where $P_G(z)$ is defined by (7) and

$$Q_G(z)=E\{\theta/z\}P_G(z). \tag{12}$$

Define

$$U_i(z)=\begin{cases}1 & \text{if } z_i=z\\ 0 & \text{if } z_i\neq z\end{cases} \qquad P_n(z)=\frac{1}{n}\sum_{i=1}^{n}U_i(z) \tag{13}$$

$$W_i=\frac{1}{i}\sum_{j=1}^{i}z_j, \qquad Q_n(z)=\frac{1}{n}\sum_{i=1}^{n}W_iU_i(z); \tag{14}$$

then by the laws of large numbers we have

$$P\lim_{n\to\infty}P_n(z)=E\{U_1(z)\}=P_G(z) \tag{15}$$

$$P\lim_{n\to\infty}Q_n(z)=E\{W_1U_1(z)\}=E\{\theta/z\}P_G(z)=Q_G(z). \tag{16}$$

The asymptotic optimal estimator $\phi^n(z)$ is now defined as

$$\phi^n(z)=\frac{Q_n(z)}{P_n(z)}=\frac{\sum_{i=1}^{n}W_iU_i(z)}{\sum_{i=1}^{n}U_i(z)}. \tag{17}$$

Then from (11), (15), and (16)

$$P\lim_{n\to\infty}\phi^n(z)=\phi^G(z). \tag{18}$$

The computational algorithm for the asymptotic optimal estimator $\phi^n(z)$ is given in (17). With each new observation z, the procedure involves simple comparison and counting of the past observations. In any practical situation, if (9) is satisfied, the convergence of the error with this estimator to Bayes error is guaranteed by (10).

V. Concluding Remarks

In this correspondence a nonparametric signal estimation scheme is considered. The lack of *a priori* statistical information is compensated for by the past independent observations on the signal. The signal itself is considered to be a random variable, although the procedure is also applicable for the detection of deterministic signals since this may be considered a special case where the parameter space Ω consists of a single point to which $G(\theta)$ assigns probability one. The empirical Bayes estimator $\phi^n(z)$ may not be the best estimator for a small number of observations since $P_n(z)$ is also small with high probability, and it may be interesting to investigate a modified sequential procedure using competing estimators like the minimax estimator for small n until $P_n(z)$ exceeds some predetermined value.

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A Test For Whiteness

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***Abstract*—In this note a test to check if a discrete random process is white or not is proposed. Its reliability is compared to that of another test commonly used in system identification [1], filtering [3], etc.**

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I. Test for Whiteness

Let $y(\cdot)$ be a discrete random process and let

$$r(i)=1/N\sum_{t=0}^{N-i}y(t)y(t+i) \tag{1}$$

be a consistent estimate of its theoretical covariance, computed from the sample $y(0),\cdots,y(N)$. If $y(\cdot)$ is white, then asymptotically [2]

$$r(i)\sim n(0,r^2(0)/N), \qquad i\geqslant 1 \tag{2}$$

$$\operatorname{cov}[r(i),r(m)]=0, \qquad i\neq m. \tag{3}$$

It follows from (2) that if $y(\cdot)$ is white, then

$$|r(i)|\leqslant 1.95r(0)/\sqrt{N} \qquad \forall i\geqslant 1 \tag{4}$$

with a risk of about 5 percent. Using (4) a simple test can be immediately conceived:

$$y(\cdot)\text{ is white iff } |r(i)|\leqslant 1.95r(0)/\sqrt{N} \qquad \text{for } i=1,\cdots,k \quad k>1. \tag{5}$$

This test is commonly used, e.g. [1].

On the other hand, if $y(\cdot)$ is white, then from (2) and (3) it follows that

$$N\sum_{i=1}^{k}r^2(i)/r^2(0)$$

has a chi-square distribution with k degrees of freedom. Further, for large k ($k\geqslant 3$), approximatively

$$N\sum_{i=1}^{k}r^2(i)/r^2(0)\sim n(k,2k). \tag{6}$$

Then if $y(\cdot)$ is white,

$$\sum_{i=1}^{k}r^2(i)\leqslant(k+1.65\sqrt{2k})r^2(0)/N \tag{7}$$

with a risk of about 5 percent. The test (7) has also been obtained, as a particular case, in [4].

II. Analysis

From (2) and (3) it follows that $r(i)$ and $r(m)$ $i\neq m$ are independent random variables. Then the first type of risk (ftr) of (5) (ftr = the probability of rejecting the hypothesis that $y(\cdot)$ is white when this is true) can be easily computed:

$$p_{11}=1-(0.95)^k. \tag{8}$$

It is noted that (8) has unacceptable values for usual values of k (e.g., $k=10\rightarrow p_{11}=41$ percent).

In order to decrease ftr, (5) is modified as follows:

$$y(\cdot)\text{ is white iff } |r(i)|\leqslant 1.95r(0)/\sqrt{N} \text{ for at least } k-p \quad (p\geqslant 1) \text{ values of } i,\ i=1,2,\cdots,k. \tag{9}$$

For (9) the ftr is

$$p_{12}=1-\sum_{i=0}^{p}C_k^i(0.05)^i(0.95)^{k-i}. \tag{10}$$

When k and p are chosen, the second type of risk (str) must also be considered (str = the probability to accept as being white a nonwhite noise). This risk cannot be determined because it depends on $E[r(i)]$, but it is intuitive that to keep a reasonable "equilibrium" between ftr and str

$$p=0.05k, \qquad p=1,2,3,\cdots. \tag{11}$$

Values p and k commonly used are (e.g., [3])

$$p=1, \qquad k=20\rightarrow p_{12}=29 \text{ percent}$$

$$p=2, \qquad k=40\rightarrow p_{12}=37 \text{ percent}.$$

TABLE I

	Test (9)	Test (7)
ftr	27 percent	7 percent
str	31 percent	8 percent

For the test (7) the ftr is 5 percent for any k. This means that (7) is obviously better than (9).

III. Numerical Example

The tests (7) and (9) are compared in the following situations:

$$\left.\begin{array}{l} y_a(t)=e(t) \\ y_b(t)=e(t)+0.25e(t-1) \end{array}\right\} \quad \begin{array}{l} e(\cdot)=\text{unit variance white noise} \\ N=500,\ k=20,\ p=1. \end{array}$$

For $e(\cdot)$ 100 realizations are used. The number of realizations in which $y_a(\cdot)$ ($y_b(\cdot)$) is found nonwhite (white) is considered percentage ftr (str).

From Table I it is noted that the values for ftr are close to those expected. The values of str depend on the considered nonwhite process. Note, however, that $y_b(\cdot)$ is "close to" white noise and Table I shows a good behavior of the test (7) in this quite difficult situation.

In conclusion, from the theoretical analysis and Table I it follows that the test (7) proposed in this note is more powerful than the commonly used tests (5), (9).

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Further Comments on "On the Evaluation of $\sum_{n=0}^{\infty} n^k x^n$ with Applications to Z Transforms"

DAVID A. SMITH

In the above paper,[1] Abed states a theorem (partially proved and partially conjectured) about the sum of the indicated series in closed form. The conjectured form is correct and is, in fact, a classical theorem, probably first proved by Leonhard Euler over 200 years ago:

$$(1-x)^{k+1}\sum_{n=0}^{\infty} n^k x^n=\sum_{j=1}^{k} A_{kj}x^j \tag{1}$$

where the coefficients A_{kj} are the *Eulerian numbers*, defined by

$$A_{k,1}=A_{k,k}=1, \tag{2}$$

$$A_{k,j}=jA_{k-1,j}+(k-j+1)A_{k-1,j-1}. \tag{3}$$

L. Carlitz [1] observed almost a generation ago that the Eulerian numbers were not as well known as they should be and that they were frequently being rediscovered. Evidently, their rediscovery goes on. J. Riordan has given two different proofs of (1) that are readily accessible in [3, Problem 2, Ch. 2, p. 38] and [4, p. 89]. There is a table of Eulerian numbers through $k=10$ in [3, p. 215].

The use of (1) for evaluation of Z-transforms is apparently not new, even to at least one of the authors cited by Abed; see [2], in particular, Table IV in the Appendix, which lists the closed-form sums of the indicated series up to $k=10$, but without mention of the recurrence (3) for generating the coefficients.

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[1] H. A. Abed, *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 835–836, Dec. 1972; see also N. A. Kheir, *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 414–415, Aug. 1973.

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Comments on "The Structure of Robust Observers"

E. A. GALPERIN

In the above paper[1] the sensitivity of conventional observers to variations of parameters is discussed, the so-called closed-loop observers driven by the observer error are introduced, and a method is proposed for robust closed-loop observer design. However, the paper suffers from serious inaccuracies and presents misleading results as outlined below.

Item 1: Stating in Theorem 1, p. 582, that conventional observer (2) fails to provide observer action for almost every perturbation $(\delta N, \delta L)$ in nominal parameters, the author, to improve the matter and "to provide for the possibility of a closed-loop performance," introduces the feedback structure (22 a, b,) p. 583, for which he formulates the "robust observer design problem" as the problem of determining the conditions under which there exists a closed-loop observer (22) well acting for any small variation δK of the matrix K_0 which is one of the four matrices (M_0, K_0, R_0, T_0) of the structure (22). Further, p. 584, the necessary condition for robustness under the variation δK is obtained: $T_0V=R_0C_0$ (33).

The author has mentioned (p. 583) that his structure (22) can be written in the conventional form (2) by identifying $N_0=M_0-K_0T_0$ (24), $L_0=K_0R_0$ (25), giving at a time some reasons for distinction, e.g., that "the sensitivity properties of the two realizations can be totally different." However, if one proceeds a little further and applies the necessary condition (33) for robustness of (22), then from (24) and (25) for the perturbed parameters (M_0, $K_0+\delta K$, R_0, T_0) one obtains $\delta N=-\delta K\cdot T_0$, $\delta L=\delta K\cdot R_0$ and, making use of Lemma 1, we see that condition (13) for conventional observers to be robust is satisfied for every δK because $\delta N\cdot V+\delta L\cdot C_0=-\delta K\cdot T_0\cdot V+\delta K\cdot R_0\cdot C_0=\delta K(R_0C_0-T_0V)=0$ for any δK since $T_0V=R_0C_0$. Namely, for the plant

$$dx(t)/dt=A_0x(t) \qquad t\geqslant 0, \quad x(0)=x_0 \text{ unknown} \tag{1a}$$

$$y(t)=C_0x(t) \tag{1b}$$

the conventional observer

$$dz(t)/dt=N_0z(t)+L_0y(t) \qquad t\geqslant 0, \quad z(0)=z_0 \tag{2}$$

evaluating the set of linear functionals $Vx(t)$ (in the sense $z(t)\to Vx(t)$

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[1] S. P. Bhattacharyya, *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 581–588, Aug. 1976.