Linear models

Example dataset: paper planes flying experiment

```
library(paperplanes)
head(paperplanes)
```

```
  id    hour   person gender age       plane paper distance
1  1 [17,18)   Roland   male  30  Standard80    80      7.8
2  2 [17,18)   Astrid female  30 Concorde120   120      2.7
3  3 [17,18)   Roland   male  30 Standard120   120      9.2
4  4 [17,18) Isabella female  48 Standard120   120      6.0
5  5 [17,18) Fabienne female  17 Standard120   120      7.3
6  6 [17,18) Fabienne female  17 Standard120   120      7.8
```

Questions

- What is the relationship between age and distance flown?

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- Do adults achieve longer distances?

Questions

- What is the relationship between age and distance flown?
- Do adults achieve longer distances?
- Can we predict distance flown from participant's age? How well?

Always plot your data first!

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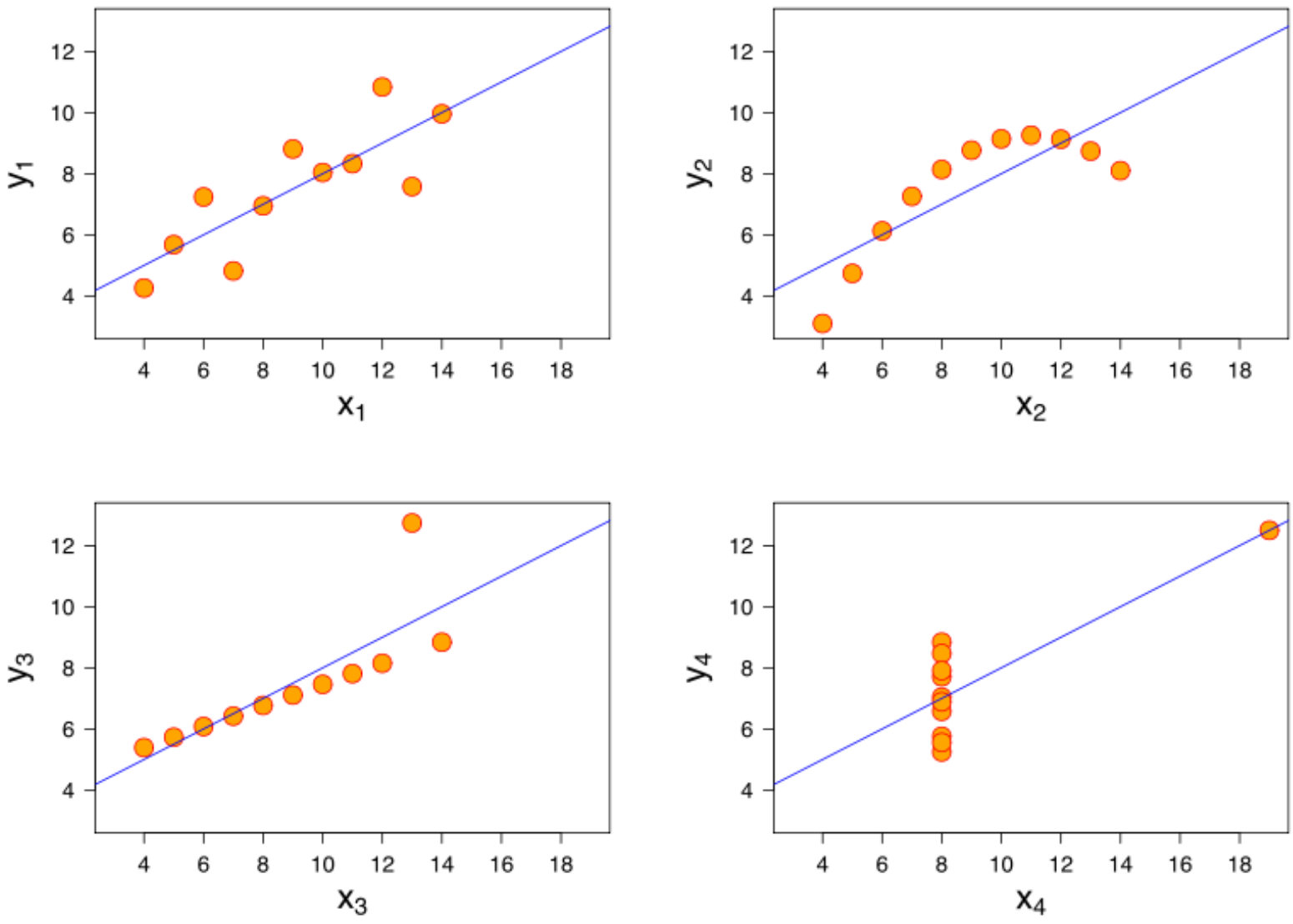
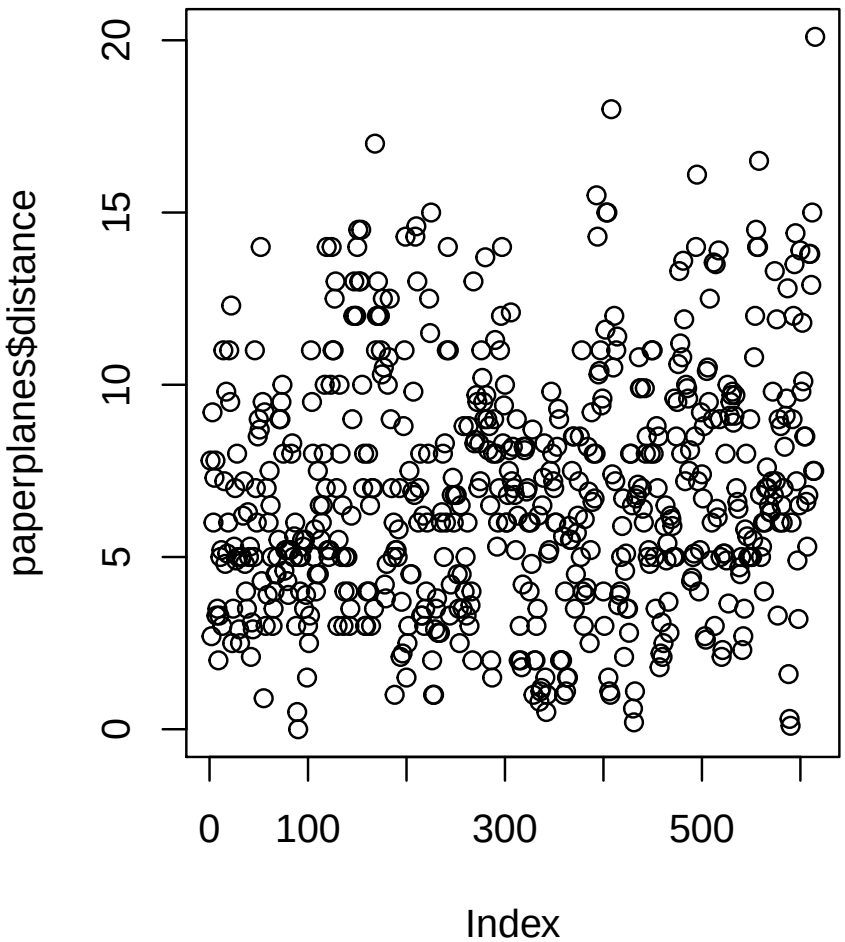


Figure 1

Exploratory Data Analysis (EDA)

Outliers

```
plot(paperplanes$distance)
```



Outliers impact on regression

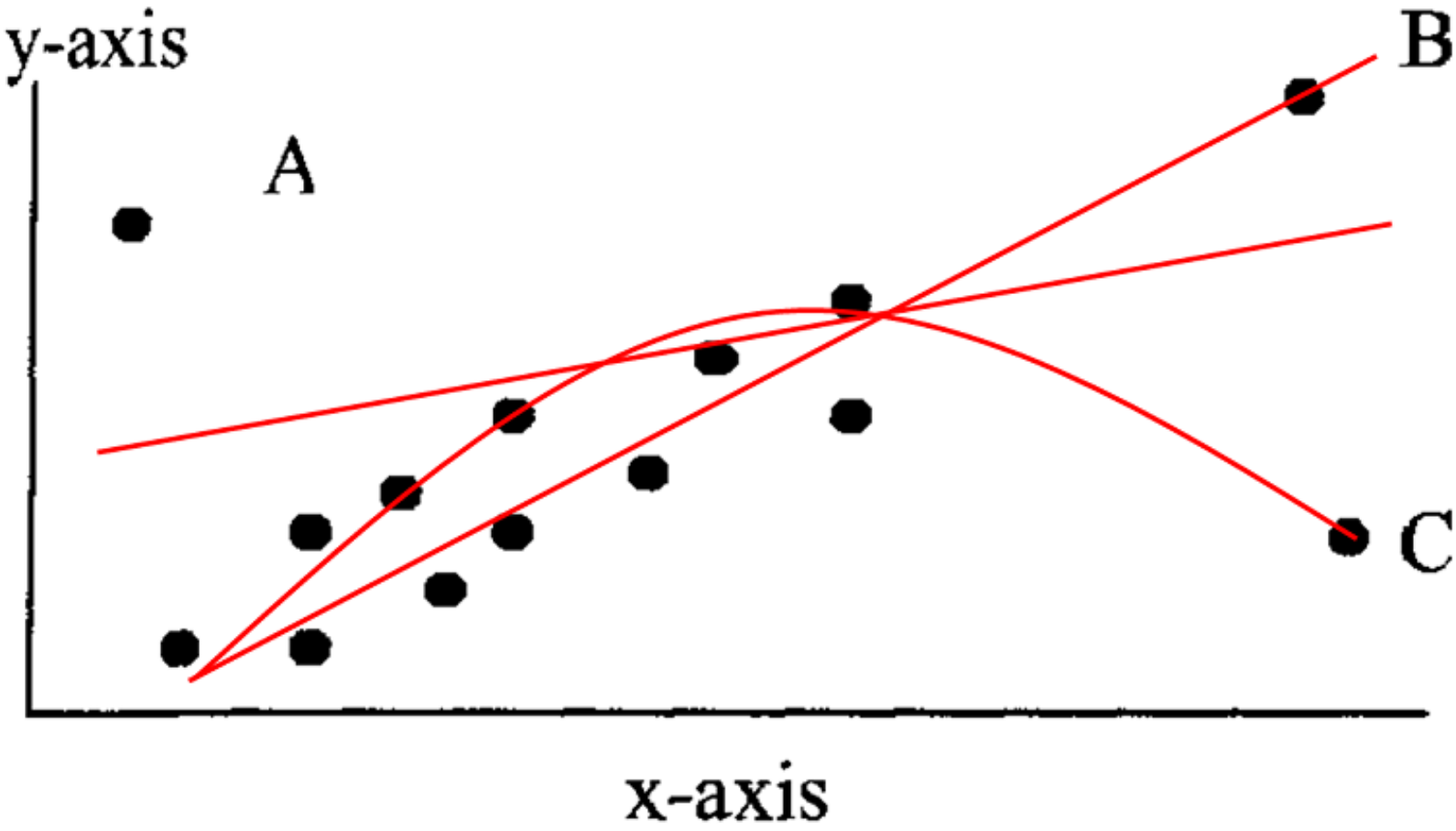
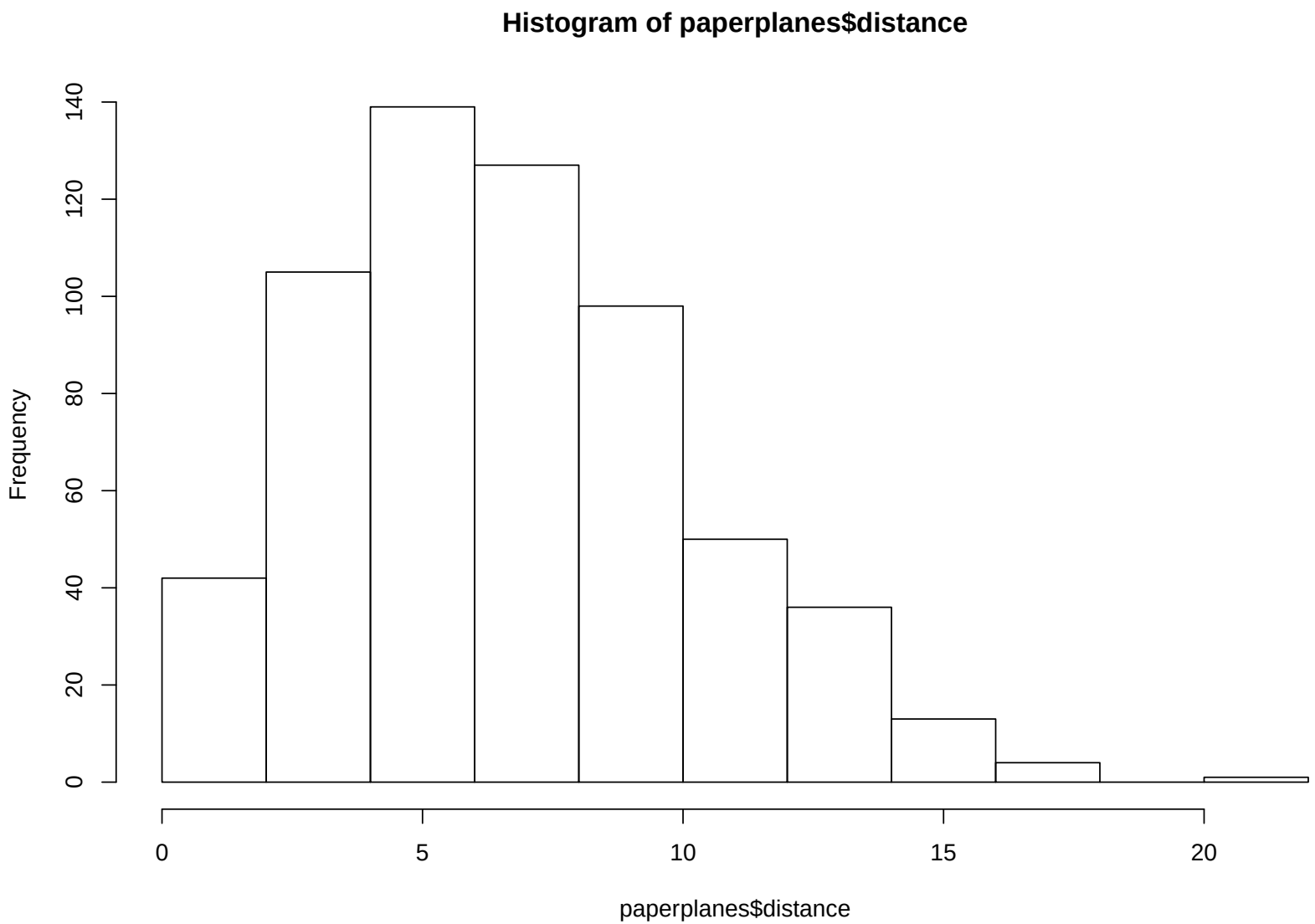


Figure 2

See http://rpsychologist.com/d3/correlation/

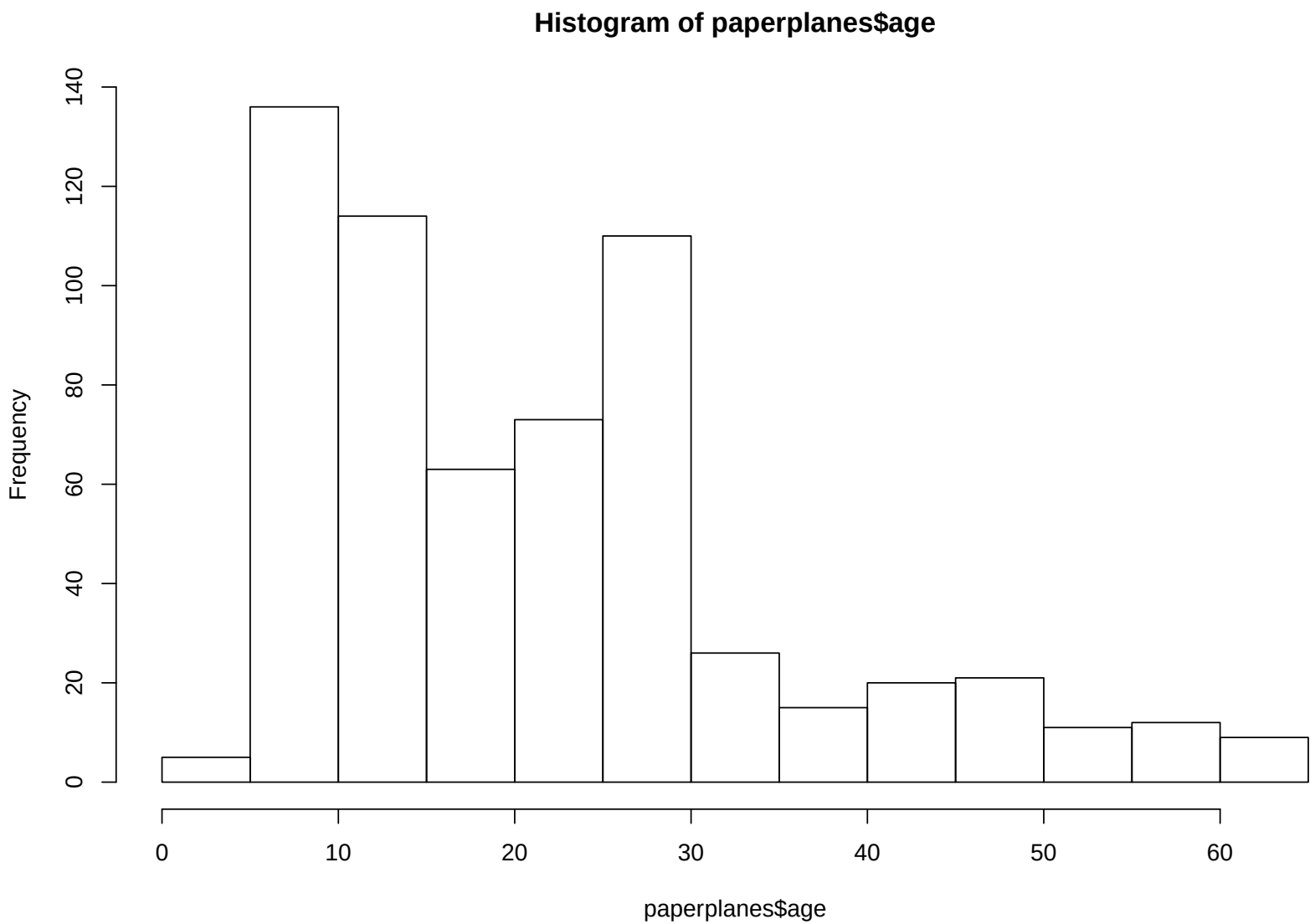
Histogram of response variable

```
hist(paperplanes$distance)
```



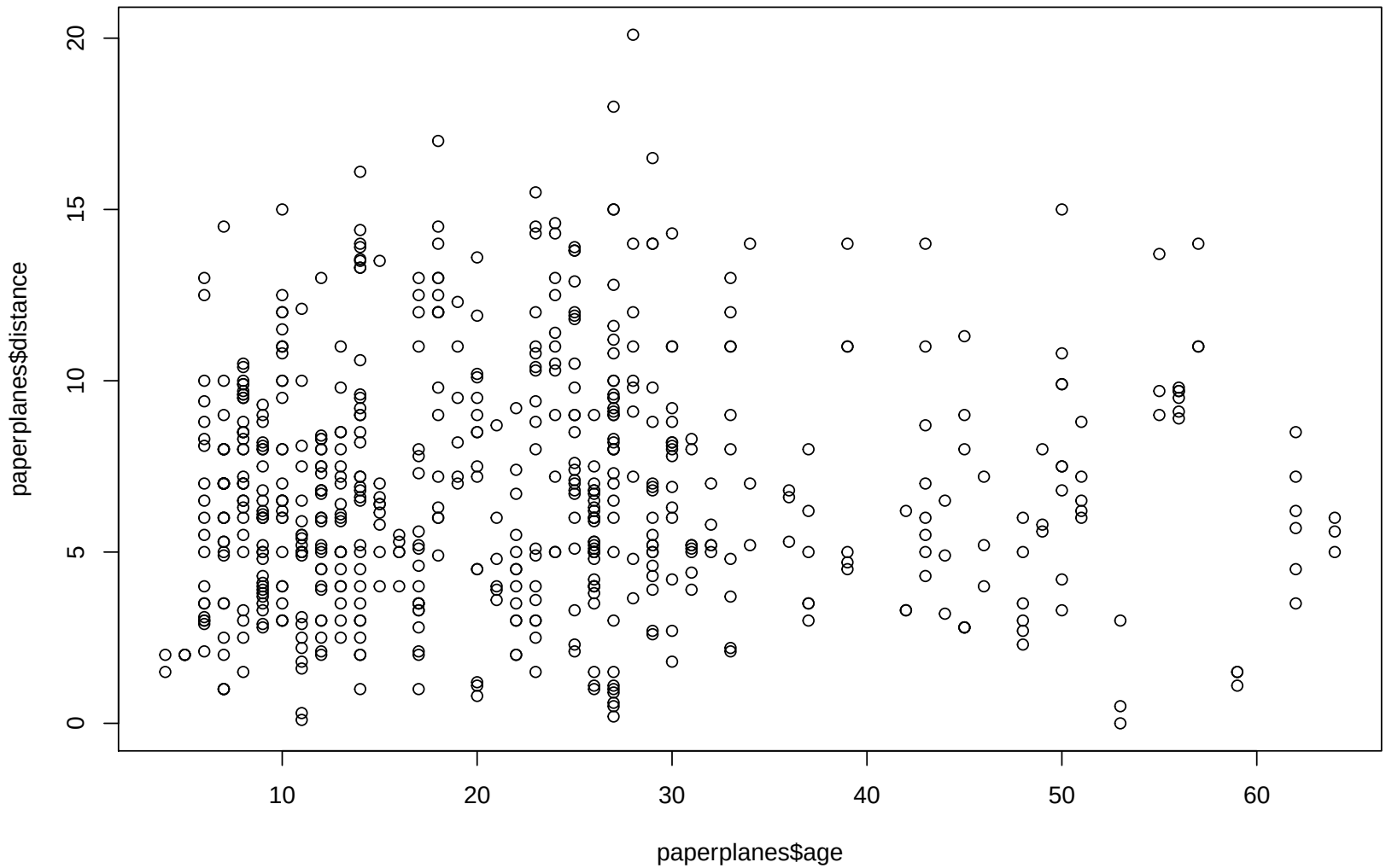
Histogram of predictor variable

```
hist(paperplanes$age)
```



Scatterplot

```
plot(paperplanes$age, paperplanes$distance)
```



Model fitting

Now fit model

Hint: `lm`

Now fit model

```
m1 <- lm(distance ~ age, data = paperplanes)
```

which corresponds to

$$Distance_i = a + b \cdot age_i + \varepsilon_i$$
$$\varepsilon_i \sim N\left(0, \sigma^2\right)$$

Model interpretation

What does this mean?

```
Call:
lm(formula = distance ~ age, data = paperplanes)

Residuals:
    Min      1Q  Median      3Q     Max
-7.1929 -2.6014 -0.3789  2.1572 13.1658

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  6.64440    0.26982  24.626   <2e-16 ***
age          0.01035    0.01040   0.996     0.32
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.504 on 613 degrees of freedom
Multiple R-squared:  0.001614,  Adjusted R-squared:  -1.434e-05
F-statistic: 0.9912 on 1 and 613 DF,  p-value: 0.3198
```

Presenting model results

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	6.64	0.27	24.63	0.00
age	0.01	0.01	1.00	0.32

Presenting model results

	Model 1
(Intercept)	6.64 (0.27)***
age	0.01 (0.01)
R^2	0.00
Adj. R^2	-0.00
Num. obs.	615
RMSE	3.50

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 2: Statistical models

Retrieving model coefficients

```
coef(m1)
```

```
(Intercept)         age
 6.64439782  0.01034968
```

Tidy up model coefficients with broom

```
library(broom)
tidy(m1)
```

```
# A tibble: 2 x 5
  term        estimate std.error statistic  p.value
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>
1 (Intercept)   6.64      0.270     24.6   1.29e-93
2 age           0.0103    0.0104     0.996 3.20e- 1
```

```
glance(m1)
```

```
# A tibble: 1 x 11
  r.squared adj.r.squared sigma statistic p.value    df logLik
      <dbl>         <dbl> <dbl>     <dbl>   <dbl> <int>  <dbl> <
1   0.00161    -0.0000143  3.50     0.991   0.320     2 -1643. 3
# ... with 2 more variables: deviance <dbl>, df.residual <int>
```

Confidence intervals

```
confint(m1)
```

```
                   2.5 %     97.5 %
(Intercept)  6.11452177 7.17427388
age         -0.01006553 0.03076489
```

Using effects package

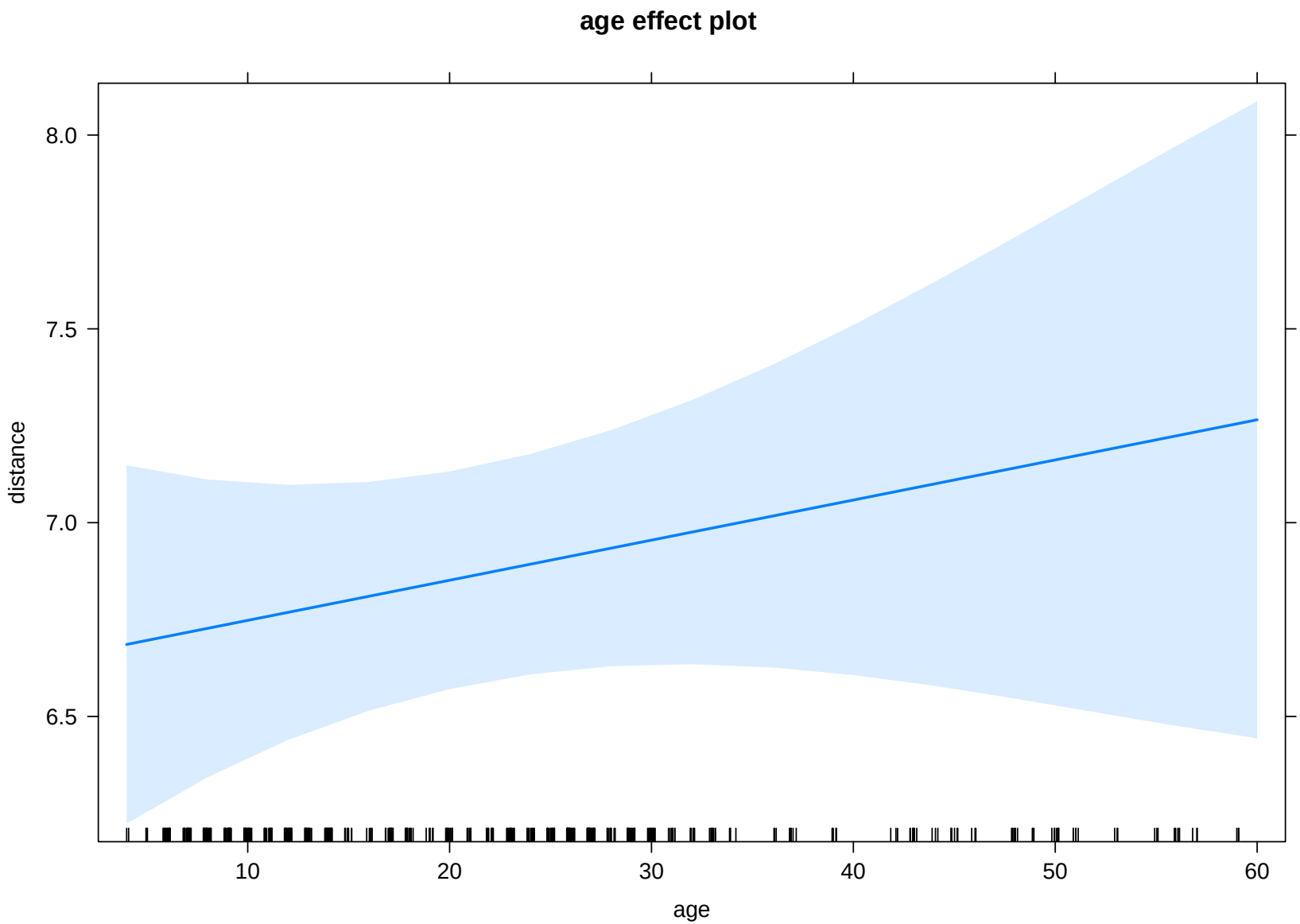
```
library(effects)
summary(allEffects(m1))
```

```
 model: distance ~ age

 age effect
age
       4        20        30        50        60
6.685797 6.851391 6.954888 7.161882 7.265379

 Lower 95 Percent Confidence Limits
age
       4        20        30        50        60
6.223509 6.570601 6.634085 6.528536 6.443633

 Upper 95 Percent Confidence Limits
age
       4        20        30        50        60
7.148084 7.132182 7.275692 7.795228 8.087125
```

Visualising fitted model

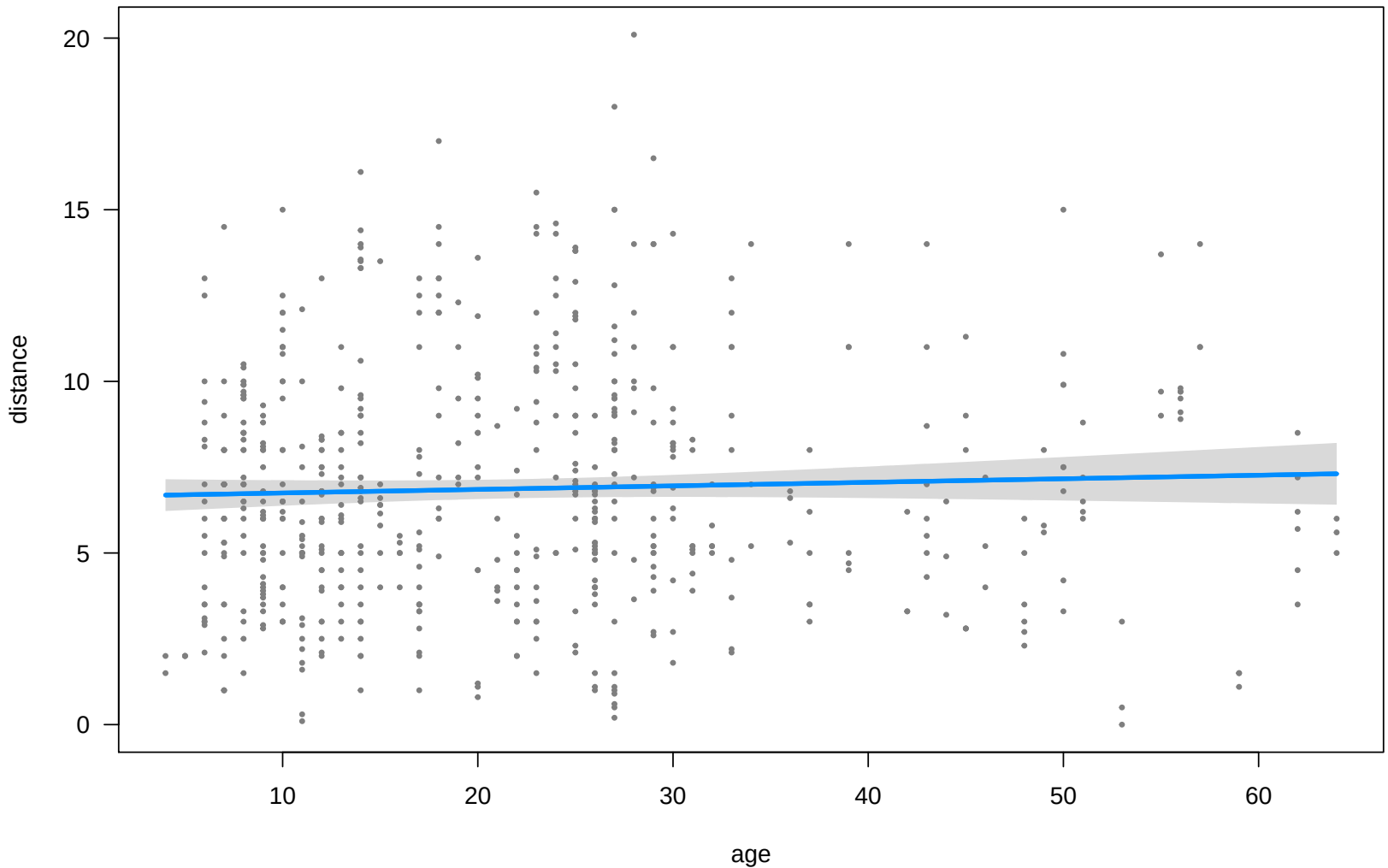
Plot effects

```
plot(allEffects(m1))
```



Plot model (visreg)

```
library(visreg)
visreg(m1)
```



Model checking

Linear model assumptions

- Linearity (transformations, GAM...)

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- Residuals:

Linear model assumptions

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- Residuals:
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Linear model assumptions

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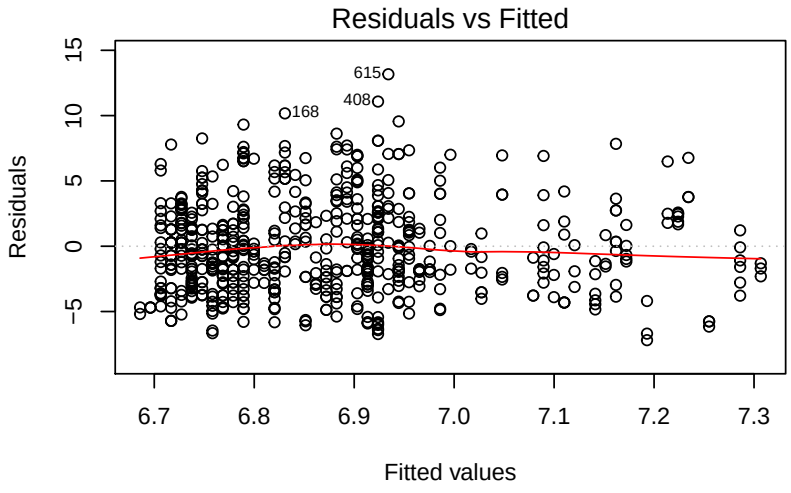
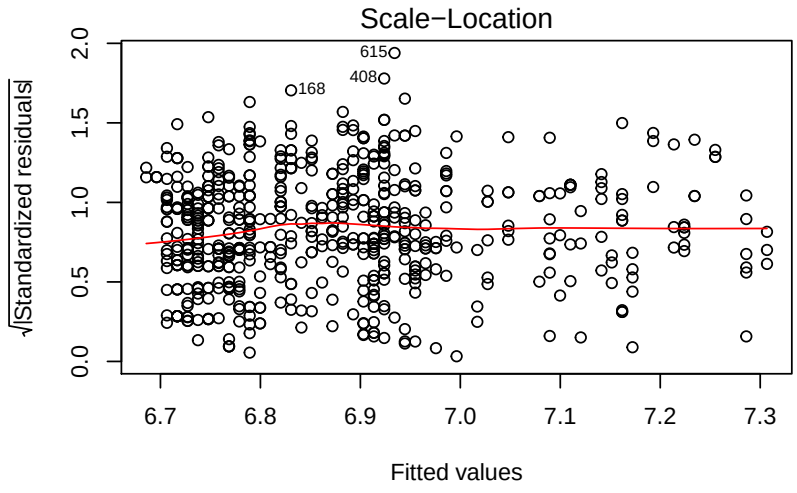
- Linearity (transformations, GAM...)
- Residuals:
  - Independent
  - Equal variance
  - Normal

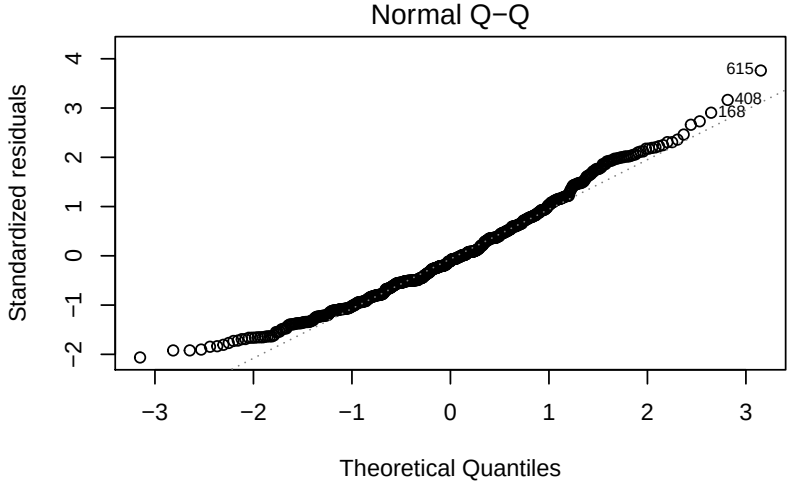
Linear model assumptions

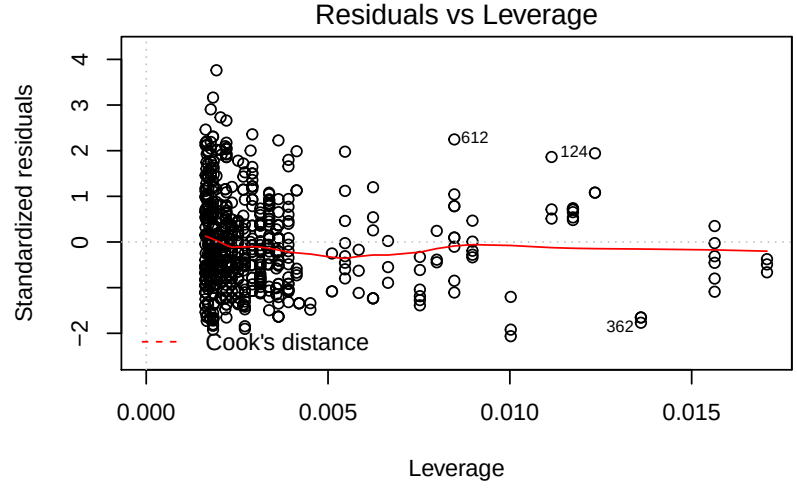
- Linearity (transformations, GAM...)
- Residuals:
  - Independent
  - Equal variance
  - Normal
- No measurement error in predictors

Model checking: residuals



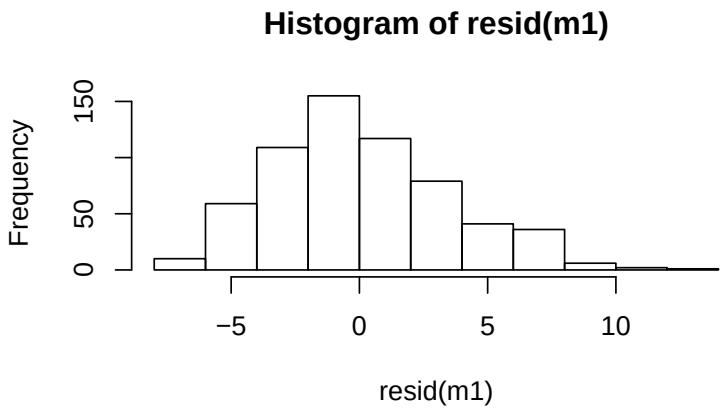






Are residuals normal?

```
hist(resid(m1))
```




```
lm(formula = distance ~ age, data = paperplane
            coef.est coef.se
(Intercept) 6.64     0.27
age         0.01     0.01
---
n = 615, k = 2
residual sd = 3.50, R-Squared = 0.00
```

SD of residuals = 3.5 coincides with estimate of `sigma`.

Using model for prediction

How good is the model in predicting distance?

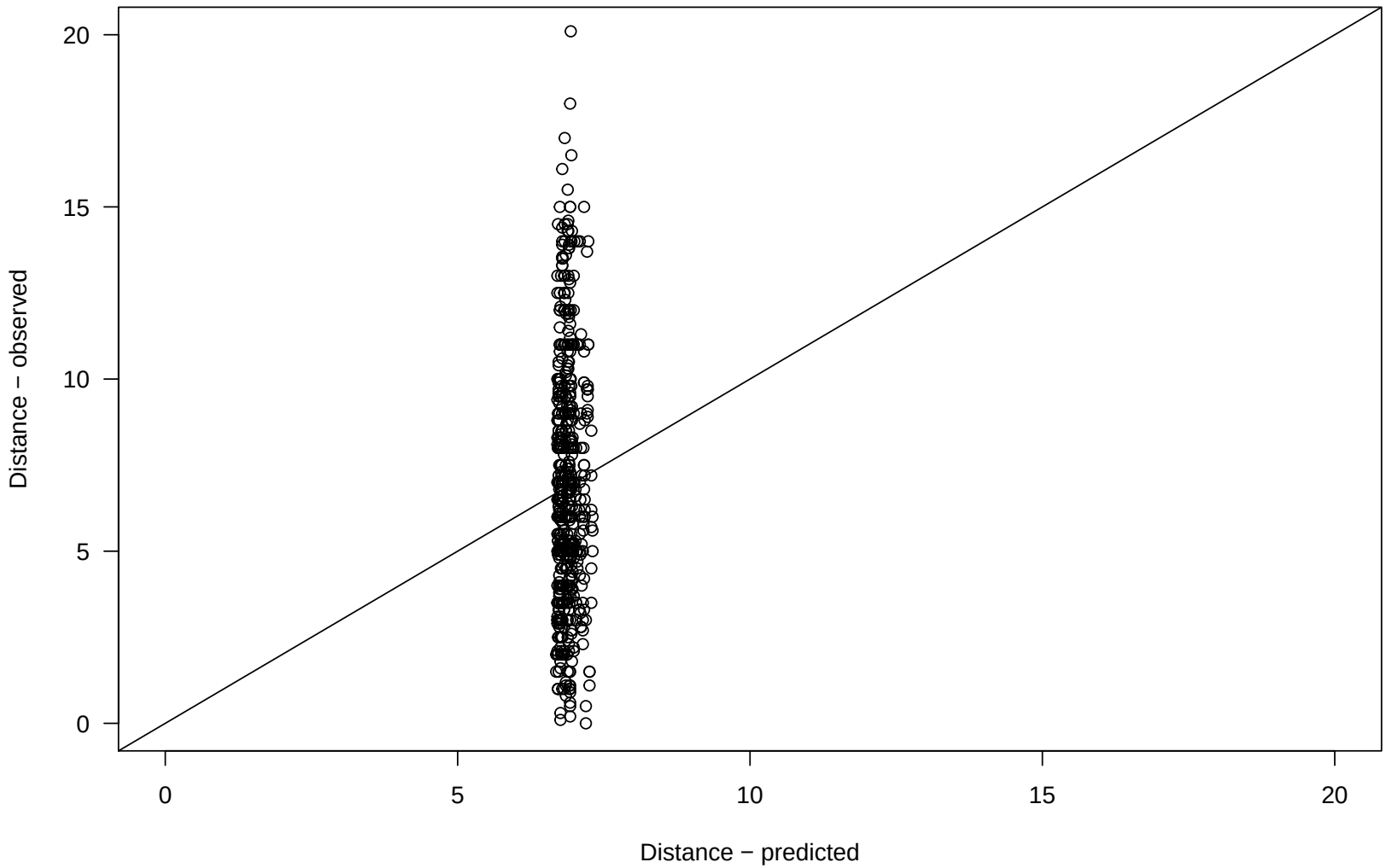
`fitted` gives predictions for each observation

```
paperplanes$distance.pred <- fitted(m1)
head(paperplanes)
```

```
# A tibble: 6 x 9
     id hour    person  gender   age plane     paper distance distance.pred
  <int> <fct>   <chr>   <fct>  <dbl> <chr>     <int>    <dbl>          <dbl>
1     1 [17,18) Roland  male      30 Standard~    80      7.8           6.95
2     2 [17,18) Astrid  female    30 Concorde~   120      2.7           6.95
3     3 [17,18) Roland  male      30 Standard~   120      9.2           6.95
4     4 [17,18) Isabel~ female    48 Standard~   120      6             7.14
5     5 [17,18) Fabien~ female    17 Standard~   120      7.3           6.82
6     6 [17,18) Fabien~ female    17 Standard~   120      7.8           6.82
```

Calibration plot: Observed vs Predicted values

```
plot(paperplanes$distance.pred, paperplanes$distance, xlab = "Di
```



Using fitted model for prediction

Q: Expected distance if age = 30?

```
new.age <- data.frame(age = c(30))
predict(m1, new.age, se.fit = TRUE)
```

```
$fit
       1
6.954888

$se.fit
[1] 0.1633552

$df
[1] 613

$residual.scale
[1] 3.503736
```

Using fitted model for prediction

Q: Expected distance if age = 30?

```
new.age <- data.frame(age = c(30))
predict(m1, new.age, se.fit = TRUE, interval = "confidence", lev
```

```
$fit
       fit      lwr      upr
1 6.954888 6.634085 7.275692

$se.fit
[1] 0.1633552

$df
[1] 613

$residual.scale
[1] 3.503736
```

Using fitted model for prediction

Q: Expected distance if age = 30?

```
new.age <- data.frame(age = c(30))
predict(m1, new.age, se.fit = TRUE, interval = "prediction", lev
```

```
$fit
       fit         lwr      upr
1 6.954888 0.06663211 13.84314

$se.fit
[1] 0.1633552

$df
[1] 613

$residual.scale
[1] 3.503736
```

Important functions

- `plot`

Important functions

- `plot`
- `summary`

Important functions

- `plot`
- `summary`
- `coef`

Important functions

- `plot`
- `summary`
- `coef`
- `confint`

Important functions

- `plot`
- `summary`
- `coef`
- `confint`
- `fitted`

Important functions

- `plot`
- `summary`
- `coef`
- `confint`
- `fitted`
- `resid`

Important functions

- `plot`
- `summary`
- `coef`
- `confint`
- `fitted`
- `resid`
- `allEffects`

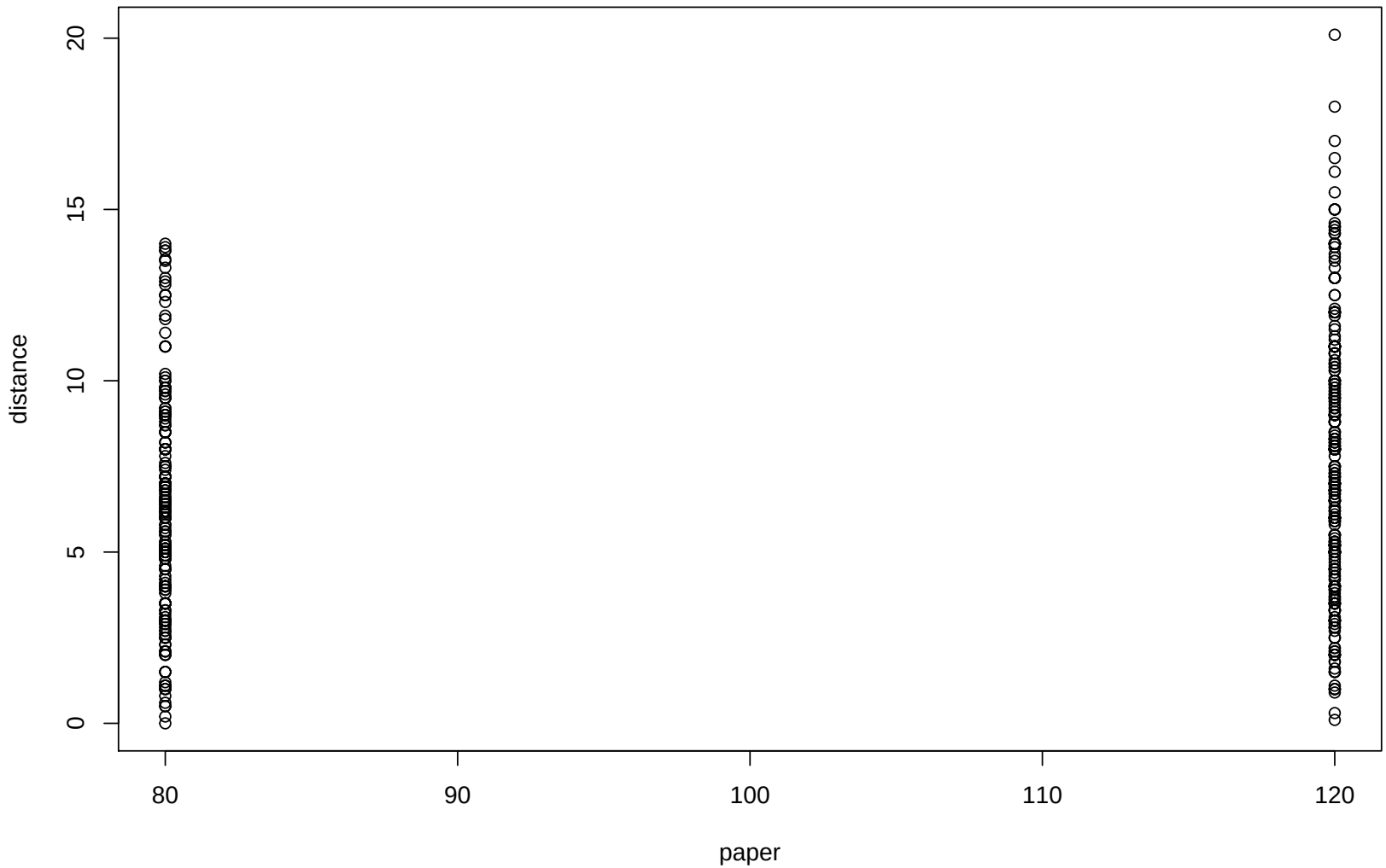
Important functions

- `plot`
- `summary`
- `coef`
- `confint`
- `fitted`
- `resid`
- `allEffects`
- `predict`

Categorical predictors (factors)

Q: Does distance vary with paper type?

```
plot(distance ~ paper, data = paperplanes)
```



Model distance ~ paper

All right here?

```
m2 <- lm(distance ~ paper, data = paperplanes)
```

```
Call:
lm(formula = distance ~ paper, data = paperplanes)

Residuals:
    Min      1Q  Median      3Q     Max
-7.2756 -2.3756 -0.3756  2.2244 12.7244

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 3.638290   0.750041   4.851 1.56e-06 ***
paper       0.031144   0.007095   4.389 1.34e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.453 on 613 degrees of freedom
Multiple R-squared:  0.03047,   Adjusted R-squared:  0.02889
```

Model distance ~ paper

Paper is a factor!

```
paperplanes$paper <- as.factor(paperplanes$paper)
```

```
       id              hour        person             gender
 Min.   :  1.0   [19,20) :139   Length:615         female:213
 1st Qu.:154.5   [22,23) :108   Class :character   male  :402
 Median :308.0   [21,22) : 89   Mode  :character
 Mean   :308.0   [18,19) : 86
 3rd Qu.:461.5   [23,Inf): 78
 Max.   :615.0   [17,18) : 75
                 (Other) : 40
      age            plane            paper        distance
 Min.   : 4.00   Length:615         80 :248   Min.   : 0.000
 1st Qu.:11.00   Class :character   120:367   1st Qu.: 4.350
 Median :20.00   Mode  :character             Median : 6.500
 Mean   :22.11                               Mean   : 6.873
 3rd Qu.:28.00                               3rd Qu.: 9.000
 Max.   :64.00                               Max.   :20.100

 distance.pred
```

Model distance ~ paper

```
m2 <- lm(distance ~ paper, data = paperplanes)
```

```
Call:
lm(formula = distance ~ paper, data = paperplanes)

Residuals:
    Min      1Q  Median      3Q     Max
-7.2756 -2.3756 -0.3756  2.2244 12.7244

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   6.1298     0.2192  27.958  < 2e-16 ***
paper120      1.2458     0.2838   4.389 1.34e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.453 on 613 degrees of freedom
Multiple R-squared:  0.03047,   Adjusted R-squared:  0.02889
F-statistic: 19.27 on 1 and 613 DF,  p-value: 1.339e-05
```

Linear model with categorical predictors

```
m2 <- lm(distance ~ paper, data = paperplanes)
```

which corresponds to

$$y_i = a + bx_i + \varepsilon_i$$

$$distance_i = a + b_{paper120} + \varepsilon_i$$

Model distance ~ paper

```
m2 <- lm(distance ~ paper, data = paperplanes)
```

```
Call:
lm(formula = distance ~ paper, data = paperplanes)

Residuals:
    Min      1Q  Median      3Q     Max
-7.2756 -2.3756 -0.3756  2.2244 12.7244

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   6.1298     0.2192  27.958  < 2e-16 ***
paper120      1.2458     0.2838   4.389 1.34e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.453 on 613 degrees of freedom
Multiple R-squared:  0.03047,   Adjusted R-squared:  0.02889
F-statistic: 19.27 on 1 and 613 DF,  p-value: 1.339e-05
```

Effects: Estimated Distance ~ paper

```
summary(allEffects(m2))
```

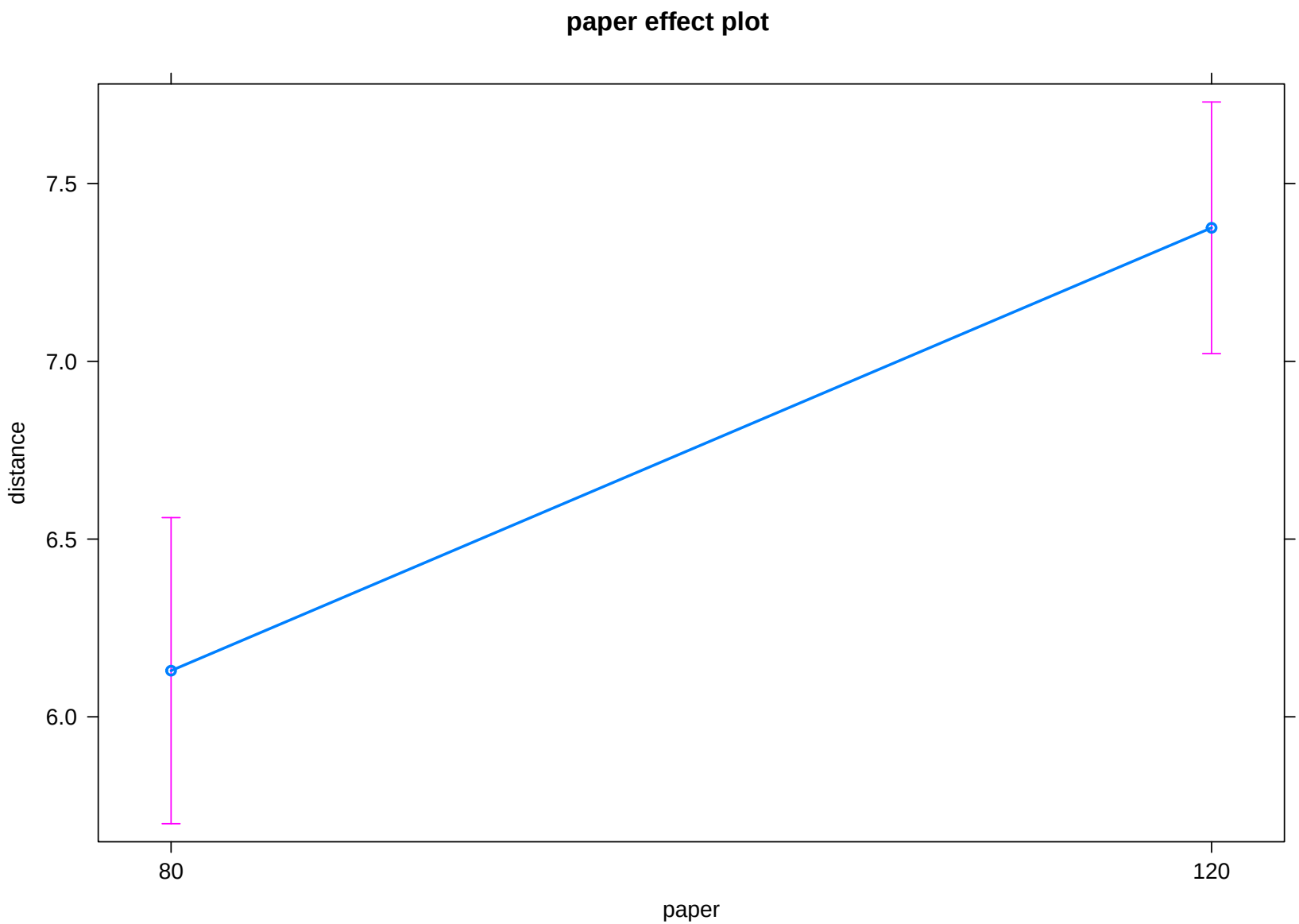
```
 model: distance ~ paper

 paper effect
paper
      80       120
6.129839 7.375613

 Lower 95 Percent Confidence Limits
paper
      80       120
5.699269 7.021668

 Upper 95 Percent Confidence Limits
paper
      80       120
6.560408 7.729558
```

Presenting model results

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	6.13	0.22	27.96	0
paper120	1.25	0.28	4.39	0

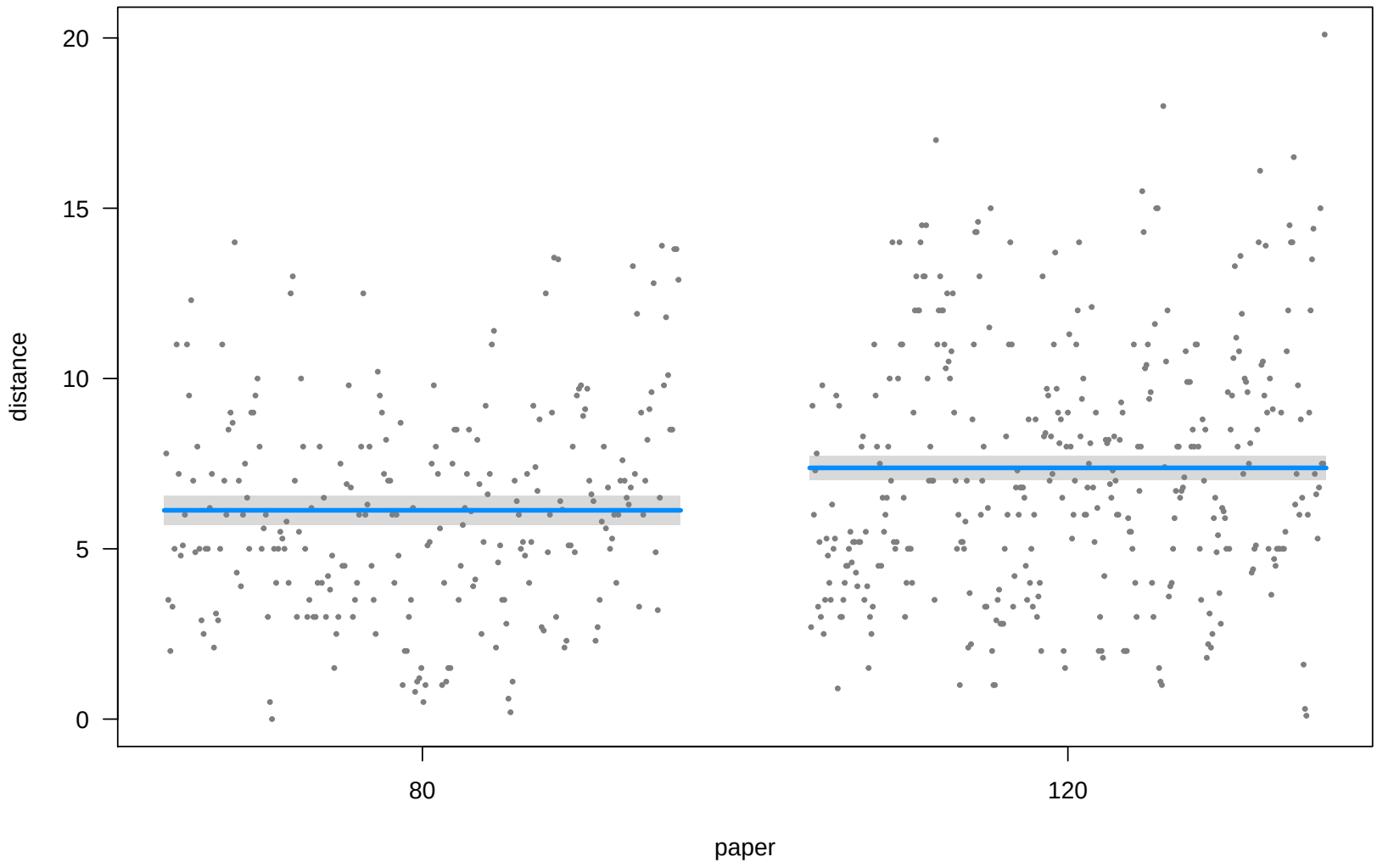
Plot

```
plot(allEffects(m2))
```

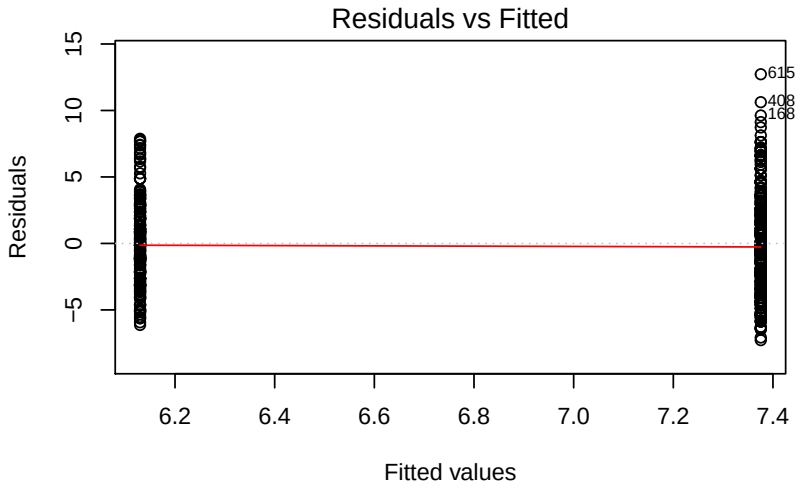


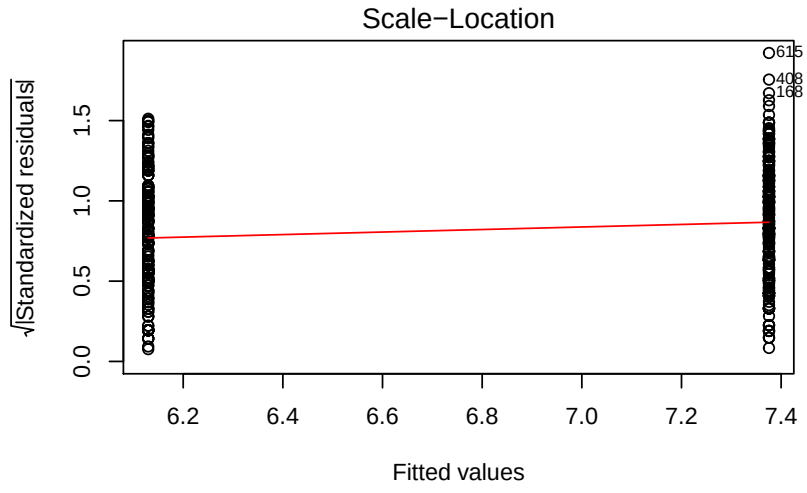
Plot (visreg)

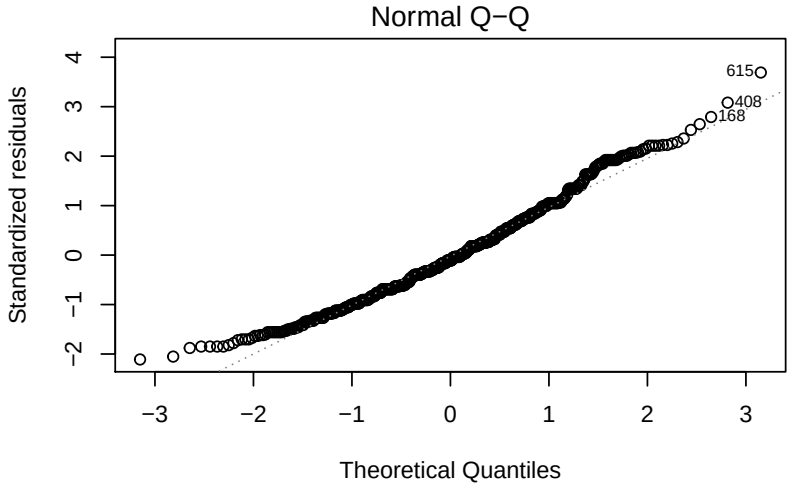
```
visreg(m2)
```

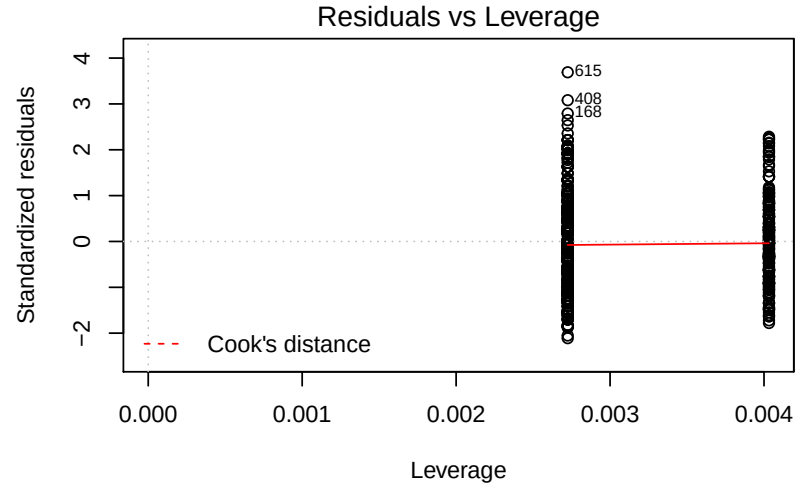


Model checking: residuals



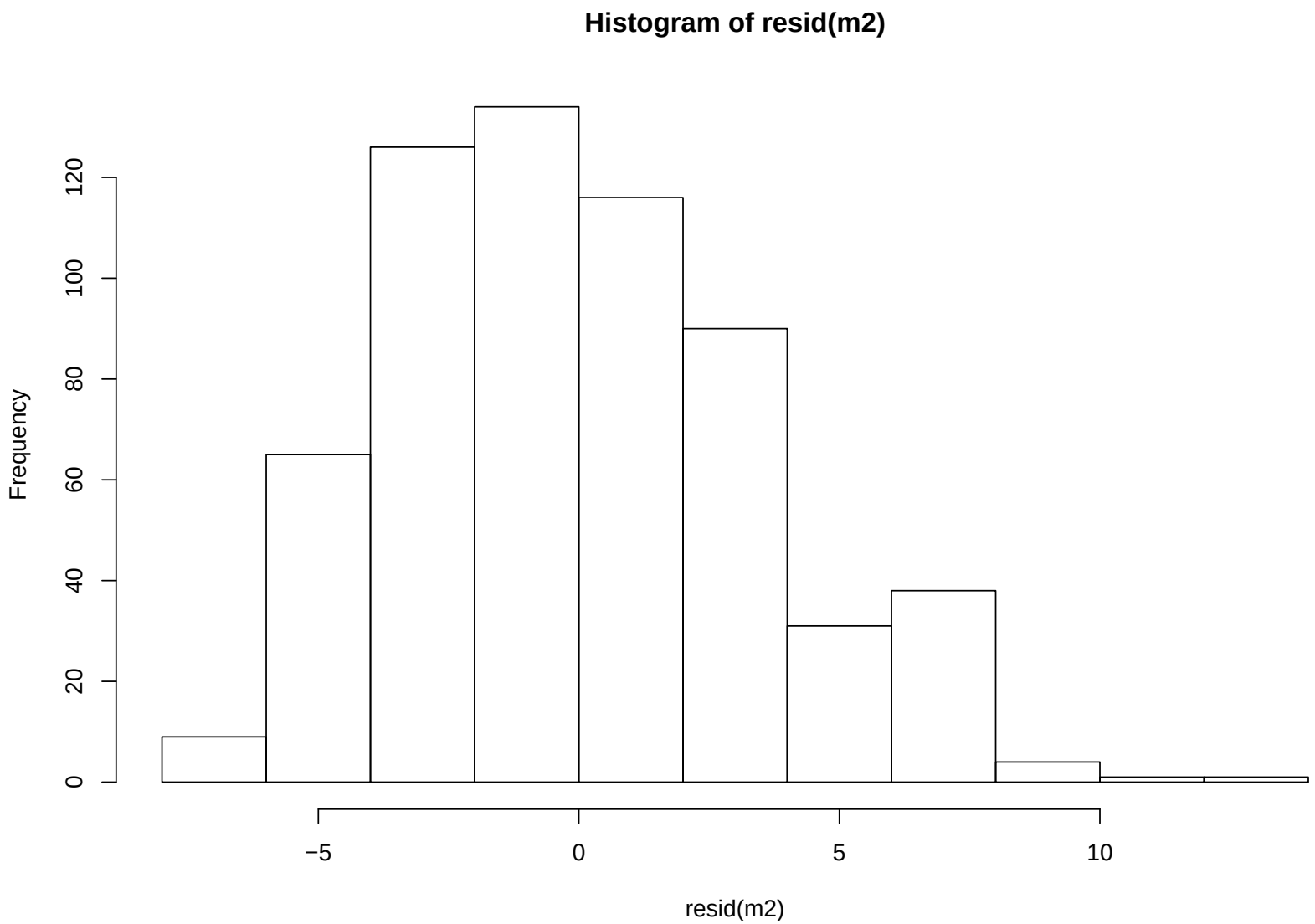






Model checking: residuals

```
hist(resid(m2))
```



Exercise: Does distance vary with gender?

Combining continuous and categorical predictors

Predicting distance based on age and paper type

```
lm(distance ~ paper + age, data = paperplanes)
```

$$y_i = a + bx_i + \varepsilon_i$$

$$distance_i = a + b_{paper120} + c \cdot age_i + \varepsilon_i$$

Predicting distance based on age and paper type

```
Call:
lm(formula = distance ~ age + paper, data = paperplanes)

Residuals:
    Min      1Q  Median      3Q     Max
-7.1092 -2.4753 -0.3576  2.2523 12.5892

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.69210    0.33641  16.920  < 2e-16 ***
age          0.01774    0.01035   1.714   0.0871 .
paper120     1.32192    0.28683   4.609 4.93e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.447 on 612 degrees of freedom
Multiple R-squared:  0.0351,    Adjusted R-squared:  0.03195
F-statistic: 11.13 on 2 and 612 DF,  p-value: 1.784e-05
```

Presenting model results

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	5.69	0.34	16.92	0.00
age	0.02	0.01	1.71	0.09
paper120	1.32	0.29	4.61	0.00

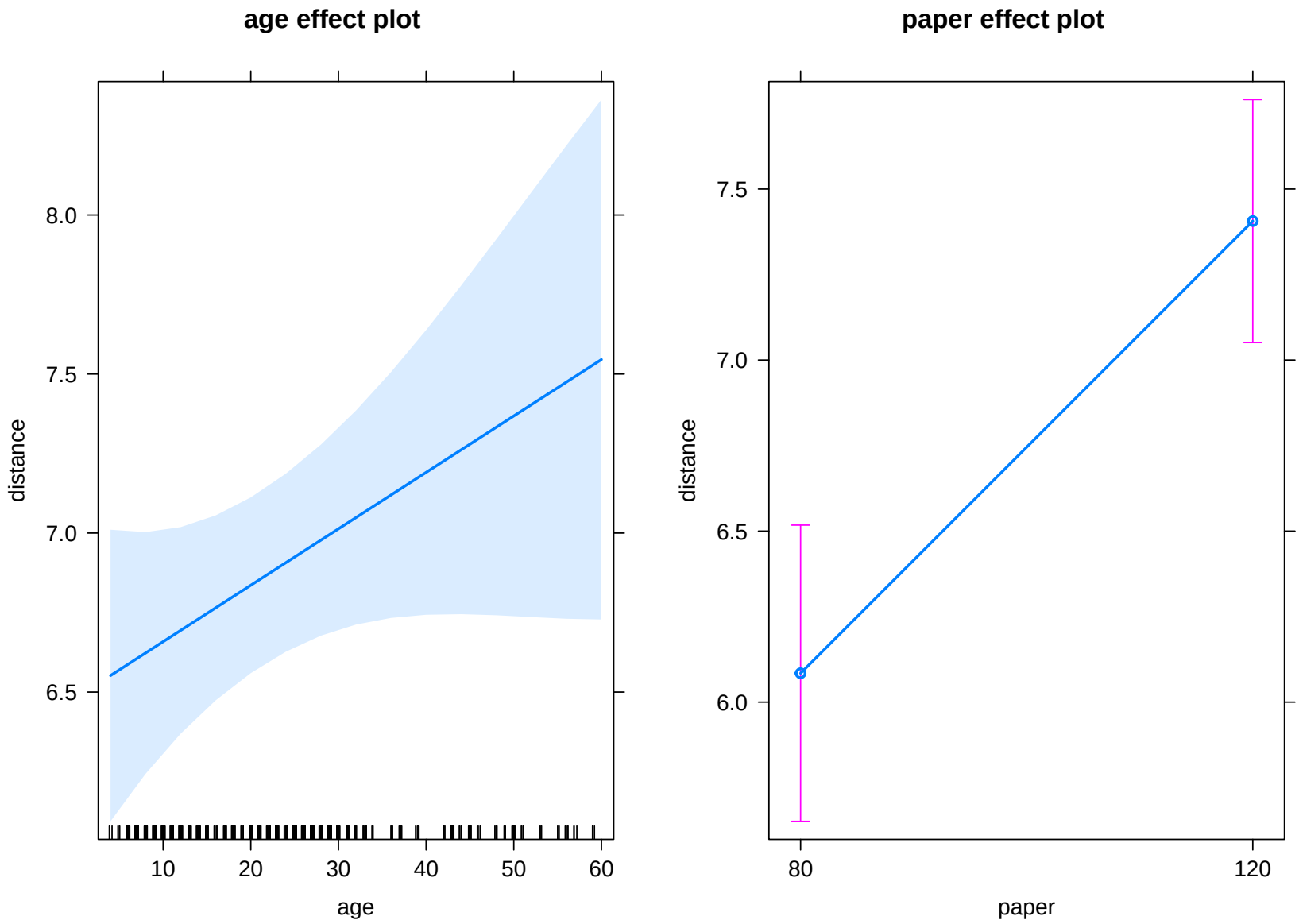
Estimated distance

```
summary(allEffects(multreg))
```

```
 model: distance ~ age + paper

 age effect
age
       4        20        30        50        60
6.551921 6.835779 7.013191 7.368014 7.545425

 Lower 95 Percent Confidence Limits
age
       4        20        30        50        60
6.093516 6.559431 6.696578 6.738709 6.728156

 Upper 95 Percent Confidence Limits
age
       4        20        30        50        60
7.010326 7.112127 7.329803 7.997318 8.362694

 paper effect
paper
      80      120
6.084400 7.406318

 Lower 95 Percent Confidence Limits
paper
      80      120
5.651366 7.051182

 Upper 95 Percent Confidence Limits
```

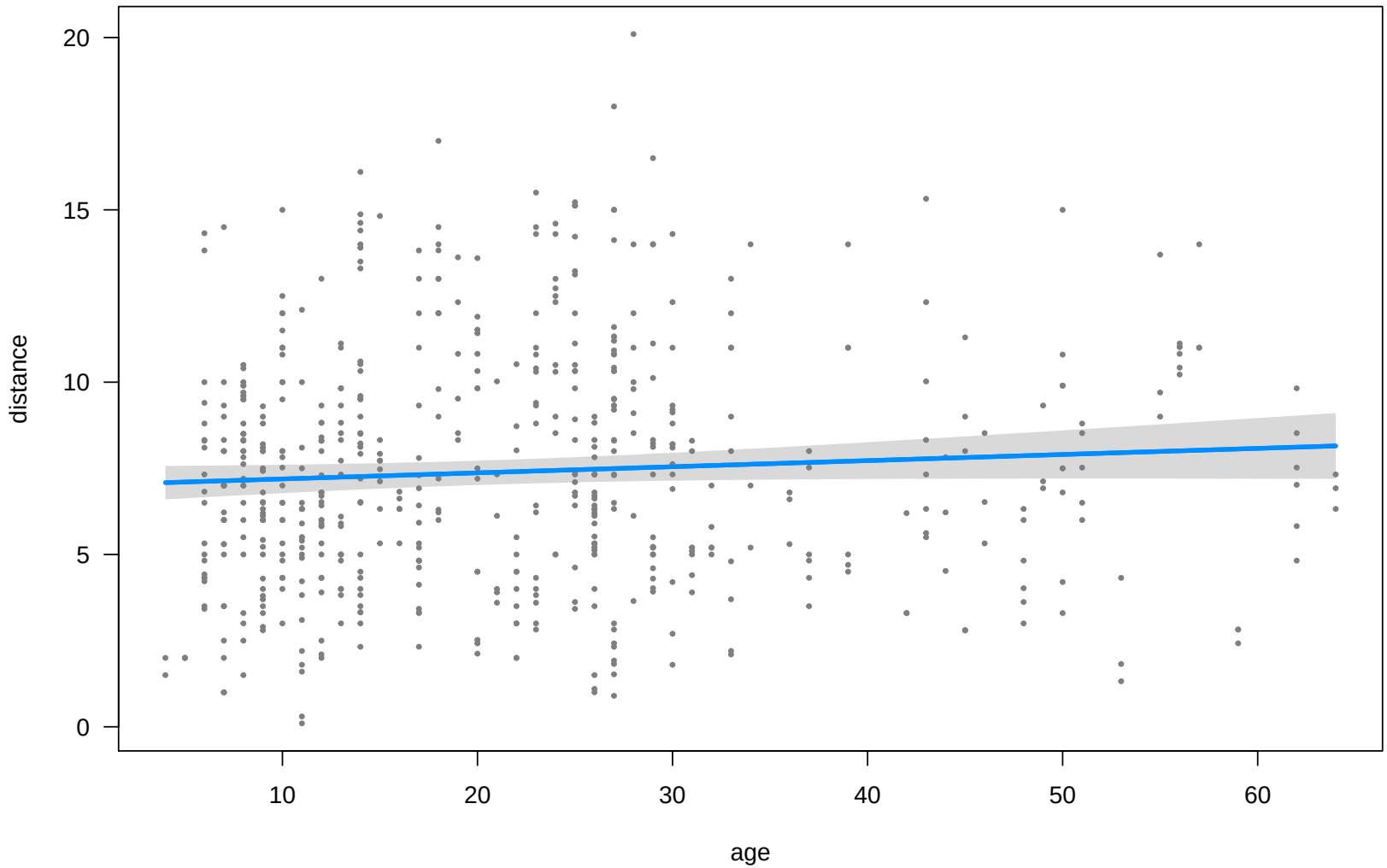
Plot

```
plot(allEffects(multreg))
```

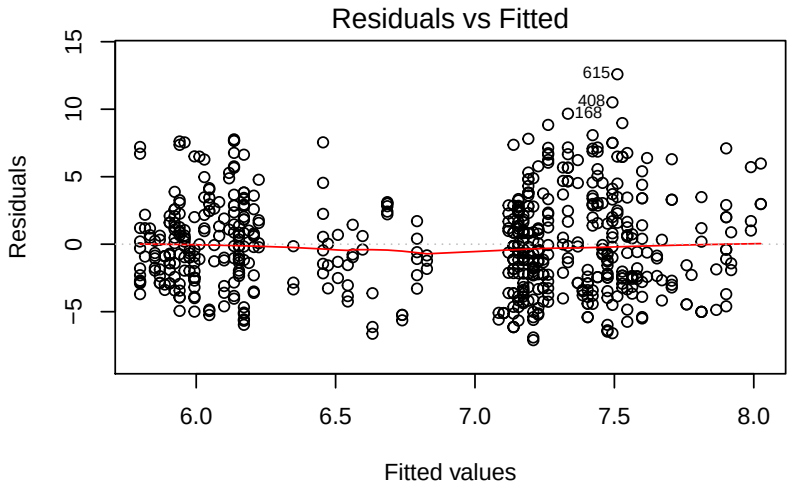


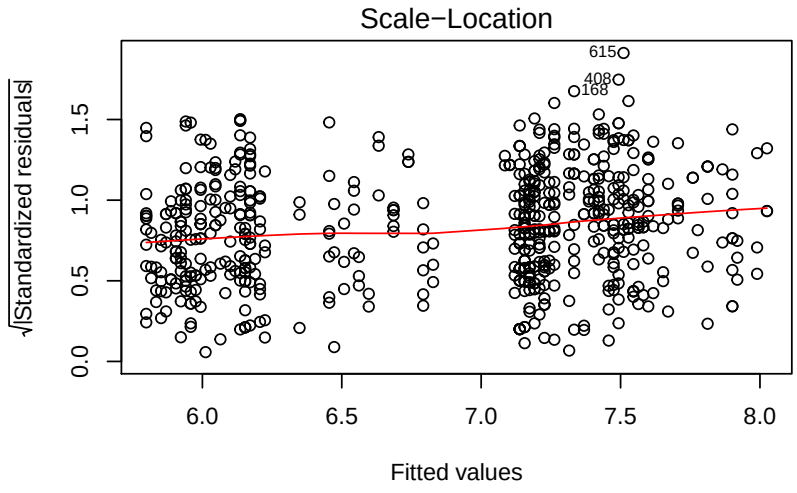
Plot (visreg)

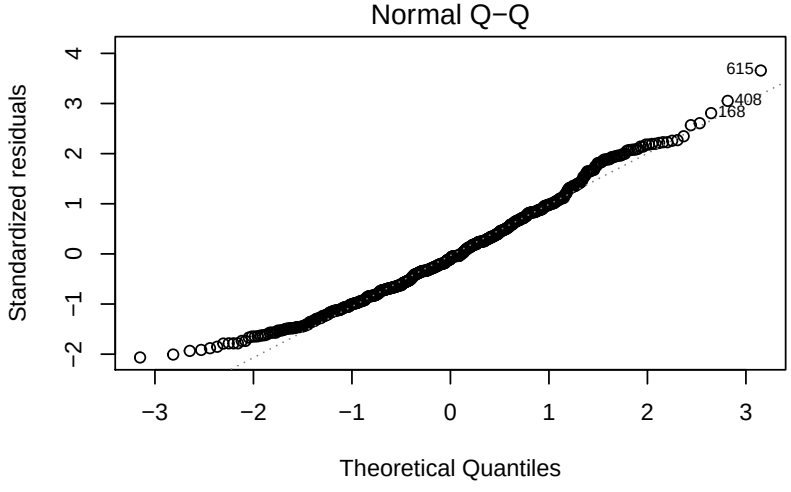
```
visreg(multreg)
```

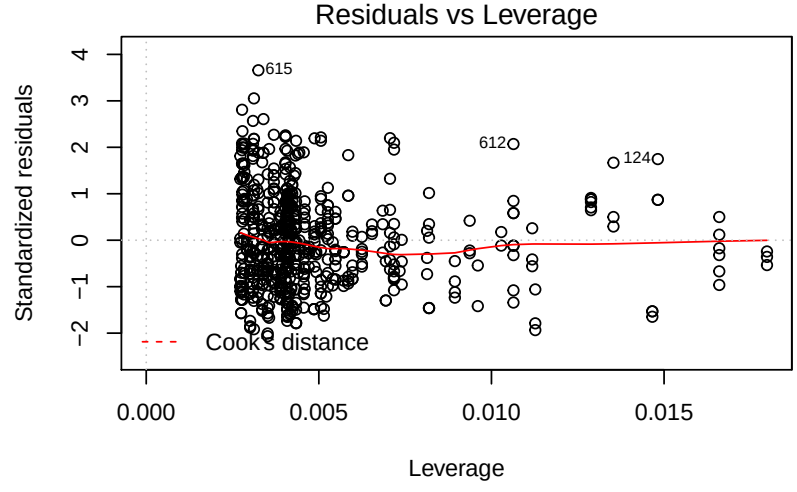


Model checking: residuals



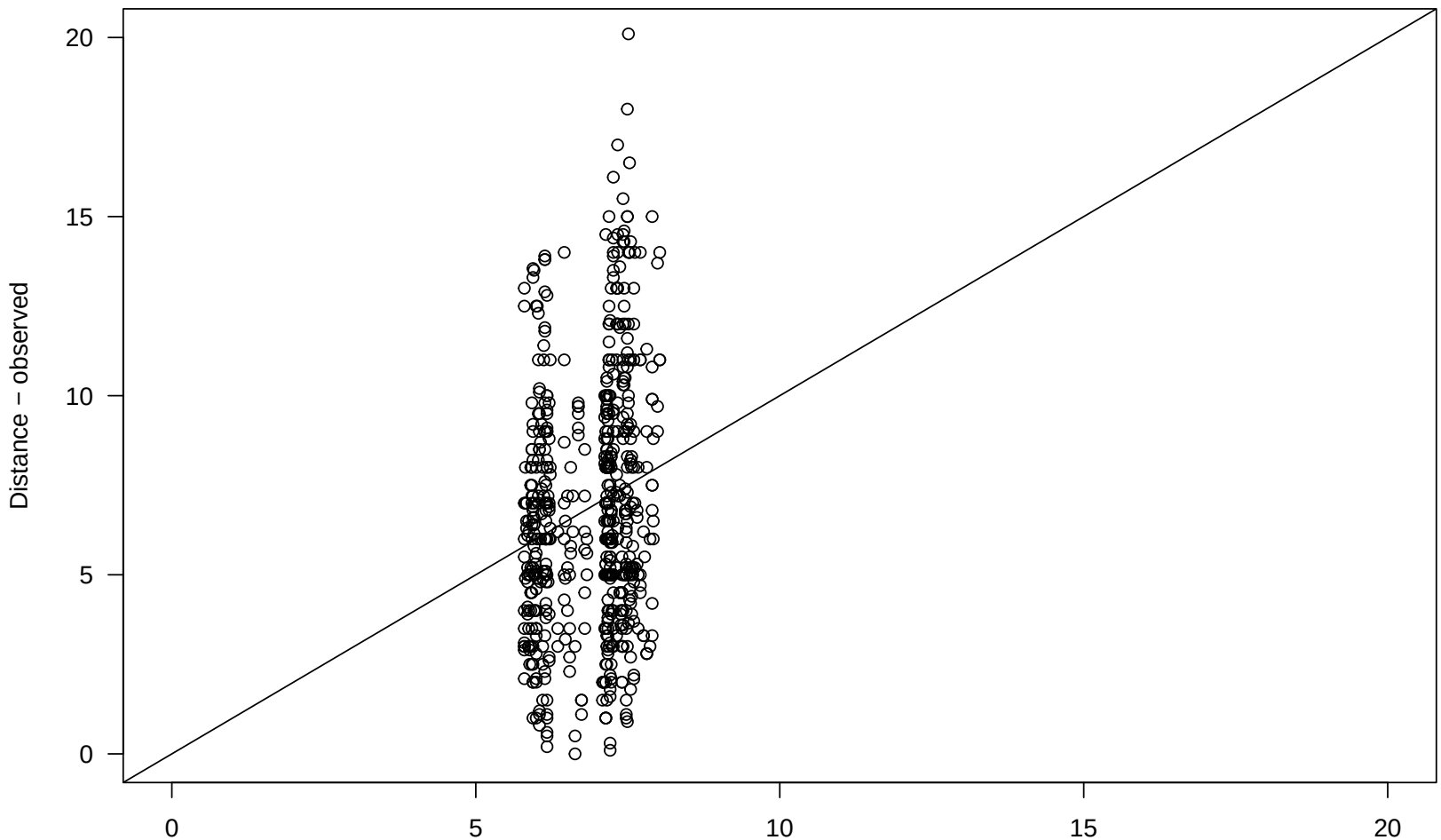






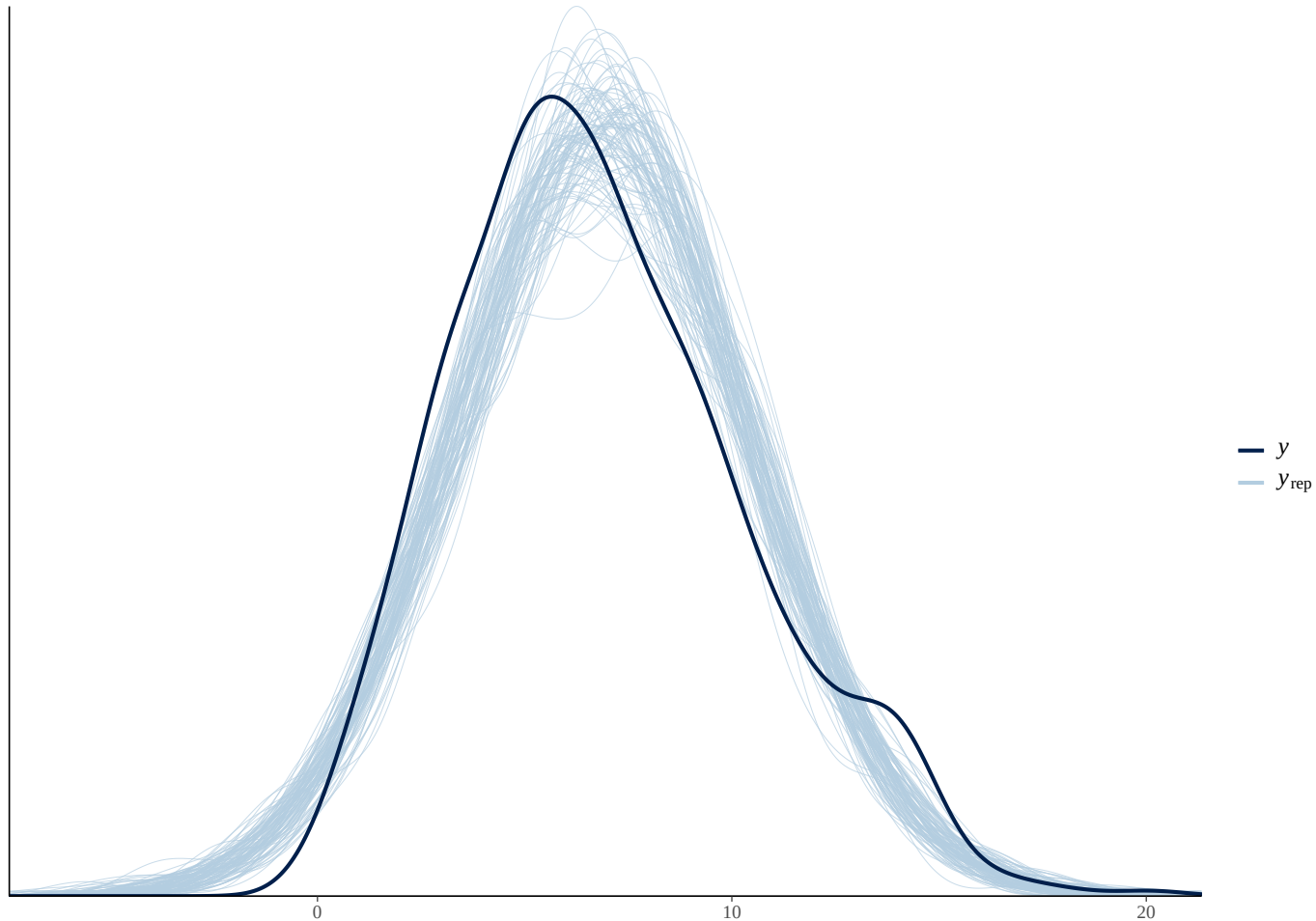
How good is this model? Calibration plot

```
paperplanes$distance.pred <- fitted(multreg)
plot(paperplanes$distance.pred, paperplanes$distance, xlab = "Di
abline(a = 0, b = 1)
```



Model checking with simulated data

```
library(bayesplot)
sims <- simulate(multreg, nsim = 100)
ppc_dens_overlay(paperplanes$distance, yrep = t(as.matrix(sims)))
```



Extra exercises

- mammal sleep: Are sleep patterns related to diet?

Extra exercises

- mammal sleep: Are sleep patterns related to diet?
- iris: Predict petal length $\sim$ petal width and species