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# Discontinuous Galerkin Finite Element Method for Wave Equations with Time-dependent Coefficients

Master Thesis

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### **Abstract**

The symmetric interior penalty discontinuous Galerkin finite element method is presented for a second order elliptical problem in 1d which yields a symmetric, positive definite bilinear form for a sufficiently positive penalty parameter. This guarantees existence of and uniqueness of the discrete problem. Implementation and theoretical details are discussed. Numerical results confirm the optimal convergence rate of  $\mathcal{O}(h^{r+1})$  in the  $L^2$ -norm for  $\mathcal{P}^r$ -elements.

## **0.1 Introduction**

The classical continuous Finite Element Method (FEM) is a widely used and very powerful

## **0.2 Overview**

# Chapter 1

## DG for Elliptic Problem

First we consider a time-independent elliptic problem. Not only is it useful for initiation to the subject to first consider a simpler elliptic problem, but it is also an essential preparational step in deriving the SIPG bilinear form for the elliptic part of the hyperbolic problem as well.

The goal of this chapter is to build all the necessary theoretical and practical tools to solve a given elliptic problem numerically and then experimentally test the method for different parameters. We will define the necessary notation and derive the SIPG variational formulation as well as in detail describe further implementation steps as for example what basis of the finite element space we chose and how to derive local matrices. Finally this chapter will also include some well established theoretical results in the context of discontinuous Galerkin methods. The derivation of the bilinear form is inspired by Chapter 1 in [6] as well as [2] and [3] for cross reference.

### 1.1 Problem

We consider the following elliptic model problem:

$$-(c(x)u'(x))' = f(x) \quad \forall x \in \Omega \quad (1.1)$$

$$u(0) = g_0, u(1) = g_1 \quad (1.2)$$

where  $\Omega = (0, 1)$  is the domain,  $g_0, g_1 \in \mathbb{R}$  are Dirichlet boundary conditions,  $f \in L^2(\Omega)$  and  $c : \Omega \rightarrow \mathbb{R}$  satisfies:

$$c_{\min} \leq c(x) \leq c_{\max} \quad \forall x \in \Omega$$

for  $0 < c_{\min} \leq c_{\max}$ . Multiplying the solution by a test function and integrating by parts over  $\Omega$  we get the standard weak formulation:

Find  $u \in \{v \in H^1(\Omega) \mid v(0) = g_0, v(1) = g_1\}$  such that:

$$a(u, v) = (f, v)_{L^2(\Omega)} \quad \forall v \in C_c^\infty(\Omega) \quad (1.3)$$

Where

$$a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}, \quad (u, v) \mapsto \int_{\Omega} c(x)u'(x)v'(x)dx$$

defines the standard elliptic bilinear form on  $H^1(\Omega)$  and

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} uv \, dx$$

denotes the  $L^2$ -inner product.

If we assume  $f \in L^2(\Omega)$  and  $c \in C^1(\Omega)$  we know by *Lax-Milgram* that there exists a unique (weak) solution of (1.3) and from the *elliptic regularity theory* we can conclude higher regularity of the weak solution, namely  $u \in H^2(\Omega)$ . For reference see chapter 6.2, 6.3 in [1].

## 1.2 Discretization

We will now discretize the domain  $\Omega$ , note that the notation will persist throughout this thesis. Let  $0 = x_0 < \dots < x_{N+1} = 1$  be the mesh faces,  $I_n = (x_n, x_{n+1})$  for  $n = 0, \dots, N$  be the elements and  $\mathcal{T}_h = \{I_n\}_{n=0}^N$  a partition of  $\Omega$  for some fixed  $N \in \mathbb{N}$ . We denote the element length by  $h_n = x_{n+1} - x_n$  for  $n = 0, \dots, N$  and the global meshsize by  $h = \max_n h_n$ . Next we define the discontinuous finite element space

$$V_h^r(\mathcal{T}_h) = \{v \in L^2(\Omega) \mid v|_{I_n} \in \mathcal{P}^r(I_n)\} \quad (1.4)$$

where  $\mathcal{P}^r(I_n)$  denotes the space of polynomials  $p : I_n \rightarrow \mathbb{R}$  of degree  $r$  for  $r \in \mathbb{N}$ . When the context allows it, we will denote the finite element space with just  $V_h$  for simplicity.  $V_h$  is our final approximation space in which the numerical solution lays. We observe that in contrast to a continuous finite element approximation space here the resulting solution is a priori discontinuous by construction. Furthermore we have here  $V_h \not\subset H^1(\Omega)$ . This is especially apparent in 1d due to the Sobolev embedding  $H^1(\Omega) \subset C^0(\Omega)$ . Any discontinuous element of  $V_h$  can therefore not be in  $H^1(\Omega)$ .

To proceed we will require the following trace operators:

**Definition 1.1.** Let  $v : \Omega \rightarrow \mathbb{R}$  be piecewise continuous and let  $n \in \{1, \dots, N\}$

(i) We denote  $v(x_n^+) := \lim_{x \searrow x_n} v(x)$ ,  $v(x_n^-) := \lim_{x \nearrow x_n} v(x)$  the limit from above/below.

(ii) We define the **jump** at  $x_n$  as

$$[v(x_n)] := v(x_n^-) - v(x_n^+)$$

and the **average** at  $x_n$  as

$$\{v(x_n)\} := \frac{v(x_n^+) + v(x_n^-)}{2}$$

furthermore by convention we set:

$$[v(x_0)] := -v(x_0^+), \quad [v(x_{N+1})] := v(x_{N+1}^-), \quad \{v(x_0)\} := v(x_0^+), \quad \{v(x_{N+1})\} := v(x_{N+1}^-)$$

## 1.3 Variational Formulation

To derive the SIPG variational formulation, let  $v \in V_h$  be a test function. As mentioned in section 1.1 we can assume that the coefficient  $c \in C^1(\Omega)$  and the exact solution  $u \in H^2(\Omega) \subset C^1(\Omega)$ . Due to the discontinuity of the test function in contrast to continuous FEM we multiply  $u$  with  $v$  on each element  $I_n$  and integrate by parts locally

$$\int_{x_n}^{x_{n+1}} f v \, dx = - \int_{x_n}^{x_{n+1}} (cu')' v \, dx = \int_{x_n}^{x_{n+1}} cu' v' \, dx - cu' v \Big|_{x_n}^{x_{n+1}} \quad \forall n = 0, \dots, N$$

then sum over all elements

$$(f, v)_{L^2(\Omega)} = \sum_{n=0}^N \int_{I_n} cu' v' \, dx - \sum_{n=0}^{N+1} [c(x_n)u'(x_n)v(x_n)] \quad (1.5)$$

where we have used that  $\sum_{n=0}^N w \Big|_{x_n}^{x_{n+1}} = w(x_{N+1}^-) - w(x_N^+) + w(x_N^-) - \dots - w(x_1^+) + w(x_1^-) - w(x_0^+) = \sum_{n=0}^{N+1} [w(x_n)]$  for any piece-wise continuous function  $w$ .

By our construction are  $c, u'$  continuous on  $\Omega$ , this means

$$[c(x_n)u'(x_n)v(x_n)] = c(x_n)u'(x_n)[v(x_n)] = \{c(x_n)u'(x_n)\}[v(x_n)] \quad \forall n = 0, \dots, N+1 \quad (1.6)$$

and

$$[u(x_n)] = 0 \quad \forall n = 1, \dots, N \quad (1.7)$$

To derive the final variational form we will now have to add two additional terms to (1.5):

**Step 1.** Firstly we need to symmetrize our currently non-symmetrical right hand side which will correspond to the SIPG bilinear form. To do so we add the symmetry term  $-\sum_{n=0}^{N+1} \{c(x_n)v'(x_n)\}[u(x_n)]$  on both sides of (1.5) so we get

$$\begin{aligned} & (f, v)_{L^2(\Omega)} - g_1 c(x_{N+1}^-) v'(x_{N+1}^-) + g_0 c(x_0^+) v'(x_0^+) \\ &= \sum_{n=0}^N \int_{I_n} cu'v' \, dx - \sum_{n=0}^{N+1} \{c(x_n)u'(x_n)\}[v(x_n)] + \{c(x_n)v'(x_n)\}[u(x_n)] \end{aligned}$$

where on the left hand side of the equation we have applied (1.7) for the interior node contributions of the sum (which therefore vanish), and the boundary condition (1.2) ensuring the left hand side to be solely dependent on  $v$ .

**Step 2.** The bilinear form we seek to create will (for now) be defined on  $V_h \times V_h$  meaning it will intake discontinuous functions. In particular the numerical solution will be a discontinuous function whereas the exact solution is continuous. To counterweigh this discrepancy we need to integrate a penalization mechanism, seeking to minimize discontinuous behaviors. Technically speaking this penalization term will guarantee coercivity of the bilinear form (see section 1.8).

Let  $\sigma > 0$  constant, we define:

$$\mathbf{c}_n := \begin{cases} \max(c(x_n^+), c(x_n^-)), & n = 1, \dots, N \\ c(x_n^+), & n = 0 \\ c(x_n^-), & n = N + 1 \end{cases}, \quad \mathbf{h}_n := \begin{cases} \min(h_n, h_{n-1}), & n = 1, \dots, N \\ h_n, & n \in \{0, N + 1\} \end{cases}$$

with this we define our penalization parameter

$$\mathbf{a}_n := \frac{\sigma \mathbf{c}_n}{\mathbf{h}_n} > 0 \quad \forall n = 0 \dots, N + 1 \quad (1.8)$$

Similarly to Step 1 we can now add the penalty term  $\sum_{n=0}^{N+1} \mathbf{a}_n [u(x_n)][v(x_n)]$  on both sides of (1.5) and get the final *discrete* SIPG variational formulation.

Find  $u_h \in V_h$  such that:

$$b_h(u_h, v) = \ell_h(v), \quad \forall v \in V_h \quad (1.9)$$

where

$$\begin{aligned} b_h(u, v) &= \sum_{n=0}^N \int_{I_n} cu'v' \, dx - \sum_{n=0}^{N+1} \{c(x_n)u'(x_n)\}[v(x_n)] + \{c(x_n)v'(x_n)\}[u(x_n)] + \sum_{n=0}^{N+1} \mathbf{a}_n [u(x_n)][v(x_n)] \\ \ell_h(v) &= (f, v)_{L^2(\Omega)} - g_1 c(x_{N+1}^-) v'(x_{N+1}^-) + g_0 c(x_0^+) v'(x_0^+) + \mathbf{a}_{N+1} g_1 v(x_{N+1}^-) + \mathbf{a}_0 g_0 v(x_0^+) \end{aligned}$$

for  $u, v \in V_h$ .

## 1.4 Boundary Conditions

By adding the terms  $-\sum_{n=0}^{N+1} \{c(x_n)v'(x_n)\}[u(x_n)]$ ,  $\sum_{n=0}^{N+1} \mathbf{a}_n [u(x_n)][v(x_n)]$  on both sides of (1.5) we *weakly* imposed the Dirichlet boundary conditions into the variational form. This stands in contrast to how boundary conditions are usually imposed in continuous FEM. Indeed one could also impose them strongly, meaning we could define

$$V_h^r(\mathcal{T}_h) = \{v \in L^2(\Omega) \mid v|_{I_n} \in \mathcal{P}^r(I_n), v(x_0) = g_0, v(x_{N+1}) = g_1\}$$

but this solely as a side note, we will continue to work with purely weakly imposed boundary conditions.

One could alternatively desire to implement *Neumann* boundary conditions, this slightly changes the variational formulation. We illustrate the idea on the following example boundary condition. A solution  $u$  should satisfy:

$$u(0) = g_0, u'(1) \cdot n_1 = g_1$$

where again  $g_0, g_1 \in \mathbb{R}$  are the boundary values and  $n_1$  denotes the outward normal of the domain at the upper boundary. In 1d we trivially have  $n_1 = 1, n_0 = -1$ , where  $n_0$  denotes the outward normal at the lower boundary.

Now recall the initial incomplete formulation (1.5). First we take the Neumann boundary contribution  $\{c(x_{N+1})u'(x_{N+1})\}[v(x_{N+1})]$  to the other side of the equation. We get

$$\sum_{n=0}^N \int_{I_n} cu'v' \, dx - \sum_{n=0}^N \{c(x_n)u'(x_n)\}[v(x_n)] = (f, v)_{L^2(\Omega)} + g_1 c(x_{N+1}^-)v'(x_{N+1}^-)$$

We have used  $\{c(x_{N+1})u'(x_{N+1})\}[v(x_{N+1})] = c(x_{N+1}^-)u'(x_{N+1}^-)v(x_{N+1}^-) \cdot n_1 = g_1 c(x_{N+1}^-)v(x_{N+1}^-)$

From here on we proceed similarly as in the Dirichlet case. The main difference is that we always omit the boundary face with the Neumann boundary condition.

We add the symmetry and penalty terms

$$- \sum_{n=0}^N \{c(x_n)v'(x_n)\}[u(x_n)] + \sum_{n=0}^N \mathbf{a}_n[u(x_n)][v(x_n)]$$

to both sides, using again that the real solution has zero jump on the interior faces and applying the boundary conditions we finally derive the variational form

$$\begin{aligned} & \sum_{n=0}^N \int_{I_n} cu'v' \, dx - \sum_{n=0}^N \{c(x_n)u'(x_n)\}[v(x_n)] + \{c(x_n)v'(x_n)\}[u(x_n)] + \sum_{n=0}^N \mathbf{a}_n[u(x_n)][v(x_n)] \\ &= (f, v)_{L^2(\Omega)} + g_0 c(x_0^+)v'(x_0^+) + \mathbf{a}_0 g_0 v(x_0^+) + g_1 c(x_{N+1}^-)v(x_{N+1}^-) \end{aligned}$$

## 1.5 Matrix-Vector System

We will now derive the fully discrete Matrix-Vector system given by the variational form (1.9). To do so let  $r \in \mathbb{N}$  denote the polynomial degree and consequently the element degree of freedom. Note that in this thesis we will only consider global polynomial degrees, meaning one set polynomial degree for all elements. Next let  $\{\Phi_0, \dots, \Phi_M\}$  be a basis of  $V_h$ , where  $M = \dim(V_h)$ . We can represent the sought Galerkin approximation as  $u_h = \sum_{j=0}^M \alpha_j \Phi_j \in V_h$  for coefficients  $\alpha_j \in \mathbb{R}$ . Then (1.9) is equivalent to:

$$\sum_{j=0}^M \alpha_j b_h(\Phi_j, \Phi_i) = \ell_h(\Phi_i) \quad \forall i = 0, \dots, M$$

which corresponds to the system:

$$\mathbf{B}\mathbf{u} = \mathbf{l} \tag{1.10}$$

for  $\mathbf{B} \in \mathbb{R}^{M \times M}$ ,  $[\mathbf{B}]_{i,j} = b_h(\Phi_j, \Phi_i)$ ,  $\mathbf{u} \in \mathbb{R}^M$ ,  $[\mathbf{u}]_j = \alpha_j$ ,  $\mathbf{l} \in \mathbb{R}^M$ ,  $[\mathbf{l}]_j = \ell_h(\Phi_j)$ .

## 1.6 Basis of Finite Element Space

There are many ways of choosing basis functions for finite element spaces. In this thesis we solely focus on elementwise nodal Lagrangian basis functions, although there are alternatives which are just as valid, like the modal Legendre basis. When choosing basis nodes for a lagrangian basis we have a couple of choices to make.

1. *Should the quadrature nodes coincide with the basis nodes?*



Having the quadrature nodes coincide with the basis nodes simplifies the calculations, because by the property of a Lagrangian basis the values of the basis functions at the quadrature nodes, i.e the basis nodes coincide with the Kroneker delta. So a value matrix composed of each value at the quadrature nodes for each basis function (per element) is just the identity matrix. Specifically we have:

Let  $\xi_0, \dots, \xi_r$  be the basis- and quadrature nodes and let  $\hat{\phi}_0, \dots, \hat{\phi}_r$  be the Lagrangian basis, then

$$\hat{\phi}_i(\xi_j) = \delta_{i,j} := \begin{cases} 1 & , \text{ for } i = j \\ 0 & , \text{ for } i \neq j \end{cases}$$

therefore  $\mathbf{I} = \mathbf{L}$  where  $\mathbf{I}$  is the identity and  $[\mathbf{L}]_{i,j} = [\hat{\phi}_i(\xi_j)]$  the basis shape-function value-matrix. This reduces the computational cost, since there is no need to actually evaluate/interpolate the polynomials at the quadrature nodes. The clear downside is that we couple the exactness of the quadrature to the polynomial degree of the basis function. If for example we fix a polynomial degree  $r \in \mathbb{N}$ . The Gauss-Lobatto quadrature with  $r + 1$  nodes is exact for polynomials of degree  $2r - 1$ . When computing the mass matrix we have to calculate an integral of the form  $\int_I \Phi_j \Phi_i dx$  where  $I \in \mathcal{T}_h$  is a element and  $\Phi_i|_I, \Phi_j|_I \in \mathcal{P}^r(I)$  are basis functions. This means  $\Phi_j \cdot \Phi_i \in \mathcal{P}^{2r}$  and therefore we introduce an error.

## 2. Should there be basis nodes on the element boundaries?

In contrast to continuous FEM we do not need to impose continuity over the element boundaries, so we have the option of allowing the nodes to be exclusively in the interior of the elements. This allows us to use Gauss-Legendre quadrature nodes in 1d for example, which is exact for polynomials of degree  $2r + 1$  for  $r + 1$  quadrature nodes. This would remedy the issue of approximating the mass matrix inexactly as described before when choosing the same quadrature nodes as basis nodes. The downside here is that we still require boundary values to incorporate boundary conditions and calculating the penalty, consistency and symmetry terms (see matrices  $\mathbf{B}_{\text{cons}}$ ,  $\mathbf{B}_{\text{penal}}$  in section 1.7), so the boundary values have to be interpolated.

In this thesis we will not try to justify the choice of basis function too much and use Gauss-Lobatto quadrature nodes as basis nodes. This is a commonly used nodal basis in continuous 1d-FEM and we will also use it for our DG method. In Appendix A.2.3 of [4] the *Fekete points*, which in 1d coincide with the Gauss-Lobatto points, are recommended with the argument that this nodal basis yields a mass matrix with an optimal condition number. For more detailed information on choosing basis functions and alternative approaches see for example Appendix A.2 in [4].

Let  $n \in \{1, \dots, N\}$  and  $I_n \in \mathcal{T}_h$  be an arbitrary element. We denote  $\hat{I} = (-1, 1)$  the *reference element* and  $F_n : \hat{I} \rightarrow I_n, \xi \mapsto \frac{x_n + x_{n+1}}{2} + \frac{h_n}{2}\xi$  the *element map*. This now allows us to define a basis on the reference element and extend it to all elements using the element map.

For a fixed polynomial degree  $r \geq 2$  let  $\xi_0, \dots, \xi_r \in [-1, 1]$  be the Gauss-Lobatto nodes.

$$\begin{array}{c|c} r = 2 & \{-1, 1\} \\ r = 3 & \{-1, 0, 1\} \\ r = 4 & \{-1, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1\} \\ \vdots & \vdots \end{array}$$

The inner nodes are given by the roots of  $L'_{r-1}$ , the derivative of the  $r - 1$ -th Legendre polynomial. We define the basis on the reference element as the Lagrangian nodal basis

$$\hat{\phi}_i(\xi) := \prod_{\substack{j=0 \\ j \neq i}}^r \frac{\xi - \xi_j}{\xi_i - \xi_j}, \quad \text{for } i = 0, \dots, r \quad (1.11)$$

and define the basis functions on the element  $I_n$  as

$$\phi_i^n : I_n \rightarrow \mathbb{R}, \quad \phi_i^n(x) := \hat{\phi}_i(F_n^{-1}(x))$$

as a last step we extend the basis functions to the whole domain  $\Omega$  by zero

$$\Phi_i^n : \Omega \rightarrow \mathbb{R}, \quad \Phi_i^n(x) := \begin{cases} \phi_i^n(x), & \text{for } x \in I_n \\ 0, & \text{else} \end{cases} \quad (1.12)$$

for  $n = 0, \dots, N$  and  $i = 0, \dots, r$ . Clearly we have  $\text{span}(\hat{\phi}_0, \dots, \hat{\phi}_r) = \mathcal{P}^r(\hat{I})$  and by extension  $\text{span}(\Phi_0^0, \dots, \Phi_r^N) = V_h^r(\mathcal{T}_h)$ . It is essential that our basis has only local support, meaning the basis functions are zero on most of the domain. This is the key property which allows the final matrices to be sparse. Choosing basis functions with global support, the computational cost would be unfeasible for small mesh sizes.

By having chosen a Lagrangian nodal basis the mesh nodes exactly coincide with the Gauss-Lobatto nodes on each element. To simplify the notation we introduce a *local-to-global* index map

$$T : \{0, \dots, N\} \times \{0, \dots, r\} \rightarrow \{1, \dots, M\} \quad (1.13)$$

where  $M = (r+1)(N+1) = \dim(V_h)$ .  $T$  takes an element index  $n$  and a local basis function index  $i$  as inputs and returns the globally assigned node index  $T(n, i)$ .  $T$  corresponds to the *connectivity matrix*. In the simplest case we have the global index ordered from left to right and get  $T(n, i) = nr + i$ .

## 1.7 Stiffness Matrix Assembly

With the basis functions in (1.12) defined we can now in detail investigate how to assemble the matrix  $\mathbf{B}$  in (1.10). To do so we firstly separate the bilinear form  $b_h$  into different components

$$\begin{aligned} a_h(u, v) &:= \sum_{n=0}^N \int_{I_n} cu'v' \, dx \\ b_h^{\text{cons}}(u, v) &:= \sum_{n=0}^{N+1} \{c(x_n)u'(x_n)\}[v(x_n)] + \{c(x_n)v'(x_n)\}[u(x_n)] \\ b_h^{\text{penal}}(u, v) &:= \sum_{n=0}^{N+1} \mathbf{a}_n[u(x_n)][v(x_n)] \end{aligned}$$

Let  $u_h = \sum_{m=0}^N \sum_{j=0}^r \alpha_j^m \Phi_j^m \in V_h$  denote the Galerkin approximation, then as discussed in section 1.6 the discrete variational formulation (1.9) is equivalent to

$$\sum_{m=0}^N \sum_{j=0}^r \alpha_j^m \left( a_h(\Phi_j^m, \Phi_i^n) - b_h^{\text{cons}}(\Phi_j^m, \Phi_i^n) + b_h^{\text{penal}}(\Phi_j^m, \Phi_i^n) \right) = \ell_h(\Phi_i^n), \quad \forall n = 0, \dots, N, i = 0, \dots, r \quad (1.14)$$

which corresponds to the matrix vector system (1.10) where we can write

$$\mathbf{B} = \mathbf{A} - \mathbf{B}_{\text{cons}} + \mathbf{B}_{\text{penal}}$$

we will assemble the three (symmetric) matrices separately.

$$\begin{aligned} [\mathbf{B}]_{T(n,i), T(m,j)} &= b_h(\Phi_j^m, \Phi_i^n), & [\mathbf{A}]_{T(n,i), T(m,j)} &= a_h(\Phi_j^m, \Phi_i^n) \\ [\mathbf{B}_{\text{cons}}]_{T(n,i), T(m,j)} &= b_h^{\text{cons}}(\Phi_j^m, \Phi_i^n) & [\mathbf{B}_{\text{penal}}]_{T(n,i), T(m,j)} &= b_h^{\text{penal}}(\Phi_j^m, \Phi_i^n) \end{aligned}$$

where  $T$  is given by (1.13)

### 1.7.1 Assembly of $\mathbf{A}$

$\mathbf{A}$  is assembled similarly to the standard stiffness matrix in continuous finite element. The main difference is that there is no overlap in the elementwise contributions. Each set of local (element) basis functions only contributes to the integrals over said element. We can rewrite  $\mathbf{A} = \sum_{s=0}^N \mathbf{A}^{(s)}$ , where  $[\mathbf{A}^{(s)}]_{T(n,i),T(m,j)} = \int_{I_s} c(\Phi_j^m)'(\Phi_i^n)' dx$ . Now since we have  $\text{supp}(\Phi_i^n) \subset I_n$  the only non-zero entries of  $\mathbf{A}^{(s)}$  are the ones where both  $n = m = s$ . Pulling back the integral to the reference element using the chain rule and the substitution  $F_s^{-1}(x) = \xi$  we find

$$\int_{I_s} c(x) (\Phi_j^s)'(x) (\Phi_i^s)'(x) dx = \frac{2}{h_s} \int_{\hat{I}} c(F_s(\xi)) \hat{\phi}_j'(\xi) \hat{\phi}_i'(\xi) d\xi$$

This integral now only depends on the reference shape functions, the element length  $h_s$  and the values of the coefficient  $c$ . Using the Gauss-Lobatto quadrature rule we can approximate the integral only requiring the values of  $c$  at the nodes, whilst having the values of  $\phi, \phi'$  preloaded. The total assembly of  $\mathbf{A}$  can therefore be achieved by calculating a local contribution matrix  $\hat{\mathbf{A}}^{(s)} \in \mathbb{R}^{(r+1) \times (r+1)}$  for each element  $I_s$  and adding it into  $\mathbf{A}$ .

**Example 1.2.** For  $c \equiv 1$  with  $\mathcal{P}^1$ -elements ( $r = 1$ ) we have

$$\hat{\mathbf{A}}^{(s)} = \frac{1}{h_s} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

### 1.7.2 Assembly of $\mathbf{B}$ consistency part

As before we rewrite  $\mathbf{B}_{\text{cons}} = \sum_{s=0}^{N+1} \mathbf{B}_{\text{cons}}^{(s)}$ , where

$$[\mathbf{B}_{\text{cons}}^{(s)}]_{T(n,i),T(m,j)} = \{c(x_s) \Phi_j^{m'}(x_s)\} [\Phi_i^n(x_s)] + \{c(x_s) \Phi_i^{n'}(x_s)\} [\Phi_j^m(x_s)]$$

#### Interior Faces

First let  $s \in \{1, \dots, N\}$  denote an interior face, we observe again, that the entries of  $\mathbf{B}_{\text{cons}}^{(s)}$  can only be non-zero for an index tuple  $(T(n,i), T(m,j))$  if  $n, m \in \{s, s-1\}$  by the local support of the basis functions. Therefore assembling  $\mathbf{B}_{\text{cons}}$  again comes down to calculating a local contribution matrix

$$\hat{\mathbf{B}}_{\text{cons}}^{(s)} = \begin{bmatrix} \mathbf{C}_{\text{cons}}^{(s-1,s-1)} & \mathbf{C}_{\text{cons}}^{(s-1,s)} \\ \mathbf{C}_{\text{cons}}^{(s,s-1)} & \mathbf{C}_{\text{cons}}^{(s,s)} \end{bmatrix} \in \mathbb{R}^{2(r+1) \times 2(r+1)}$$

for each interior face  $x_s$  consisting of four blocks we will now lay out in more detail. We discuss the boundary case separately. Using again the local support of the basis functions we find

$$\begin{aligned} [\mathbf{C}_{\text{cons}}^{(s-1,s-1)}]_{i,j} &= \frac{c(x_s^-)}{2} \Phi_j^{s-1'}(x_s^-) \Phi_i^{s-1}(x_s^-) + \frac{c(x_s^-)}{2} \Phi_i^{s-1'}(x_s^-) \Phi_j^{s-1}(x_s^-) \\ [\mathbf{C}_{\text{cons}}^{(s-1,s)}]_{i,j} &= \frac{c(x_s^+)}{2} \Phi_j^{s'}(x_s^+) \Phi_i^{s-1}(x_s^-) - \frac{c(x_s^-)}{2} \Phi_i^{s-1'}(x_s^-) \Phi_j^s(x_s^+) \\ [\mathbf{C}_{\text{cons}}^{(s,s-1)}]_{i,j} &= -\frac{c(x_s^+)}{2} \Phi_j^{s'}(x_s^+) \Phi_i^s(x_s^+) - \frac{c(x_s^+)}{2} \Phi_i^{s'}(x_s^+) \Phi_j^s(x_s^+) \end{aligned}$$

where we have used the definitions of jump and average in (1.1). Note that  $\mathbf{C}_{\text{cons}}^{(s-1,s)} = (\mathbf{C}_{\text{cons}}^{(s,s-1)})^T$  by the symmetry of the bilinear form  $b_h^{\text{cons}}$ . Next we represent the values of the basis functions  $\Phi$  at the element boundary by the values of the reference shape functions  $\hat{\phi}$

$$\begin{aligned} \Phi_i^s(x_s^+) &= \hat{\phi}_i(-1), & \Phi_i^{s-1}(x_s^-) &= \hat{\phi}_i(1) \\ \Phi_i^{s'}(x_s^+) &= \frac{2}{h_s} \hat{\phi}_i'(-1), & \Phi_i^{s-1'}(x_s^-) &= \frac{2}{h_{s-1}} \hat{\phi}_i'(1) \end{aligned} \tag{1.15}$$

which finally yields

$$\begin{aligned} [\mathbf{C}_{\text{cons}}^{(s-1,s-1)}]_{i,j} &= \frac{c(x_s^-)}{h_{s-1}} \hat{\phi}'_j(1) \hat{\phi}_i(1) + \frac{c(x_s^-)}{h_{s-1}} \hat{\phi}'_i(1) \hat{\phi}_j(1) \\ [\mathbf{C}_{\text{cons}}^{(s-1,s)}]_{i,j} &= \frac{c(x_s^+)}{h_s} \hat{\phi}'_j(-1) \hat{\phi}_i(1) - \frac{c(x_s^-)}{h_{s-1}} \hat{\phi}'_i(1) \hat{\phi}_j(-1) \\ [\mathbf{C}_{\text{cons}}^{(s,s)}]_{i,j} &= -\frac{c(x_s^+)}{h_s} \hat{\phi}'_j(-1) \hat{\phi}_i(-1) - \frac{c(x_s^+)}{h_s} \hat{\phi}'_i(-1) \hat{\phi}_j(-1) \end{aligned}$$

**Example 1.3.** Consider  $c \equiv 1$  for  $\mathcal{P}^1$ -elements ( $r = 1$ ) with an equidistant mesh with meshsize  $h$  we have

$$\hat{\mathbf{B}}_{\text{cons}}^{(s)} = \frac{1}{h} \begin{bmatrix} 0 & -1/2 & 1/2 & 0 \\ -1/2 & 1 & -1 & 1/2 \\ 1/2 & -1 & 1 & -1/2 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix}$$

### Boundary Faces

For  $s \in \{0, N+1\}$  and  $x_s$  a boundary face we now have a smaller local contribution matrix since the contribution can only come from the one element to which  $x_s$  belongs. Meaning we have  $\hat{\mathbf{B}}_{\text{cons}}^{(0)}, \hat{\mathbf{B}}_{\text{cons}}^{(N+1)} \in \mathbb{R}^{(r+1) \times (r+1)}$  with

$$\begin{aligned} [\hat{\mathbf{B}}_{\text{cons}}^{(0)}]_{i,j} &= -\frac{2c(x_0^+)}{h_0} \hat{\phi}'_j(-1) \hat{\phi}_i(-1) - \frac{2c(x_0^+)}{h_0} \hat{\phi}'_i(-1) \hat{\phi}_j(-1) \\ [\hat{\mathbf{B}}_{\text{cons}}^{(N+1)}]_{i,j} &= \frac{2c(x_{N+1}^-)}{h_{N+1}} \hat{\phi}'_j(1) \hat{\phi}_i(1) + \frac{2c(x_{N+1}^-)}{h_{N+1}} \hat{\phi}'_i(1) \hat{\phi}_j(1) \end{aligned}$$

**Example 1.4.** For  $c \equiv 1$  with  $\mathcal{P}^1$ -elements ( $r = 1$ ) we have

$$\hat{\mathbf{B}}_{\text{cons}}^{(0)} = \frac{1}{h_0} \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \quad \hat{\mathbf{B}}_{\text{cons}}^{(N+1)} = \frac{1}{h_{N+1}} \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

### 1.7.3 Assembly of B penalty part

Again we rewrite  $\mathbf{B}_{\text{penal}} = \sum_{s=0}^{N+1} \mathbf{B}_{\text{penal}}^{(s)}$ , where

$$[\mathbf{B}_{\text{penal}}^{(s)}]_{T(n,i), T(m,j)} = \mathbf{a}_s [\Phi_j^m(x_s)] [\Phi_i^n(x_s)]$$

We proceed analogously to 1.7.2.

### Interior Faces

Let  $s \in \{1, \dots, N\}$  and  $x_s$  denote an interior face. As before we have that the entries of  $\mathbf{B}_{\text{penal}}^{(s)}$  can only be nonzero for an index tuple  $(T(n,i), T(m,j))$  if  $n, m \in \{s, s-1\}$ , similar to 1.7.2 we find that the assembly boils down to adding up local contributions represented in a local contribution matrix

$$\hat{\mathbf{B}}_{\text{penal}}^{(s)} = \begin{bmatrix} \mathbf{C}_{\text{penal}}^{(s-1,s-1)} & \mathbf{C}_{\text{penal}}^{(s-1,s)} \\ \mathbf{C}_{\text{penal}}^{(s,s-1)} & \mathbf{C}_{\text{penal}}^{(s,s)} \end{bmatrix} \in \mathbb{R}^{2(r+1) \times 2(r+1)}$$

and using (1.15), and the definition of the penalization parameter (1.8) we specifically find

$$\begin{aligned} [\mathbf{C}_{\text{penal}}^{(s-1,s-1)}]_{i,j} &= \mathbf{a}_s \hat{\phi}_j(1) \hat{\phi}_i(1) \\ [\mathbf{C}_{\text{penal}}^{(s-1,s)}]_{i,j} &= -\mathbf{a}_s \hat{\phi}_j(-1) \hat{\phi}_i(1) \\ [\mathbf{C}_{\text{penal}}^{(s,s)}]_{i,j} &= \mathbf{a}_s \hat{\phi}_j(-1) \hat{\phi}_i(-1) \end{aligned}$$

where again by symmetry of the penalty term we have  $\mathbf{C}_{\text{penal}}^{(s-1,s)} = (\mathbf{C}_{\text{penal}}^{(s,s-1)})^T$  and  $\mathbf{a}_s$  only depends on the two adjacent elements  $I_{s-1}, I_s$ .

**Example 1.5.** Consider  $c \equiv 1$  for  $\mathcal{P}^1$ -elements ( $r = 1$ ) with an equidistant mesh with meshsize  $h$  we have

$$\widehat{\mathbf{B}}_{\text{penal}}^{(s)} = \frac{\sigma}{h} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

### Boundary Faces

For  $s \in \{0, N+1\}$  we again have only the respective boundary element contributing. So the local contribution matrices  $\widehat{\mathbf{B}}_{\text{penal}}^{(s)} \in \mathbb{R}^{(r+1) \times (r+1)}$  satisfy

$$[\widehat{\mathbf{B}}_{\text{penal}}^{(0)}]_{i,j} = \mathbf{a}_0 \widehat{\phi}_j(-1) \widehat{\phi}_i(-1), \quad [\widehat{\mathbf{B}}_{\text{penal}}^{(N+1)}]_{i,j} = \mathbf{a}_{N+1} \widehat{\phi}_j(1) \widehat{\phi}_i(1)$$

**Example 1.6.** For  $c \equiv 1$  with  $\mathcal{P}^1$ -elements ( $r = 1$ ) we have

$$\widehat{\mathbf{B}}_{\text{penal}}^{(0)} = \frac{\sigma}{h_0} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{\mathbf{B}}_{\text{penal}}^{(N+1)} = \frac{\sigma}{h_{N+1}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

### 1.7.4 System Vector Assembly

We divide assembling the vector  $\mathbf{l}$  in (1.10) into two parts.

$$\mathbf{l} = \mathbf{l}_{\text{load}} + \mathbf{l}_{\text{bc}}$$

First we recall the assembly of the load vector  $\mathbf{l}_{\text{load}}$ , i.e. the vector containing the contributions of the forcing term  $f$  and secondly we will describe how to add the Dirichlet boundary condition contributions ( $\mathbf{l}_{\text{bc}}$ ).

#### Load Vector

The assembly of the load vector is completely analogous to the continuous finite element case. Using the local support of  $\Phi_i^n$  we can rewrite

$$\int_{\Omega} f \Phi_i^n dx = \sum_{s=0}^N \int_{I_s} f \Phi_i^n dx = \int_{I_n} f(x) \Phi_i^n(x) dx = \frac{h_n}{2} \int_{-1}^1 f(F_n(\xi)) \widehat{\phi}_i(\xi) d\xi$$

meaning as before we can assemble  $\mathbf{l}_{\text{load}} = \sum_{s=0}^N \mathbf{l}_{\text{load}}^{(s)}$  where

$$[\mathbf{l}_{\text{load}}^{(s)}]_{T(n,i)} = \int_{I_s} f \Phi_i^n dx = \delta_{n,s} \frac{h_n}{2} \int_{-1}^1 f(F_n(\xi)) \widehat{\phi}_i(\xi) d\xi$$

which can be characterized by the local contribution vector  $\widehat{\mathbf{l}}_{\text{load}}^{(s)} \in \mathbb{R}^{r+1}$  defined as

$$[\widehat{\mathbf{l}}_{\text{load}}^{(s)}]_i = \frac{h_s}{2} \int_{-1}^1 f(F_s(\xi)) \widehat{\phi}_i(\xi) d\xi$$

In practice we approximate the integral using the Gauss-Lobatto quadrature rule.

**Example 1.7.** For  $f$  piecewise constant (i.e.  $f|_{I_s} \equiv f_s \in \mathbb{R} \quad \forall s = 0, \dots, N$ ) with  $\mathcal{P}^1$ -elements ( $r = 1$ ) we have

$$\widehat{\mathbf{l}}_{\text{load}}^{(s)} = \frac{f_s h_s}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

## Dirichlet Boundary Condition Vector

We have

$$[\mathbf{l}_{bc}]_{T(n,i)} = -g_1 c(x_{N+1}^-) \Phi_i^{n'}(x_{N+1}^-) + g_0 c(x_0^+) \Phi_i^{n'}(x_0^+) + \mathbf{a}_{N+1} g_1 \Phi_i^n(x_{N+1}^-) + \mathbf{a}_0 g_0 \Phi_i^n(x_0^+)$$

where the entries can clearly only be non-zero at indices corresponding to boundary elements. We characterize the assembly using the local contribution vectors  $\hat{\mathbf{l}}_{bc}^{(N+1)}, \hat{\mathbf{l}}_{bc}^{(0)} \in \mathbb{R}^{r+1}$  where

$$[\hat{\mathbf{l}}_{bc}^{(0)}]_i = \frac{g_0}{h_0} c(x_0^-) \hat{\phi}_i(-1) + \mathbf{a}_0 g_0 \hat{\phi}_i(-1), \quad [\hat{\mathbf{l}}_{bc}^{(N+1)}]_i = -\frac{g_1}{h_N} c(x_{N+1}^-) \hat{\phi}_i(1) + \mathbf{a}_{N+1} g_1 \hat{\phi}_i(1)$$

## 1.8 Existence of Discrete Solution

Firstly we will recall some basic definitions:

**Definition 1.8.** Let  $V$  be a normed vector space and  $b : V \times V \rightarrow \mathbb{R}$  be a bilinear form.

(i) We say  $b$  is **continuous** if  $\exists C_{cont} > 0$ , such that

$$|b(u, v)| \leq C_{cont} \|u\| \|v\| \quad \forall u, v \in V$$

(ii) We say  $b$  is **symmetric** if

$$b(u, v) = b(v, u) \quad \forall u, v \in V$$

(iii) We say  $b$  is **coercive** if  $\exists C_{coer} > 0$ , such that

$$b(u, u) \geq C_{coer} \|u\|^2 \quad \forall u \in V$$

Since (1.9) corresponds to the finite dimensional system (1.10) uniqueness and existence of a solution are equivalent. The bilinear form  $b_h$  is *symmetric* by construction the goal of this section is to show that  $b_h$  is also *coercive*. From the coercivity of  $b_h$  it will follow that the matrix  $\mathbf{B}$  in (1.10) is positive definite and hence invertible, which means there exists a (unique) solution of (1.9).

**Lemma 1.9.** Let  $V = \text{span}(\varphi_1, \dots, \varphi_M)$  be a finite dimensional normed vector space with  $\dim(V) = M \in \mathbb{N}$  and let  $b : V \times V \rightarrow \mathbb{R}$  be a symmetric, coercive bilinear form, then the matrix  $[\mathbf{B}]_{i,j} = [b(\varphi_j, \varphi_i)]_{i,j} \in \mathbb{R}^{N \times N}$  is symmetric positive definite.

*Proof.* Clearly  $\mathbf{B}$  is symmetric.

Let  $\mathbf{v} = (v_1, \dots, v_M) \in \mathbb{R}^M$  then  $v = \sum_{i=1}^M v_i \varphi_i \in V$  and we have:

$$\mathbf{v}^T \mathbf{B} \mathbf{v} = \sum_{i,j=1}^M v_i v_j b(\varphi_j, \varphi_i) = b(v, v) \geq C_{coer} \|v\|^2$$

where we have used the bilinearity and the coercivity of  $b$ . □

Next we will require a usefull tool often used in FEM proofs to bound a boundary integral with the integral over the interior domain. These kind of inequalities are in the literature often called *inverse (trace) inequalities* and are in essence trace inequalities on finite dimensional subspaces. We will here rely on a result and proof as presented in [7].

**Lemma 1.10** (Inverse inequality). Let  $r \in \mathbb{N}$  be the polynomial degree,  $a, b \in \mathbb{R}$  with  $a < b$  and let  $\mathcal{P}^r([a, b])$  denote the space of polynomials of degree  $r$  defined on  $[a, b]$ .

For any  $v \in \mathcal{P}^r([a, b])$  we have:

1.  $|v(a)|^2 \leq \frac{(r+1)^2}{|b-a|} \|v\|_{L^2([a,b])}^2$
2.  $|v(b)|^2 \leq \frac{(r+1)^2}{|b-a|} \|v\|_{L^2([a,b])}^2$

*Proof.* We will prove the statements first for the reference element  $\hat{I} = [-1, 1]$  and then use a scaling argument to show the general case by applying a simple substitution.

**Step 1 (Setup).**

We will make use of the Legendre orthonormal basis of  $\mathcal{P}^r(\hat{I})$ : Let  $P_0, \dots, P_r$  denote the Legendre polynomials on  $\mathcal{P}^r(\hat{I})$ . Recall the following well known facts (see for example [5]):

1.  $\{P_0, \dots, P_r\}$  form an orthogonal basis of  $\mathcal{P}^r(\hat{I})$  under the  $L^2(\hat{I})$  inner product. Meaning:

$$\text{span}(P_0, \dots, P_r) = \mathcal{P}^r(\hat{I}), \quad \int_{-1}^1 P_i P_j d\xi = \begin{cases} \frac{2}{2i+1}, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

2.  $P_i(1) = 1, P_i(-1) = (-1)^i, \quad \forall i = 0, \dots, r$

Let  $\psi_i = \sqrt{\frac{2i+1}{2}} P_i$  for  $i = 0, \dots, r$  denote the normed basis function. Clearly we now have

$$\psi_i(-1) = (-1)^i \sqrt{\frac{2i+1}{2}}, \quad \psi_i(1) = \sqrt{\frac{2i+1}{2}}, \quad \int_{-1}^1 \psi_i \psi_j d\xi = \delta_{i,j}, \quad \forall i = 0, \dots, r$$

where  $\delta_{i,j} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$ , and hence  $\{\psi_0, \dots, \psi_r\}$  form an orthonormal basis.

**Step 2 (Proof on reference element).**

For any  $v \in \mathcal{P}^r(\hat{I})$  there exist coefficients  $v_0, \dots, v_r \in \mathbb{R}$ , such that  $v = \sum_{i=0}^r v_i \psi_i$ . By applying Cauchy-Schwarz we find

$$|v(-1)|^2 = \left| \sum_{i=0}^r v_i \psi_i(-1) \right|^2 \leq \left( \sum_{i=0}^r v_i^2 \right) \left( \sum_{i=0}^r \psi_i(-1)^2 \right) = \left( \sum_{i=0}^r v_i^2 \right) \left( \sum_{i=0}^r \frac{2i+1}{2} \right) = \left( \sum_{i=0}^r v_i^2 \right) \frac{(r+1)^2}{2}$$

and finally the orthonormality of the  $\psi_i$  yields

$$\frac{(r+1)^2}{2} \sum_{i=0}^r v_i^2 = \frac{(r+1)^2}{2} \sum_{i,j=0}^r v_i v_j \delta_{i,j} = \frac{(r+1)^2}{2} \|v\|_{L^2(\hat{I})}^2$$

This yields the first inequality for the reference element. The second inequality can be proven analogously.

**Step 3 (Scaling argument).**

Now we assume that  $v \in \mathcal{P}^r([a, b])$ . Using the affine (element) map

$$F : [-1, 1] \rightarrow [a, b], \xi \mapsto \frac{a+b}{2} + \frac{|b-a|}{2} \xi$$

we can pull  $v$  back to the reference element by defining  $\hat{v}(\xi) := v(F(\xi))$  for all  $\xi \in \hat{I}$ . Clearly  $\hat{v} \in \mathcal{P}^r(\hat{I})$  hence, by Step 2 we obtain

$$|v(a)|^2 = |\hat{v}(F^{-1}(a))|^2 = |\hat{v}(-1)|^2 \leq \frac{(r+1)^2}{2} \int_{-1}^1 \hat{v}(\xi)^2 d\xi = \frac{(r+1)^2}{2} \frac{2}{|b-a|} \|v\|_{L^2([a,b])}^2$$

where in the last equality we have applied a change of variable  $x = F(\xi)$  to the integral. Applying the same line of reasoning to  $|v(b)|^2$  proves both inequalities and so we are done.  $\square$

Recall the in previous sections established notations, let  $r \in \mathbb{N}$  denote the polynomial degree and  $V_h^r(\mathcal{T}_h)$  be the discrete subspace.

**Definition 1.11.** We define the *energy norm* on  $V_h$  by

$$\|v\|_h^2 := \sum_{n=0}^N \int_{I_n} c(x) v'(x)^2 dx + \sum_{n=0}^{N+1} \mathbf{a}_n [v(x_n)]^2 \quad (1.16)$$

where  $\mathbf{a}$  denotes the penalization term in (1.8).

**Lemma 1.12.**  $\|\cdot\|_h$  defines a norm on  $V_h$ .

*Proof.* Clearly we have  $\|\lambda v\|_h = |\lambda| \|v\|_h$  for all  $\lambda \in \mathbb{R}, v \in V_h$ .

By definition we have  $\mathbf{a}, c > 0$  and by extension  $\|v\|_h \geq 0$  for all  $v \in V_h$ . Suppose now that  $\|v\|_h = 0$  for some  $v \in V_h$ , then we must have  $v|_{I_n} \equiv \text{const}$  and  $[v(x_n)] = 0$  for all  $n$ . So  $v$  must be constant on all elements and have a jump of zero at the element boundaries. These two facts combined imply that  $v$  is constant on all of  $\Omega$ . By the definition of the jump at the boundary nodes of  $\Omega$  it immediately follows that  $v = 0$ . Clearly  $\|0\|_h = 0$ , therefore  $\|\cdot\|_h$  is positive definite.

Using  $[v(x_n) + w(x_n)] = [v(x_n)] + [w(x_n)] \quad \forall v, w \in V_h, n = 0, \dots, N+1$  we find

$$\begin{aligned} \|v + w\|_h &\leq \left( \sum_{n=0}^N (\|\sqrt{c}v'\|_{L^2(I_n)} + \|\sqrt{c}w'\|_{L^2(I_n)})^2 + \sum_{n=0}^{N+1} (\sqrt{\mathbf{a}_n}([v(x_n)] + [w(x_n)]))^2 \right)^{1/2} \\ &\leq \|v\|_h + \|w\|_h \end{aligned}$$

where in the last inequality we have used the triangle inequality of the euclidian vector norm on  $\mathbb{R}^{2N+3}$ , with the vector given as

$$\mathbf{v} = [\|\sqrt{c}v'\|_{L^2(I_0)}, \dots, \|\sqrt{c}v'\|_{L^2(I_N)}, \sqrt{\mathbf{a}_0}[v(x_0)], \dots, \sqrt{\mathbf{a}_{N+1}}[v(x_{N+1})]]^T$$

this shows the triangle inequality for  $\|\cdot\|_h$  and hence it is a norm.  $\square$

**Theorem 1.13.** Let  $r \in \mathbb{N}$ , the bilinear form  $b_h$  in (1.9) is continuous on  $V_h^r(\mathcal{T}_h)$  and if furthermore  $\sigma \geq \frac{6(r+1)^2 c_{\max}}{c_{\min}}$ ,  $b_h$  is also coercive on  $V_h^r(\mathcal{T}_h)$ . The coercivity and continuity constants are given by

$$C_{\text{coer}} = \frac{1}{2}, \quad C_{\text{cont}} = (3 + \frac{5}{4}C_\sigma)$$

where  $C_\sigma = \frac{(r+1)^2 c_{\max}}{\sigma c_{\min}}$

*Proof. Step 1 (Coercivity).*

Let  $w \in V_h$ . Note that

$$b_h(w, w) = \|w\|_h^2 - 2 \sum_{n=0}^{N+1} \{c(x_n)w'(x_n)\}[w(x_n)] \quad (1.17)$$



To derive the coercivity of  $b_h$  we will estimate the term  $2 \sum_{n=0}^{N+1} \{c(x_n)w'(x_n)\}[w(x_n)]$  from above applying Lemma 1.10 and additional smaller tools:

Using the general fact  $2ab \leq a^2 + b^2, \forall a, b \in \mathbb{R}$  we estimate

$$\begin{aligned} 2 \sum_{n=0}^{N+1} \{c(x_n)w'(x_n)\}[w(x_n)] &= 2 \sum_{n=0}^{N+1} \{c(x_n)w'(x_n)\} \left(\frac{\mathbf{a}_n}{2}\right)^{-1/2} \left(\frac{\mathbf{a}_n}{2}\right)^{1/2} [w(x_n)] \\ &\leq 2 \sum_{n=0}^{N+1} \frac{\{c(x_n)w'(x_n)\}^2}{\mathbf{a}_n} + \frac{1}{2} \sum_{n=0}^{N+1} \mathbf{a}_n [w(x_n)]^2 \end{aligned} \quad (1.18)$$

Recalling  $\mathbf{a}_n = \sigma c_n \mathbf{h}_n^{-1}$  from (1.8) and noting the relations  $\mathbf{h}_n \leq h_n, c_n^{-1} \leq c(x_n^-)^{-1}, c(x_n^+)^{-1}$  we find

$$\begin{aligned} \mathbf{a}_n^{-1} c(x_n^+) &\leq \frac{h_n}{\sigma}, \quad \mathbf{a}_n^{-1} c(x_n^-) \leq \frac{h_{n-1}}{\sigma}, \quad \forall n = 1, \dots, N \\ \mathbf{a}_0^{-1} c(x_0^+) &= \frac{h_0}{\sigma}, \quad \mathbf{a}_{N+1}^{-1} c(x_{N+1}^-) = \frac{h_N}{\sigma} \end{aligned}$$

applying this and the usefull inequality  $(a+b)^2 \leq 2a^2 + 2b^2$  yields

$$\begin{aligned} &2 \sum_{n=0}^{N+1} \frac{\{c(x_n)w'(x_n)\}^2}{\mathbf{a}_n} \\ &= 2 \sum_{n=1}^N \frac{1}{4\mathbf{a}_n} \left( c(x_n^-)w'(x_n^-) + c(x_n^+)w'(x_n^+) \right)^2 + \frac{2}{\mathbf{a}_0} \left( c(x_0^+)w'(x_0^+) \right)^2 + \frac{2}{\mathbf{a}_{N+1}} \left( c(x_{N+1}^-)w'(x_{N+1}^-) \right)^2 \\ &\leq 2 \sum_{n=1}^N \frac{1}{2\sigma} \left( h_{n-1} c(x_n^-)w'(x_n^-)^2 + h_n c(x_n^+)w'(x_n^+)^2 \right) + \frac{2h_0}{\sigma} c(x_0^+)w'(x_0^+)^2 + \frac{2h_N}{\sigma} c(x_{N+1}^-)w'(x_{N+1}^-)^2 \\ &\leq \frac{c_{\max}}{\sigma} \sum_{n=1}^N \left( h_{n-1} w'(x_n^-)^2 + h_n w'(x_n^+)^2 \right) + \frac{2c_{\max}h_0}{\sigma} w'(x_0^+)^2 + \frac{2c_{\max}h_N}{\sigma} w'(x_{N+1}^-)^2 \end{aligned} \quad (1.19)$$

Since  $w \in V_h$  is a (broken) polynomial, we can apply Lemma 1.10 elementwise and find

$$w'(x_n^+)^2, w'(x_{n+1}^-)^2 \leq \frac{(r+1)^2}{h_n} \|w'\|_{L^2(I_n)}^2 \quad \forall n = 0, \dots, N \quad (1.20)$$

By combining (1.19), (1.20) and inserting  $1 = c_{\min} c_{\min}^{-1} \leq c(x) c_{\min}^{-1} \quad \forall x \in \Omega$  we find

$$2 \sum_{n=0}^{N+1} \frac{\{c(x_n)w'(x_n)\}^2}{\mathbf{a}_n} \leq 3C_\sigma \sum_{n=0}^N \|\sqrt{c}w'\|_{L^2(I_n)}^2 \quad (1.21)$$

for  $C_\sigma := \frac{(r+1)^2 c_{\max}}{\sigma c_{\min}} > 0$ .

Finally putting together (1.17), (1.18) and (1.21) yields

$$\begin{aligned} b_h(w, w) &\geq \|w\|_h^2 - 3C_\sigma \sum_{n=0}^N \|\sqrt{c}w'\|_{L^2(I_n)}^2 - \frac{1}{2} \sum_{n=0}^{N+1} \mathbf{a}_n [w(x_n)]^2 \\ &= (1 - 3C_\sigma) \sum_{n=0}^N \|\sqrt{c}w'\|_{L^2(I_n)}^2 + \frac{1}{2} \sum_{n=0}^{N+1} \mathbf{a}_n [w(x_n)]^2 \\ &\geq \frac{1}{2} \|w\|_h^2 \end{aligned}$$

for  $\sigma \geq \frac{6(r+1)^2 c_{\max}}{c_{\min}}$ , which proves the coercivity of  $b_h$  on  $V_h$ .

**Step 2 (Continuity).**

The proof the continuity of  $b_h$  uses similar ideas as the coercivity proof. Let  $u, v \in V_h$ , by using Cauchy-Schwarz we immediately get

$$\begin{aligned}
|b_h(u, v)| &\leq \sum_{n=0}^N \|\sqrt{c}u'\|_{L^2(I_n)} \|\sqrt{c}v'\|_{L^2(I_n)} + \sum_{n=0}^{N+1} |\{c(x_n)u'(x_n)\}[v(x_n)]| \\
&\quad + \sum_{n=0}^{N+1} |\{c(x_n)v'(x_n)\}[u(x_n)]| + \sum_{n=0}^{N+1} \mathbf{a}_n |[u(x_n)][v(x_n)]| \\
&=: T_{\text{ell}} + T_{\text{cons}}^{(u)} + T_{\text{cons}}^{(v)} + T_{\text{penal}}
\end{aligned} \tag{1.22}$$

The goal is now to estimate the consistency terms  $T_{\text{cons}}$  from above by something of the form  $\sum_{n=0}^{N+1} t_n(u)s_n(v) + \sum_{n=0}^{N+1} t_n(v)s_n(u)$ , such that together with the terms  $T_{\text{ell}}, T_{\text{penal}}$  we can use discrete Cauchy-Schwarz on the sums and hence separate them into a product of the two energy norms  $C_{\text{cont}}\|u\|_h\|v\|_h$  scaled by a positive constant.

We will show the estimate of  $T_{\text{cons}}^{(u)}$ , the procedure to estimate  $T_{\text{cons}}^{(v)}$  is analogous. First rewrite

$$T_{\text{cons}}^{(u)} = \sum_{n=0}^{N+1} |\{c(x_n)u'(x_n)\}\mathbf{a}_n^{-1/2}\mathbf{a}_n^{1/2}[v(x_n)]| \tag{1.23}$$

Next again using the definition of  $\mathbf{a}$  and estimates as in Step 1 we find for interior faces  $n = 1, \dots, N$

$$|\{c(x_n)u'(x_n)\}\mathbf{a}_n^{-1/2}| \leq \frac{1}{2}\sqrt{\frac{\mathbf{h}_n}{\sigma}}\sqrt{c_{\max}}(|u'(x_n^-)| + |u'(x_n^+)|)$$

and for the boundary faces

$$|\{c(x_0)u'(x_0)\}\mathbf{a}_0^{-1/2}| \leq \sqrt{\frac{\mathbf{h}_0}{\sigma}}\sqrt{c_{\max}}|u'(x_0^+)|, \quad |\{c(x_{N+1})u'(x_{N+1})\}\mathbf{a}_{N+1}^{-1/2}| \leq \sqrt{\frac{\mathbf{h}_N}{\sigma}}\sqrt{c_{\max}}|u'(x_{N+1}^-)|$$

Applying Lemma (1.10) yields for  $\beta_n(u) := \sqrt{C_\sigma}\|\sqrt{c}u'\|_{L^2(I_n)}, n = 0, \dots, N$

$$\begin{aligned}
|\{c(x_n)u'(x_n)\}\mathbf{a}_n^{-1/2}| &\leq \frac{\beta_{n-1}(u)}{2} + \frac{\beta_n(u)}{2} \quad \text{for } n = 1, \dots, N \\
|\{c(x_0)u'(x_0)\}\mathbf{a}_0^{-1/2}| &\leq \beta_0(u) \\
|\{c(x_{N+1})u'(x_{N+1})\}\mathbf{a}_{N+1}^{-1/2}| &\leq \beta_N(u)
\end{aligned}$$

which we can now plug back into (1.23) to get

$$T_{\text{cons}}^{(u)} \leq \beta_0(u)\gamma_0(v) + \beta_N(u)\gamma_{N+1}(v) + \sum_{n=1}^N \frac{\beta_{n-1}(u)}{2}\gamma_n(v) + \sum_{n=1}^N \frac{\beta_n(u)}{2}\gamma_n(v) \tag{1.24}$$

for  $\gamma_n(v) := \sqrt{\mathbf{a}_n}[v(x_n)] \forall n = 0, \dots, N+1$ . By furthermore denoting  $\alpha_n(u) := \|\sqrt{c}u'\|_{L^2(I_n)}$  we can represent

$$T_{\text{ell}} = \sum_{n=0}^N \alpha_n(u)\alpha_n(v), \quad T_{\text{penal}} = \sum_{n=0}^{N+1} \gamma_n(u)\gamma_n(v)$$

and in total for

$$\begin{aligned}\mathbf{u} &:= [\alpha_0(u), \dots, \alpha_N(u), \beta_0(u), \beta_N(u), \frac{\beta_0(u)}{2}, \dots, \frac{\beta_{N-1}(u)}{2}, \frac{\beta_1(u)}{2}, \dots, \frac{\beta_N(u)}{2}, \\ &\quad \gamma_0(u), \gamma_{N+1}(u), \gamma_1(u), \dots, \gamma_N(u), \gamma_1(u), \dots, \gamma_N(u), \gamma_0(u), \dots, \gamma_{N+1}(u)]^T \in \mathbb{R}^{6N+7} \\ \mathbf{v} &:= [\alpha_0(v), \dots, \alpha_N(v), \gamma_0(v), \gamma_{N+1}(v), \gamma_1(v), \dots, \gamma_N(v), \gamma_1(v), \dots, \gamma_N(v), \\ &\quad \beta_0(v), \beta_N(v), \frac{\beta_0(v)}{2}, \dots, \frac{\beta_{N-1}(v)}{2}, \frac{\beta_1(v)}{2}, \dots, \frac{\beta_N(v)}{2}, \gamma_0(v), \dots, \gamma_{N+1}(v)]^T \in \mathbb{R}^{6N+7}\end{aligned}$$

we get

$$\begin{aligned}T_{\text{ell}} + T_{\text{cons}}^{(u)} + T_{\text{cons}}^{(v)} + T_{\text{penal}} &\leq \mathbf{u}^T \mathbf{v} \leq |\mathbf{u}| |\mathbf{v}| \\ &\leq \left( \sum_{n=0}^N \left(1 + \frac{5}{4} C_\sigma\right) \|\sqrt{c} u'\|_{L^2(I_n)}^2 + 3 \sum_{n=0}^{N+1} \mathbf{a}_n [u(x_n)]^2 \right)^{1/2} \left( \sum_{n=0}^N \left(1 + \frac{5}{4} C_\sigma\right) \|\sqrt{c} v'\|_{L^2(I_n)}^2 + 3 \sum_{n=0}^{N+1} \mathbf{a}_n [v(x_n)]^2 \right)^{1/2} \\ &\leq C_{\text{cont}} \|u\|_h \|v\|_h\end{aligned}$$

where  $C_{\text{cont}} := (3 + \frac{5}{4} C_\sigma)$ . This last estimate together with 1.22 proves the continuity of  $b_h$ .  $\square$

## 1.9 Extension of the Bilinear Form

To simplify the theory we will continue to assume the exact solution to be in  $H^2(\Omega)$ . When we intend to extend the bilinear form from  $V_h$  to a bigger space which contains the exact solution we run into the problem, that  $V_h \not\subset H^1(\Omega)$ , by extension we also have  $V_h \not\subset H^2(\Omega)$ . So in contrast to continuous FEM we have to use a bigger space allowing for discontinuities and including both  $V_h$  and  $H^2(\Omega)$ .

A very common approach (see for example [2]) is to choose a vector space sum

$$V := V_h + H^2(\Omega) = \{v \in L^2(\Omega) \mid v = v_h + \tilde{v}, \text{ for } v_h \in V_h, \tilde{v} \in H^2(\Omega)\} \quad (1.25)$$

Alternatively Rivière proposes the usage of *broken Sobolev spaces* in [6]. That is, for a given partition  $\mathcal{T}_h$  of  $\Omega$  we can define

$$H^k(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid v|_I \in H^k(I), \forall I \in \mathcal{T}_h\}$$

for some  $k \in \mathbb{N}_0$ , this allows for discontinuities and we have  $V_h \in H^2(\mathcal{T}_h)$ . We will use  $V$  as defined in (1.25). Now we extend the bilinear form  $b_h$  and the linear functional  $\ell_h$  from  $V_h$  to  $V$  and formally get

$$b : V \times V \rightarrow \mathbb{R}, \quad \ell : V \rightarrow \mathbb{R} \quad (1.26)$$

where

$$\begin{aligned}b(u, v) &= \sum_{n=0}^N \int_{I_n} c u' v' \, dx - \sum_{n=0}^{N+1} \{c(x_n) u'(x_n)\} [v(x_n)] + \{c(x_n) v'(x_n)\} [u(x_n)] + \sum_{n=0}^{N+1} \mathbf{a}_n [u(x_n)] [v(x_n)] \\ \ell(v) &= (f, v)_{L^2(\Omega)} - g_1 c(x_{N+1}^-) v'(x_{N+1}^-) + g_0 c(x_0^+) v'(x_0^+) + \mathbf{a}_{N+1} g_1 v(x_{N+1}^-) + \mathbf{a}_0 g_0 v(x_0^+)\end{aligned}$$

for  $u, v \in V$ .

This yields the SIPG variational formulation

Find  $u \in V$  such that:

$$b(u, v) = \ell(v), \quad \forall v \in V \quad (1.27)$$

### Lower Regularity Solutions

The  $H^2$ -regularity assumption on the solution is a common one in the context of convergence theory, but still a rather strict one in general. If we assume only  $H^1$ -regularity, which is a reasonable assumption to

make, we encounter the problem of undefined trace values. Recall that from the trace theorem it is known that  $L^2$ -regularity is not enough to yield uniquely defined trace values. In 1d this becomes apparent, since point values of  $L^2$ -functions are not well defined. In particular for a solution  $u \in H^1(\Omega)$ , we have that  $u'(x_n^+), u'(x_n^-)$  are not well defined. Extending the bilinear form  $b_h$  to  $V$  requires some additional work, which is often done in one of two ways:

1. Lifting operators (see [3])
2.  $L^2$ -projection (see [2])

We will completely circumvent this issue by assuming  $H^2$ -regularity.

Recall that in the proof of Theorem 1.13 for coercivity and continuity of the bilinear form we have used the finite dimensionality of the polynomial spaces over each element in  $\mathcal{T}_h$ . This means that the coercivity and continuity properties cannot be simply applied to our extended form  $b$ . For this we would have to introduce either lifting or the  $L^2$ -projection as mentioned before.

### 1.9.1 Consistency of the SIPG Variational Formulation

The final matrix-vector system we solve (yielding our numerical solution) is equivalent to the fully discrete variational formulation (1.9), which in turn is a discretization of not the original weak problem (1.3), but rather the SIPG variational formulation (1.27). We now have to show, that any solution of the SIPG variational formulation is also a solution of the original weak problem and vice versa to ensure our Galerkin approximation approximates the wanted exact solution, i.e. that the SIPG variational formulation is consistent with the weak formulation of the problem.

**Theorem 1.14.** *The SIPG variational formulation is consistent. That is, for  $u \in H^2(\Omega)$  we have:  $u$  is a solution of (1.3) if and only if  $u$  solves (1.27)*

*Proof. Step 1 ("⇐").*

First suppose  $u \in H^2(\Omega)$  is a solution of the SIPG variational formulation (1.27), then clearly

$$b(u, v) = \ell(v) \quad \forall v \in C_c^\infty(\Omega) \subset H^2(\Omega) \quad (1.28)$$

since  $u, v$  are continuous we have

$$[u(x_n)] = [v(x_n)] = 0 \quad \forall n \in \{1, \dots, N\}$$

and  $v \in C_c^\infty(\Omega)$  implies

$$[v(x_0)] = [v(x_{N+1})] = \{c(x_0)v'(x_0)\} = \{c(x_{N+1})v'(x_{N+1})\} = 0$$

therefore (1.28) becomes

$$a(u, v) = (f, v)_{L^2(\Omega)} \quad \forall v \in C_c^\infty(\Omega) \quad (1.29)$$

where  $a$  is the elliptic bilinear form of the weak formulation in (1.3).

Now to recover the boundary conditions first note that since  $u \in H^2(\Omega)$  and  $u$  satisfies (1.29) we have

$$-(cu')' = f \quad \text{a.e. in } \Omega \quad (1.30)$$

From here we follow exactly the derivation of the discrete SIPG variational formulation in section 1.3. We multiply (1.30) by a test function  $v \in V$ , integrate over the elements  $I \in \mathcal{T}_h$  by parts, sum up over all elements and add the symmetry and penalty terms corresponding to the SIPG bilinear form, hence we find

$$b(u, v) = (f, v)_{L^2(\Omega)} - c(x_{N+1}^-)u(x_{N+1}^-)v'(x_{N+1}^-) + c(x_0^+)u(x_0^+)v'(x_0^+) + \mathbf{a}_{N+1}u(x_{N+1}^-)v(x_{N+1}^-) + \mathbf{a}_0u(x_0^+)v(x_0^+) \quad (1.31)$$

Subtracting (1.31) from (1.27) yields

$$\left(-c(x_{N+1}^-)v'(x_{N+1}^-) + \mathbf{a}_{N+1}v(x_{N+1}^-)\right)\left(u(x_{N+1}^-) - g_1\right) + \left(c(x_0^+)v'(x_0^+) + \mathbf{a}_0v(x_0^+)\right)\left(u(x_0^+) - g_0\right) = 0$$

Since the above equality holds for any  $v \in V$  we can choose fitting test functions  $v$  to show that  $u$  satisfies the boundary conditions.

**Step 2.**

Now suppose  $u \in H^2(\Omega)$  is a solution of the weak problem (1.3), as already argued in Step 1 we have

$$-(cu')' = f \quad \text{a.e. in } \Omega$$

this yields

$$b(u, v) = (f, v)_{L^2(\Omega)} - c(x_{N+1}^-)u(x_{N+1}^-)v'(x_{N+1}^-) + c(x_0^+)u(x_0^+)v'(x_0^+) + \mathbf{a}_{N+1}u(x_{N+1}^-)v(x_{N+1}^-) + \mathbf{a}_0u(x_0^+)v(x_0^+)$$

by the derivation steps recalled in Step 1. Applying the boundary conditions, which are fixed by the weak formulation (1.3), shows that  $u$  indeed satisfies the SIPG variational formulation (1.27) and we are done.  $\square$

## 1.10 Error Analysis

In this section we analyze the error between the Galerkin approximation resulting from the discrete SIPG formulation and an exact solution. We will focus on the error in the energy norm  $\|\cdot\|_h$  but also mention the resulting error measured in the  $L^2$ -norm.

We start by recalling an important polynomial approximation result for finite elements.

Fix a polynomial degree  $r \in \mathbb{N}$  and let  $I = (a, b) \subset \mathbb{R}$  be an element of length  $h = b - a$ . We denote the linear interpolation operator by

$$\mathcal{I}_h^r : L^2(I) \rightarrow \mathcal{P}^r(I), \quad (1.32)$$

where the distinct interpolation nodes are  $\xi_0, \dots, \xi_r \in I$ . For our purpose the exact positioning of these nodes can be arbitrary. For readability if the context allows it we will denote the  $i$ -th derivative of a function  $v$  by  $\frac{d^i}{dx^i}v = v^{(i)}$ .

**Lemma 1.15** (interpolant estimate). *Let  $s = r + 1$  and  $v \in H^s(I)$ , it holds that*

$$\|v - \mathcal{I}_h^r v\|_{H^m(I)} \leq \sqrt{m+1} h^{s-m} |v|_{H^s(I)} \quad \forall m \in \{0, \dots, s-1\}$$

where  $|v|_{H^s(I)} = \|v^{(s)}\|_{L^2(I)}$  denotes the  $H^s$ -seminorm.

*Proof. Step 1.*

Fix  $m \in \{0, \dots, s-1\}$ , we start by first estimating the  $H^m$ -seminorm

$$|v - \mathcal{I}_h^r v|_{H^m(I)} \leq h^{s-m} |v|_{H^s(I)} \quad (1.33)$$

Define  $e := v - \mathcal{I}_h^r v \in H^s(I) \subset C^{s-1}(I)$ , by construction of the interpolation operator we have  $e(\xi_i) = 0$  for all  $i \in \{0, \dots, r\}$ , where the  $\xi_i$  are the interpolation nodes.

By applying Rolle's theorem in succession we can find  $r$  zeroes of  $e'$ ,  $r-1$  zeroes of  $e''$ ,  $\dots$ , and one zero of  $e^{(s-1)}$ . Clearly in total there exist some points  $y_0, \dots, y_{s-1} \in I$  such that

$$e^{(i)}(y_i) = 0 \quad \forall i \in \{0, \dots, s-1\}.$$

Next we use the fundamental theorem of calculus inductively, let  $x_m \in I$  arbitrary and write

$$\begin{aligned}
e^{(m)}(x_m) &= \underbrace{e^{(m)}(y_m)}_{=0} + \int_{y_m}^{x_m} e^{(m+1)}(x_{m+1}) \, dx_{m+1} \\
&= \underbrace{\int_{y_m}^{x_m} e^{(m+1)}(y_{m+1}) \, dx_{m+1}}_{=0} + \int_{y_m}^{x_m} \int_{y_{m+1}}^{x_{m+1}} e^{(m+2)}(x_{m+2}) \, dx_{m+1} \, dx_{m+2} \\
&= \dots = \underbrace{\int_{y_m}^{x_m} \dots \int_{y_{s-2}}^{x_{s-2}} e^{(s-1)}(y_{s-1}) \, dx_{s-1} \dots dx_{m+1}}_{=0} + \int_{y_m}^{x_m} \dots \int_{y_{s-1}}^{x_{s-1}} e^{(s)}(x_s) \, dx_s \dots dx_{m+1}.
\end{aligned}$$

Squaring both sides of this equation and integrating over  $I$  yields

$$\begin{aligned}
\|e^{(m)}\|_{L^2(I)}^2 &= \int_I \left( \int_{y_m}^{x_m} 1 \cdot \int_{y_{m+1}}^{x_{m+1}} \dots \int_{y_{s-1}}^{x_{s-1}} e^{(s)}(x_s) \, dx_s \dots dx_{m+2} \, dx_{m+1} \right)^2 dx_m \\
&\leq \int_I \underbrace{|x_m - y_m|}_{\leq h} \int_{y_m}^{x_m} \left( \int_{y_{m+1}}^{x_{m+1}} \dots \int_{y_{s-1}}^{x_{s-1}} e^{(s)}(x_s) \, dx_s \dots dx_{m+2} \right)^2 dx_{m+1} \, dx_m \\
&\leq h \int_I \int_I \left( \int_{y_{m+1}}^{x_{m+1}} \dots \int_{y_{s-1}}^{x_{s-1}} e^{(s)}(x_s) \, dx_s \dots dx_{m+2} \right)^2 dx_{m+1} \, dx_m \\
&\leq h^2 \int_I \left( \int_{y_{m+1}}^{x_{m+1}} \dots \int_{y_{s-1}}^{x_{s-1}} e^{(s)}(x_s) \, dx_s \dots dx_{m+2} \right)^2 dx_{m+1} \\
&\leq \dots \leq h^{2(s-m)} \int_I |e^{(s)}(x_s)|^2 \, dx_s = h^{2(s-m)} |e|_{H^s(I)}^2,
\end{aligned}$$

where we have applied Cauchy-Schwarz inductively. But now since  $s = r + 1$  and  $\mathcal{I}_h^r v \in \mathcal{P}^r(I)$  it holds that  $(\mathcal{I}_h^r v)^{(s)} \equiv 0$  in  $I$ , which implies that  $e^{(s)} = v^{(s)}$  and therefore  $|e|_{H^s}^2 = |v|_{H^s}^2$ . This proves (1.33).

Since we have proven (1.33) for an arbitrary  $m \in \{0, \dots, s-1\}$  we can conclude that

$$|v - \mathcal{I}_h^r v|_{H^m(I)} \leq h^{s-m} |v|_{H^s(I)} \quad \forall m \in \{0, \dots, s-1\}.$$

## Step 2.

Now extending the result from the seminorm to the full norm is a quick matter. By applying step 1 and the definition of the  $H^m$ -norm we immediately get

$$\|v - \mathcal{I}_h^r v\|_{H^m(I)}^2 = \sum_{j=0}^m |v - \mathcal{I}_h^r v|_{H^j(I)}^2 \leq (m+1) h^{2(s-m)} |v|_{H^s(I)}^2.$$

□

Suppose  $u \in V$  satisfies the SIPG variational formulation (1.27) and  $u_h \in V_h$  satisfies the discrete SIPG variational formulation (1.9) (here the polynomial degree does not matter), and since  $V_h \subset V$  it holds that

$$b(u - u_h, v) = 0 \quad \forall v \in V_h \quad (1.34)$$

This is commonly known as *Galerkin orthogonality*.

## 1.11 Numerical Results

### 1.11.1 Rate of Convergence

To replicate the theoretical rate of convergence we first consider a sequence of uniform meshes. Let  $h_l = 2^{-l}$  denote the global (uniform) meshsize and  $\mathcal{T}_h^{(l)}$  denote the partitions of  $\Omega$  for  $l = 2, \dots, 9$ . As an example

we have  $\mathcal{T}_h^{(2)} = \{(0, 0.25), (0.25, 0.5), (0.5, 0.75), (0.75, 1)\}$ . To begin we worked with the constant coefficient  $c \equiv 1$ . In this case choosing  $\sigma = 10(r+1)^2$  was more than enough to guarantee positive definiteness of the system matrix. We tested our methods programmed in **MATLAB** using some very simple exact solutions  $u \in C^1(\Omega)$ , approximated them numerically finding some  $u_h \in V_h^r(\mathcal{T}_h^{(l)})$  (for an a priori fixed polynomial degree  $r$ ) and calculated the  $L^2$ -norm

$$\|u - u_h\|_{L^2(\Omega)} = \left( \int_{\Omega} |u(x) - u_h(x)|^2 dx \right)^{1/2} = \left( \sum_{n=0}^{N_l} \int_{I_n} |u(x) - u_h(x)|^2 dx \right)^{1/2}$$

and the broken  $H^1$ -norm

$$\|u - u_h\|_{H^1(\mathcal{T}_h^{(l)})} = \left( \sum_{n=0}^{N_l} \int_{I_n} |u'(x) - u_h'(x)|^2 dx \right)^{1/2}.$$

Note that even though  $u_h$  is not in  $H^1(\Omega)$  and we can therefore not calculate the normal  $H^1$ -norm, we do have

$$u, u_h \in H^1(\mathcal{T}_h^{(l)}) = \{v \in L^2(\Omega) | v|_{I_n} \in H^1(I_n) \text{ for } n = 0, \dots, N_l\}$$

so taking the broken Sobolev norm makes sense also for  $V_h$  functions.

Firstly to ensure the exactness of the method we chose the exact solution  $u(x) = x^r$  and approximated it using  $\mathcal{P}^r$  elements. As expected we got a minimal floating-point error of around  $10^{-15}$  in both  $L^2$ -, and  $H^1(\mathcal{T}_h^{(l)})$ -norms, slightly increasing for increasing  $l$  due to the growing condition number of the system matrix. This indicates exactness of the method.

Next we considered the exact solution  $u(x) = e^{-x} \sin(x)$ , we observed the with the theory conforming convergence rates  $\mathcal{O}(h^r), \mathcal{O}(h^{r+1})$ , in the  $H^1(\mathcal{T}_h)$ - and  $L^2$ -norm respectively for  $\mathcal{P}^r$ -elements. As visible in figures 1.1, 1.2

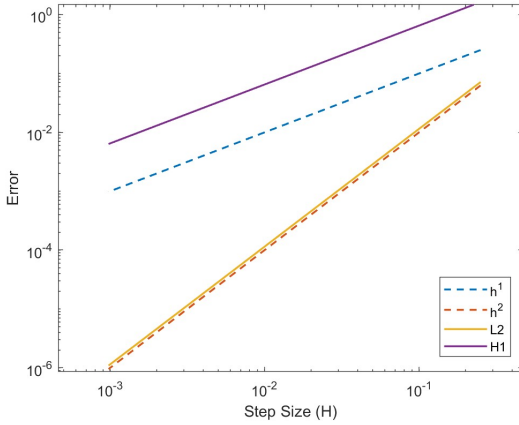


Figure 1.1: Errors of SIPG for  $P^1$ -elements

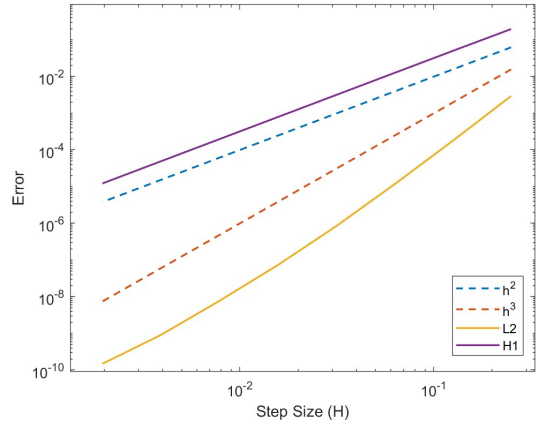


Figure 1.2: Errors of SIPG for  $P^2$ -elements

We plotted the numerical solution for  $\mathcal{P}^1, \mathcal{P}^2$ -elements in the figures 1.3, 1.4 respectively on the partition  $\mathcal{T}_h^{(3)}$ , i.e. for 8 elements with meshsize  $h = \frac{1}{8}$ . By eye the solution seems continuous, this is due to the high penalization parameter  $\sigma = 10(r+1)^2$ . In fact the solution is not continuous and the boundary conditions are not exact either, the error is just not noticeable by eye.

If we reduce the penalization parameter enough such that the bilinear form is no longer coercive we can observe the jumps growing and the discontinuity becomes apparent as is visible in figures 1.5, 1.6 where we have set  $\sigma = 1.1$ . Maybe interesting to observe is the effect of enforcing boundary conditions weakly. Clearly here the Dirichlet boundary condition  $u(0) = 0$  is not fulfilled by the numerical solutions, which would be the case for strongly enforced boundary conditions.

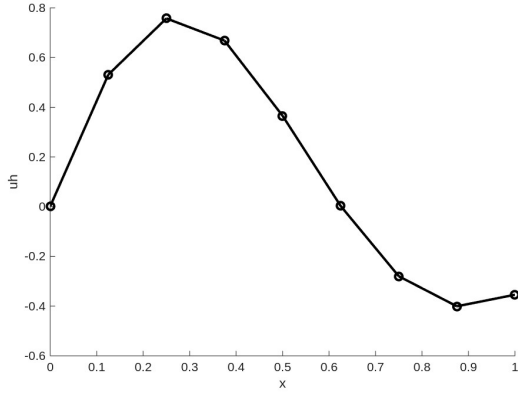


Figure 1.3: numerical SIPG-approximation for  $P^1$ -elements

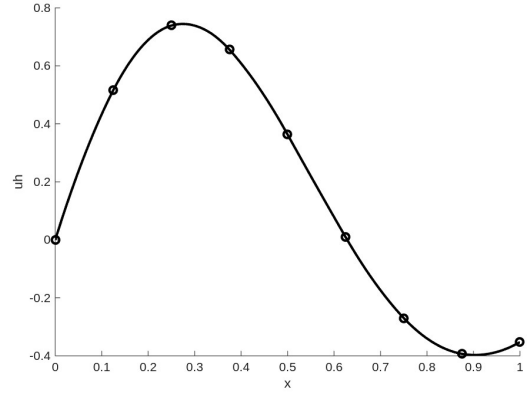


Figure 1.4: numerical SIPG-approximation for  $P^2$ -elements

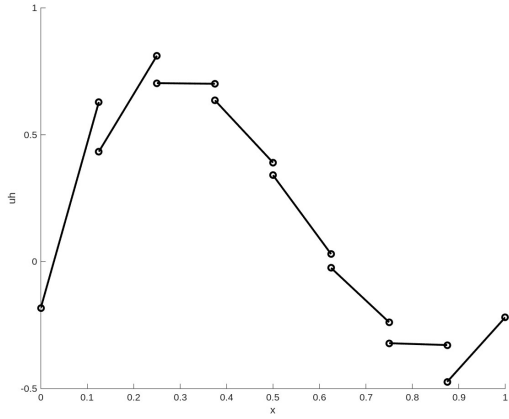


Figure 1.5: numerical SIPG-approximation for  $P^1$ -elements with small penalty

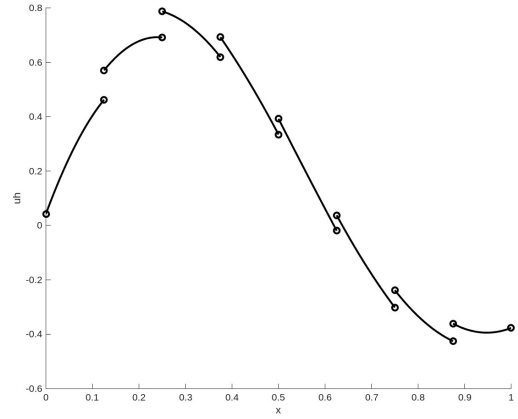


Figure 1.6: numerical SIPG-approximation for  $P^2$ -elements with small penalty

Finally we repeated the tests above using a non constant yet still smooth coefficient  $c(x) = \sin(10x) + 2$ . We observed the same convergence rates but slightly higher errors overall. This was to be expected since for a non-constant  $c$  we introduce a quadrature error.

### 1.11.2 Influence of Quadrature Rule on the Convergence Rate

As noted in section 1.6 by using  $r + 1$  Gauss-Lobatto nodes (in the context of  $\mathcal{P}^r$ -elements) as quadrature nodes and basis nodes at the same time introduces an error when assembling the mass matrix of the system. In this subsection we experimentally compare the results of using  $r + 1$  Gauss-Lobatto quadrature nodes to approximate the integrals versus using the exact integration values. To be precise we fix a polynomial degree  $r$  and our Lagrangian basis with  $r + 1$  Gauss-Lobatto nodes as specified in section 1.6 and approximate all integrals first using the Gauss-Lobatto quadrature rule with  $r + 1$  nodes, then with  $r + 2$  nodes and compare the two.

First we have to consider a slightly different elliptical problem, which requires a mass matrix.

$$-(c(x)u'(x))' + u(x) = f(x) \quad \forall x \in \Omega$$

$$u(0) = g_0, u(1) = g_1$$



We can apply the exact same tools as in the derivation of the variational formulation in section 1.3 and get the discrete SIPG variational formulation.

Find  $u_h \in V_h$  such that:

$$b_h(u_h, v) + (u_h, v)_{L^2(\Omega)} = \ell_h(v), \quad \forall v \in V_h \quad (1.35)$$

Now analogously to section 1.5 we write  $u_h = \sum_{m=0}^N \sum_{j=0}^r \alpha_j^m \Phi_j^m \in V_h$  and find that (1.35) is equivalent to

$$\sum_{m=0}^N \sum_{j=0}^r \alpha_j^m \left( b_h(\Phi_j^m, \Phi_i^n) + (\Phi_j^m, \Phi_i^n)_{L^2(\Omega)} \right) = \ell_h(\Phi_i^n) \quad \forall i \in \{0, \dots, r\}, \{n = 0, \dots, N\}$$

which in turn is equivalent to the Matrix-Vector system

$$(\mathbf{B} + \mathbf{M})\mathbf{u} = \mathbf{l} \quad (1.36)$$

where as before  $[\mathbf{B}]_{T(n,i), T(m,j)} = b_h(\Phi_j^m, \Phi_i^n)$ ,  $[\mathbf{u}]_{T(m,j)} = \alpha_j^m$ ,  $[\mathbf{l}]_{T(n,i)} = \ell_h(\Phi_i^n)$  and furthermore  $[\mathbf{M}]_{T(n,i), T(m,j)} = (\Phi_j^m, \Phi_i^n)_{L^2(\Omega)}$ .

Next we considered the same setting as described in subsection 1.11.1 and tested the convergence rates for  $\mathcal{P}^1, \mathcal{P}^2$ -elements, where we chose the exact solution  $u(x) = e^{-x} \sin(x)$  and  $c(x) = \sin(10x) + 2$ .

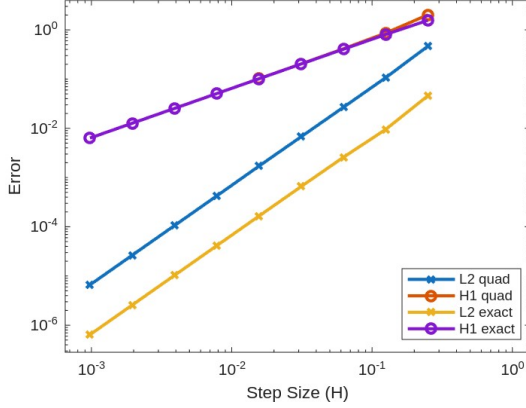


Figure 1.7: Comparison of convergence rates of higher order vs lower order quadrature for  $\mathcal{P}^1$ -elements

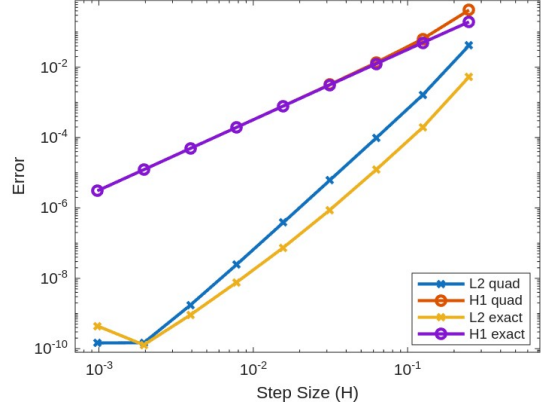


Figure 1.8: Comparison of convergence rates of higher order vs lower order quadrature for  $\mathcal{P}^2$ -elements

### Method A: "Error Quad"

This method corresponds to exactly doing what we have been doing before. We assembled the matrices of the system (1.36) as described in section 1.7 and approximated the integrals appearing in the entries of  $\mathbf{A}$ ,  $\mathbf{M}$ ,  $\mathbf{l}$  using the  $r + 1$  node Gauss-Lobatto quadrature rule, where  $r \in \{1, 2\}$ , meaning the quadrature nodes coincide with the basis nodes. We calculated the  $L^2, H^1(\mathcal{T}_h)$ -error between the numerical solution and the exact solution on all meshes and called it **L2 quad**, **H1 quad** respectively.

### Method B: "Error Exact"

For this method we assembled the matrices  $\mathbf{A}$ ,  $\mathbf{M}$ ,  $\mathbf{l}$  for the system 1.36 using  $r + 2$  Gauss-Lobatto quadrature nodes. This quadrature is exact for polynomials of order  $2r + 1$ , meaning the entries of the mass matrix  $\mathbf{M}$  are calculated exactly. Next we solved the system yielding the numerical solution and in turn calculated the  $L^2, H^1(\mathcal{T}_h)$ -errors on all meshes and called it **L2 exact**, **H1 exact** respectively.

In the figures 1.7, 1.8 we compared the error rates of Method A and B and observed no significant difference but a slight downward shift of **L2 exact** on a logarithmic scale in comparison to **L2 quad**.

## Appendix A

# Prerequisites

# Bibliography

- [1] L. C. EVANS, *Partial Differential Equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, 2 ed., 2010.
- [2] E. H. GEORGIOULIS, *Discontinuous galerkin methods for linear problems: An introduction*, in Approximation Algorithms for Complex Systems, E. Georgoulis, A. Iske, and J. Levesley, eds., vol. 3 of Springer Proc. Math., Springer, Berlin, Heidelberg, 2011, pp. 91–126.
- [3] M. GROTE, A. SCHNEEBELI, AND D. SCHÖTZNAU, *Discontinuous method for the wave equation*, SIAM Journal on Numerical Analysis, 44 (2006), pp. 2408–2431.
- [4] D. A. D. PIETRO AND A. ERN, *Mathematical Aspects of Discontinuous Galerkin Methods*, Springer, Berlin, Heidelberg, 1 ed., 2012.
- [5] A. QUARTERONIA, R. SACCO, AND F. SALERI, *Numerical Mathematics*, vol. 2, Springer, Berlin, Heidelberg, 2007.
- [6] B. RIVIÈRE, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, vol. 35 of Frontiers in Applied Mathematics, SIAM, Philadelphia, 2008.
- [7] T. WARBURTON AND J. S. HESTHAVEN, *On the constants in hp-finite element trace inverse inequalities*, Computer Methods in Applied Mechanics and Engineering, 192 (2003), pp. 2765–2773.