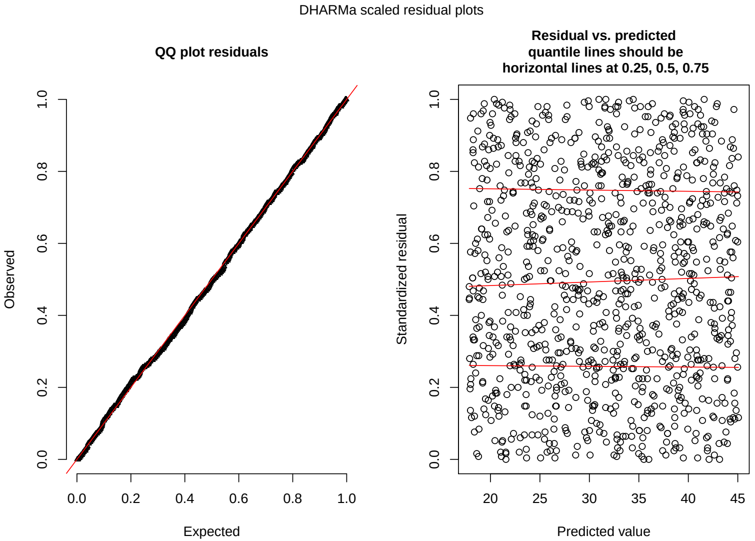
# Checking residuals

```r
plot(mixed)
```



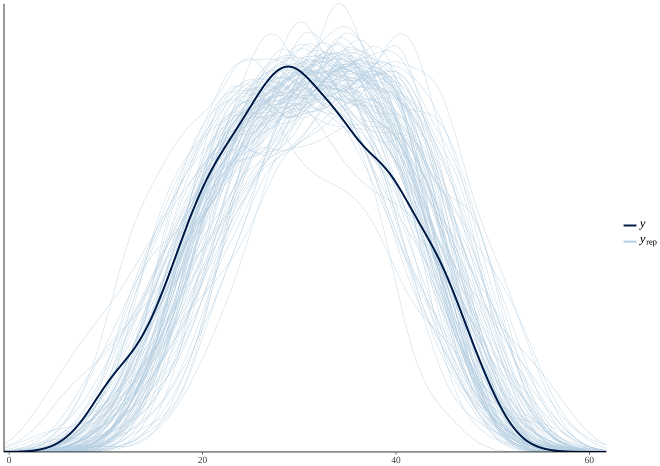
# Checking residuals (DHARMa)

```
simulateResiduals(mixed, plot = TRUE, use.u = TRUE)
```



# Model checking with simulated data

```r
library(bayesplot)
sims <- simulate(mixed, nsim = 100)
ppc_dens_overlay(trees$height, yrep = t(as.matrix(sims)))
```



# R-squared for GLMMs

Many approaches! Somewhat polemic.
Nakagawa & Schielzeth propose **marginal** (considering fixed effects only) and **conditional** $R^2$ (including random effects too):

```r
library(MuMIn)
r.squaredGLMM(mixed)
```

```
      R2m       R2c
0.6875651 0.9076325
```

# Growing the hierarchy: adding plot-level predictors

# Model with group-level predictors

We had:

$$\begin{aligned} y_i &= a_j + bx_i + \varepsilon_i \\ a_j &\sim N(0, \tau^2) \\ \varepsilon_i &\sim N(0, \sigma^2) \end{aligned}$$

Now

$$\begin{aligned} y_i &= a_j + bx_i + \varepsilon_i \\ a_j &\sim N(\mu_j, \tau^2) \\ \mu_j &= \gamma + \delta \cdot predictor_j \\ \varepsilon_i &\sim N(0, \sigma^2) \end{aligned}$$

# Merging trees and plot data

```r
plotdata <- read.csv("data-raw/plotdata.csv")
trees.full <- merge(trees, plotdata, by = "plot")
head(trees.full)
```

```
  plot   dbh height    sex dead  dbh.c temp
1    1 28.63   22.1 female    0   3.63 15.1
2    1 44.71   39.0 female    0  19.71 15.1
3    1 28.31   29.0 female    0   3.31 15.1
4    1 19.33   19.1   male    0  -5.67 15.1
5    1  9.25   12.2 female    0 -15.75 15.1
6    1 30.02   23.1 female    0   5.02 15.1
```

# Centre continuous variables

Plot temperatures referred as deviations from 15ºC

```
trees.full$temp.c <- trees.full$temp - 15
```

# Fit multilevel model

```r
group.pred <- lmer(height ~ dbh + (1 | plot) + temp.c, data = tr
arm::display(group.pred)
```

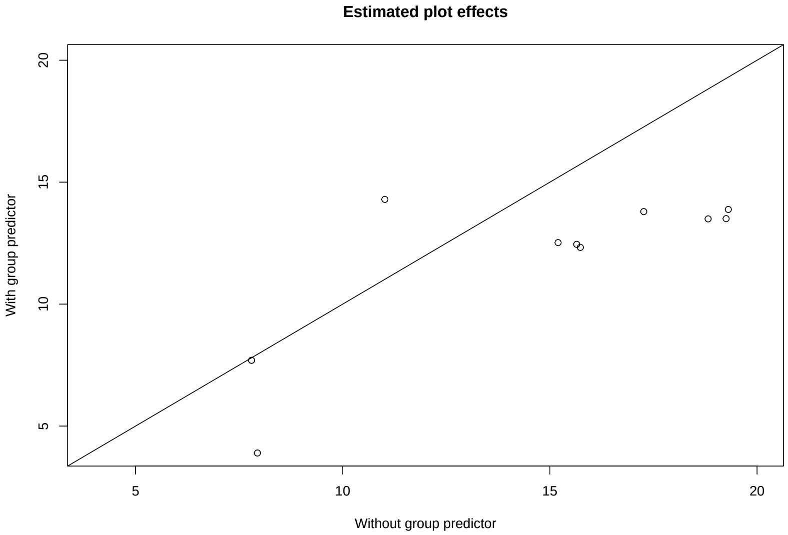
```
lmer(formula = height ~ dbh + (1 | plot) + temp.c, data = trees.
            coef.est coef.se
(Intercept) 11.79     1.75  
dbh          0.61     0.01  
temp.c       1.07     0.46  

Error terms:
 Groups   Name        Std.Dev.
 plot     (Intercept) 3.61    
 Residual             2.89    
---
number of obs: 1000, groups: plot, 10
AIC = 5012.8, DIC = 4991
deviance = 4996.9
```

# Examine model with merTools

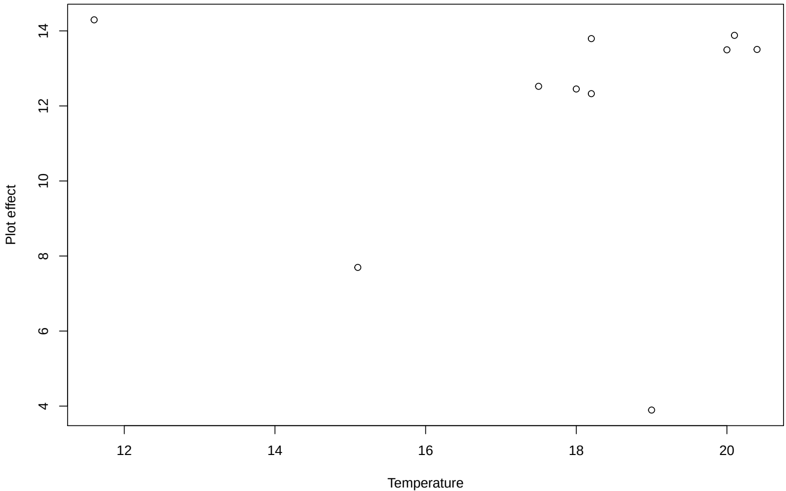
```
shinyMer(group.pred)
```

# Comparing plot effects with and without group predictor



# Are plot effects related to temperature?

```
plot(plotdata$temp, coef(group.pred)$plot[,1],
     xlab = "Temperature", ylab = "Plot effect")
```



# Varying intercepts and slopes

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- There is overall difference in height among plots (different intercepts)

```
mixed.slopes <- lmer(height ~ dbh + (1 + dbh | plot), data=trees
```

# Varying intercepts and slopes

- There is overall difference in height among plots (different intercepts)
- AND

```
mixed.slopes <- lmer(height ~ dbh + (1 + dbh | plot), data=trees
```

# Varying intercepts and slopes

- There is overall difference in height among plots (different intercepts)
- AND
- Relationship between DBH and Height varies among plots (different slopes)

```
mixed.slopes <- lmer(height ~ dbh + (1 + dbh | plot), data=trees
```

# Varying intercepts and slopes

```
lmer(formula = height ~ dbh + (1 + dbh | plot), data = trees)
            coef.est coef.se
(Intercept) 14.82     1.48
dbh          0.60     0.01

Error terms:
 Groups   Name        Std.Dev. Corr
 plot     (Intercept) 4.57
          dbh         0.01     -0.41
 Residual             2.88
---
number of obs: 1000, groups: plot, 10
AIC = 5018.6, DIC = 4995.9
deviance = 5001.3
```

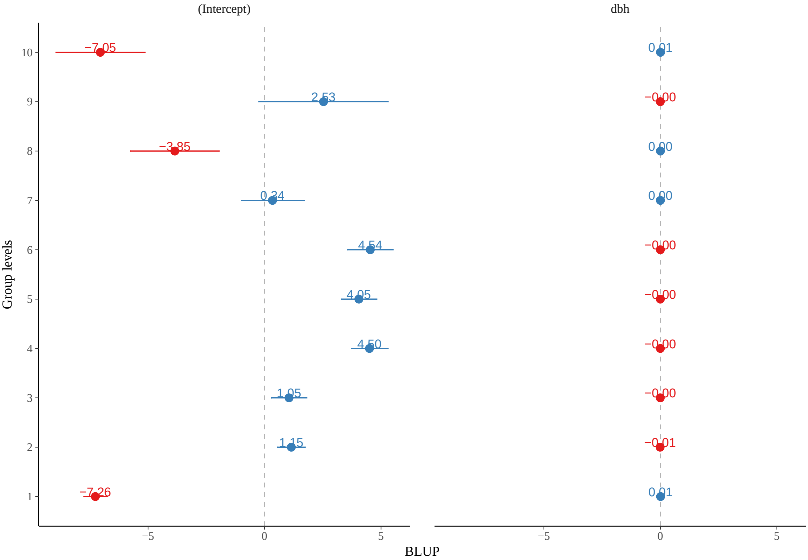
# Varying intercepts and slopes

```
$plot
   (Intercept)       dbh
1     7.554578 0.6144452
2    15.966915 0.5942836
3    15.868969 0.6008673
4    19.321161 0.6031855
5    18.866370 0.6039353
6    19.355009 0.6038332
7    15.159258 0.6067449
8    10.965429 0.6080747
9    17.348840 0.6024600
10    7.769135 0.6109349

attr(,"class")
[1] "coef.mer"
```

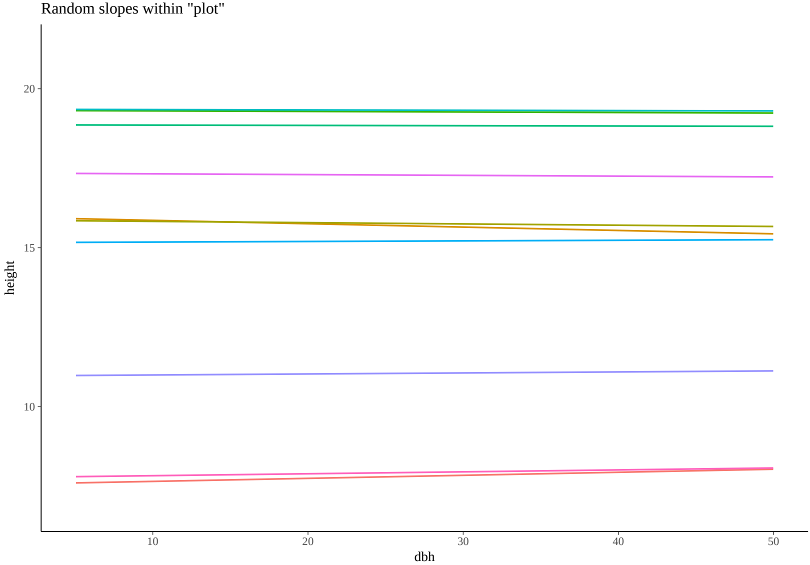
# Visualising model: sjPlot

```
sjp.lmer(mixed.slopes)
```



# Visualising model: `sjPlot`

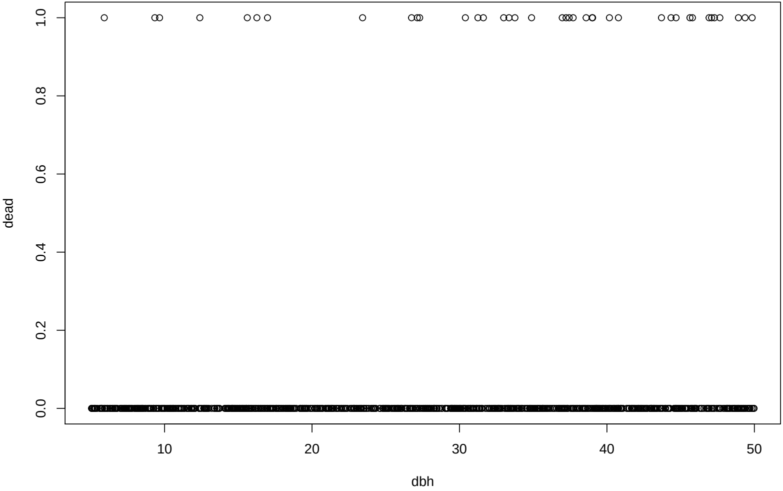
```
sjp.lmer(mixed.slopes, type = "rs.ri")
```



# Multilevel logistic regression

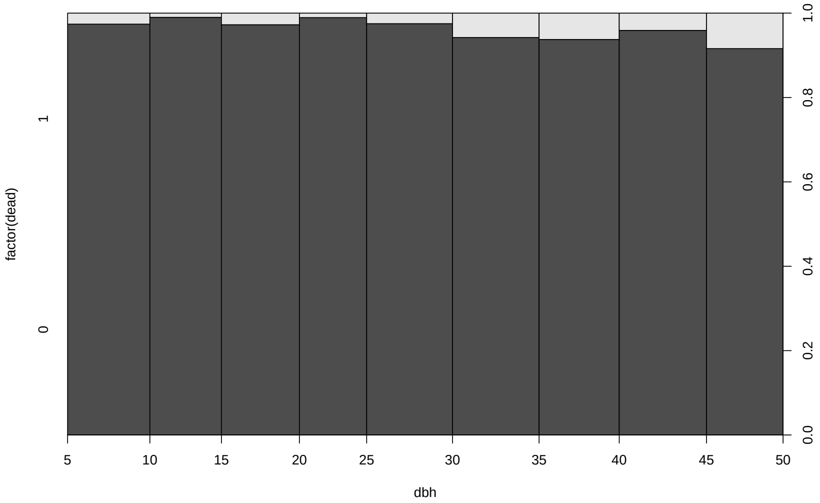
# Q: Relationship between tree size and mortality

```r
plot(dead ~ dbh, data = trees)
```



# Q: Relationship between tree size and mortality

```
plot(factor(dead) ~ dbh, data = trees)
```



# Fit simple logistic regression

```r
simple.logis <- glm(dead ~ dbh, data = trees, family=binomial)
```

```
Call:
glm(formula = dead ~ dbh, family = binomial, data = trees)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-0.4121  -0.3287  -0.2624  -0.2048   2.9127

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -4.46945    0.49445  -9.039  < 2e-16 ***
dbh          0.04094    0.01380   2.967  0.00301 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 329.51  on 999  degrees of freedom
```

# Logistic regression with *independent* plot effects

```
logis2 <- glm(dead ~ dbh + factor(plot), data = trees, family=bi
```

```
Call:
glm(formula = dead ~ dbh + factor(plot), family = binomial, data

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-0.5923  -0.3198  -0.2549  -0.1940   2.8902

Coefficients:
                Estimate Std. Error z value Pr(>|z|)
(Intercept)     -4.40106    0.52997  -8.304   <2e-16 ***
dbh              0.04060    0.01386   2.929   0.0034 **
factor(plot)2   -0.59168    0.52132  -1.135   0.2564
factor(plot)3    0.54576    0.47094   1.159   0.2465
factor(plot)4    0.05507    0.57434   0.096   0.9236
factor(plot)5   -0.38312    0.64222  -0.597   0.5508
factor(plot)6   -0.08426    0.76908  -0.110   0.9128
factor(plot)7    0.03126    1.06064   0.029   0.9765
```

# Fit multilevel logistic regression

```r
mixed.logis <- glmer(dead ~ dbh + (1|plot), data=trees, family =
```

```
glmer(formula = dead ~ dbh + (1 | plot), data = trees, family =
            coef.est coef.se
(Intercept) -4.47     0.49
dbh          0.04     0.01

Error terms:
 Groups   Name        Std.Dev.
 plot     (Intercept) 0.00
 Residual             1.00
---
number of obs: 1000, groups: plot, 10
AIC = 325.9, DIC = 319.9
deviance = 319.9
```

# Retrieve model coefficients

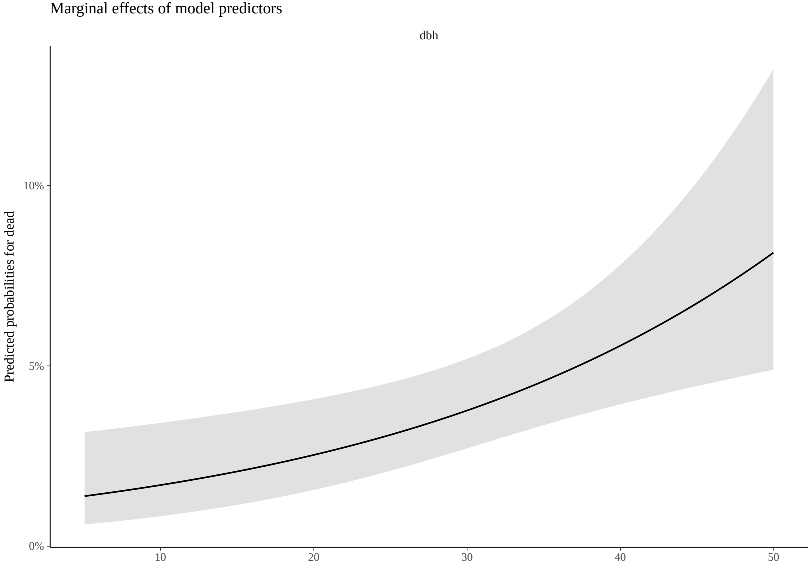
```r
coef(mixed.logis)
```

```
$plot
   (Intercept)        dbh
1    -4.469446 0.04093806
2    -4.469446 0.04093806
3    -4.469446 0.04093806
4    -4.469446 0.04093806
5    -4.469446 0.04093806
6    -4.469446 0.04093806
7    -4.469446 0.04093806
8    -4.469446 0.04093806
9    -4.469446 0.04093806
10   -4.469446 0.04093806

attr(,"class")
[1] "coef.mer"
```

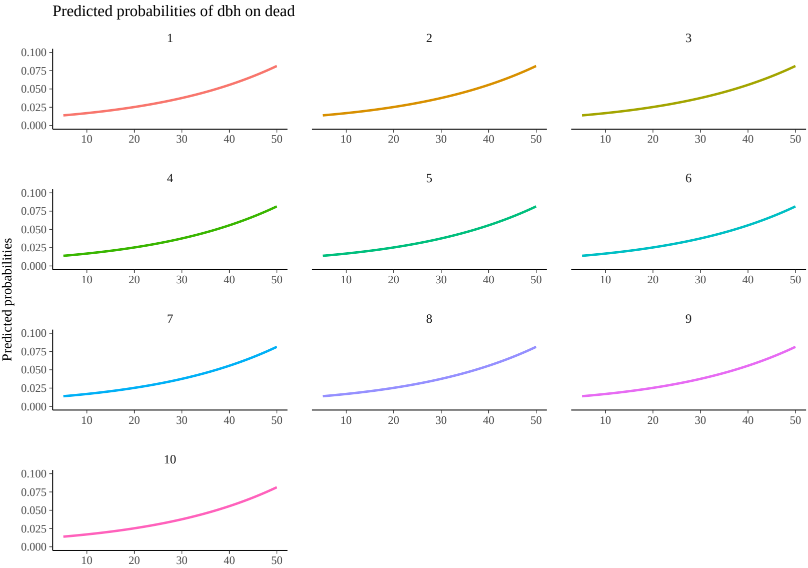
# Visualising model: `sjPlot`

```r
sjp.glmer(mixed.logis, type = "eff", show.ci = TRUE)
```



# Visualising model: sjPlot

```
sjp.glmer(mixed.logis, type = "ri.slope")
```



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- Using all the data to perform inferences for groups with small sample size

# Formula syntax for different models

```
y ~ x + (1 | group) # varying intercepts
y ~ x + (1 + x | group) # varying intercepts and slopes
y ~ x + (1 | group/subgroup) # nested
y ~ x + (1 | group1) + (1 | group2) # varying intercepts, crossed
y ~ x + (1 + x | group1) + (1 + x | group2) # varying intercepts and slopes, crossed
```