

Lecture 6

Modelling areal data

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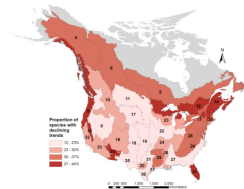
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Recall: Types of spatial data

We can distinguish three types of spatial data structures

Areal data



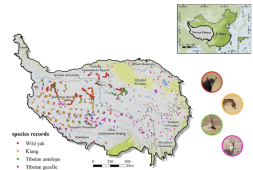
Map of bird conservation regions (BCRs) showing the proportion of bird species within each region showing a declining trend

Geostatistical data



Scotland river temperature monitoring network

Point-referenced data



Occurrence records of four ungulate species in the Tibet,

Types of spatial data

Recall:

Discrete space:

- **Data on a spatial grid (areal data)**

Continuous space:

- Geostatistical (geo-referenced) data
- Spatial point pattern data

Model components are used to reflect spatial dependence structures in discrete and continuous space.

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⇒ very similar thinking

- neighbourhood structure – neighbours in space
- e.g. we can consider random walk models in space

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- neighbourhood structure – neighbours in space
- e.g. we can consider random walk models in space
- these are special cases of **Gauss Markov random fields** (GMRF)
- recall: latent **Gaussian** models...

large latent Gaussian models

- real-world applications often involve latent fields with $n = 10^2$ – 10^5 elements
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- ~> we need simplifications to make these feasible
- GMRFs provide a computationally efficient representation

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- real-world applications often involve latent fields with $n = 10^2$ – 10^5 elements
- computation times and memory issues become a problem...

~> we need simplifications to make these feasible

- GMRFs provide a computationally efficient representation
- we will see later that these will come in handy even for continuous data (the “SPDE approach”)

The Gaussian distribution

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- The mean vector, which is set equal to 0 here, is the centre of the distribution
- The matrix Σ describes all pairwise covariances
 $\Sigma_{i,j} = \text{Cov}[x_i, x_j]$
- Analytically tractable and well understood...

Formally

$$\pi(x) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2} x^T \Sigma^{-1} x}$$

- Σ is typically large and dense
- infeasible for large n
- we need to reduce computational cost

Definition

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- Efficient numerical algorithms exist for sparse systems.
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Two Options

1. Force Σ to be sparse \Rightarrow independence...
2. Force $Q = \Sigma^{-1}$ (the precision) to be sparse \Rightarrow conditional independence

Example: AR(1) Process

$$x_1 \sim \mathcal{N}\left(0, \frac{1}{1 - \phi^2}\right), \quad x_t = \phi x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1)$$

- Σ is dense.
- But Σ^{-1} is tridiagonal (sparse).

Key Property

$$x_i \perp x_j \mid x_{-\{i,j\}} \iff Q_{ij} = 0$$

This defines the connection between conditional independence and sparsity in Q .

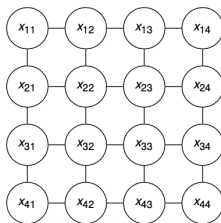
Graphical Representation

- Conditional independence structure can be represented via a graph.
- Each node corresponds to a variable x_i .
- Edges correspond to nonzero Q_{ij} .

— AR(1):



— ICAR(1):



Definition of a GMRF

Definition

A random vector x is a **Gaussian Markov Random Field (GMRF)** on a graph $G = (V, E)$ with mean μ and precision Q if:

$$\pi(x) = \frac{|Q|^{1/2}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(x - \mu)^T Q (x - \mu)\right],$$

and $Q_{ij} \neq 0 \Leftrightarrow \{i, j\} \in E$.

Example: AR(1) process

$$x_1 \sim \mathcal{N}\left(0, \frac{1}{1 - \phi^2}\right)$$

$$x_t = \phi x_{t-1} + \varepsilon_t, \quad t > 1, \quad \varepsilon_t \sim \mathcal{N}(0, 1)$$

The covariance matrix is dense since all time points are dependent, but the precision matrix Σ^{-1} is not:

$$\Sigma^{-1} = \begin{bmatrix} 1 & -\phi & 0 & \dots & 0 \\ -\phi & 1 + \phi^2 & -\phi & \dots & 0 \\ 0 & -\phi & 1 + \phi^2 & -\phi & \vdots \\ \vdots & \vdots & -\phi & 1 + \phi^2 & -\phi \\ 0 & \dots & 0 & -\phi & 1 \end{bmatrix}$$

Example: AR(1) process

- The tridiagonal form of Σ^{-1} can be exploited for quick calculations.
- This is the basis of forward filtering backward smoothing algorithms. We call $Q = \Sigma^{-1}$ the precision matrix.
- What is the key property of this example that causes Σ^{-1} to be sparse?
- Can we take advantage of it in other models?

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Theorem

$$x_i \perp x_j \mid x_{-\{i,j\}} \iff Q_{ij} = 0$$

This is the key property!

It is useful to represent the conditional independence structure through a graph — AR(1):

$x_1 \ x_2 \ x_3 \ x_4 \ x_5$

— ICAR(1):

(Informal) definition of a GMRF

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- a GMRF is a Gaussian distribution where the non-zero elements of the precision matrix are defined by a graph structure
 - in the time series AR1 example each variable is connected only to its predecessor and successor
- ~> the precision matrix is tridiagonal, i.e. sparse
- ~> efficient computation

Besag model:

- each region conditionally has a Gaussian distribution with mean equal to the average of the neighbours

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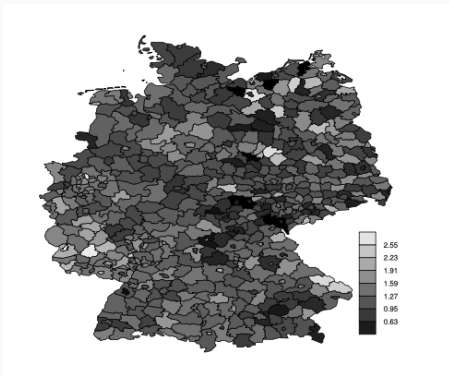
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example: larynx cancer data from Germany

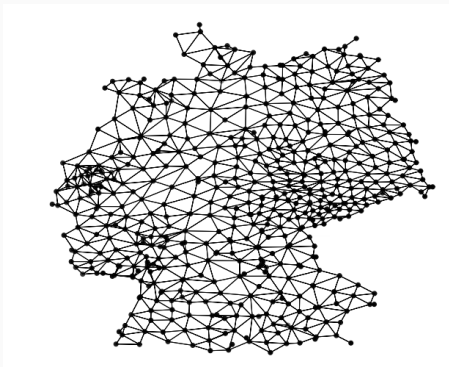


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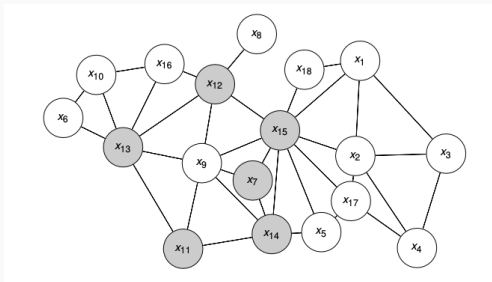


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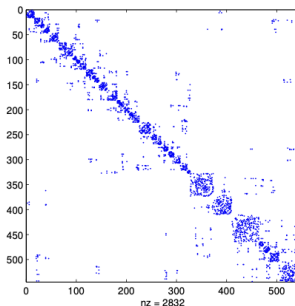
example: larynx cancer data from Germany – precision matrix of the sub graph

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Here:

$$y_i | \eta_i \sim \text{Poisson}(E_i \lambda_i)$$

$$\eta_i = \log(\lambda_i) = \beta_0 + \beta_1 pm(10) + \omega_i + z_i$$

The BYM model in inlabru

The Model

$$y_i | \eta_i \sim \text{Poisson}(E_i \lambda_i)$$

$$\log(\lambda_i) = \eta_i = \beta_0 + \beta_1 \text{pm10} + u_i + z_i$$

The code

```
1 # define model component
2 cmp = ~ Intercept(1) + pm10(pm10, model = "linear") +
3   space(space, model = "besag", graph = Q) +
4   iid(space, model = "iid")
5
6 # define model predictor
7 eta = observed ~ Intercept + space + iid
8
9 # build the observation model
10 lik = bru_obs(formula = formula,
11               family = "poisson",
12               E = expected,
13               data = resp_cases)
14
15 # fit the model
16 fit = bru(cmp, lik)
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neighbourhood structure

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1 library(spdep)
2 W.nb <- poly2nb(GGHB.IZ, queen = TRUE)
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Take-home message

- areal data are data on discretised space
- spatial relationships are represented through a neighbourhood structure
- GMRFs are computationally efficient representation
- these will also be relevant – and very useful – when it comes to approximating spatial structures in continuous space