AERSP 508 Lecture 23

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Potential flow is very good at describing ideal-fluid flow (incompressible and inviscid), but falls short in describing what happens when a fluid does not satisfy these two requirements. One example of this is the D'almebert's paradox[1]. Additionally, the potential flow is unable to tell us where it fails due to the assumptions made in order to obtain it.
So we have to go back and reconsider our previosu assumptions, especially the ones that allowed us to drop the second order derivatives in the Navier-Stoke's momentum equation, thanks to the very high value of the Reynold's number. Consequently, we again have to consider the no-slip condition.
In 1904 Ludwig Prandtl studied the viscous effects in a high Reynold's number flow and noticed that they were confined to a thin layer near the surface. This area is called the "Boundary Layer" (in the rest of the text I will use BL for it).
Inside the BL the viscous terms are as important as the inviscid terms. Looking at the Navier-Stoke's equation for momentum:

$$u\frac{du}{dx} + v\frac{du}{dy} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\nabla^2 u \tag{1}$$

the velocity gradient has to be very large in order to make up for the very low value of ν. Outside of the BL the flow behaves inviscidly.
We define a cartesian coordinate system along the surface of an object, where the x-axis is parallel to the surface and y-axis is perpendicular to the surface. The lenght of the surface is denoted by L and the distance of a point from the surface along the y-axis is denoted by ϵL. Where ϵ is a very small quantity. This is to show that the BL has very small height when compared to the length of the surface. Given that the thickness of the BL is related to the Reynold's number, ϵ will also be related to the Reynold's number.

So:

$$\frac{d}{dy} \sim \frac{1}{\epsilon L} \tag{2}$$

Using the continuity equation:

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \tag{3}$$

$$\frac{du}{dx} \sim \frac{u_\infty}{L} \tag{4}$$
$$\frac{dv}{dy} \sim \frac{[v]}{\epsilon L} \tag{5}$$
$$\Rightarrow [v] \sim \epsilon u_\infty \tag{6}$$
$$\tag{7}$$

We can also define $[v]$ in terms of L and δ, where $delta$ is the distance between the surface and the point where $u = 0.99u_e$: $v \sim u_e \frac{\delta}{L}^2$.

The normal velocity to the surface (v) is $\mathcal{O}(\epsilon)^3$, which is consistent with the earlier flow tangency condition, where [v] should approach 0 quickly as it gets further away from the surface.

Consequently ψ (the streamfunction) is $\mathcal{O}(\epsilon)$.

$$u = \frac{d\psi}{dy} \tag{8}$$
$$v = -\frac{d\psi}{dx} \tag{9}$$
$$\tag{10}$$

Analyzing the x-momentum equation for a steady flow:

$$u\frac{du}{dx} + v\frac{du}{dy} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\Big) \tag{11}$$

We can rewrite it as:

$$u_\infty\frac{u_\infty}{L} + \cancel{\epsilon}u_\infty\frac{u_\infty}{\cancel{\epsilon}L} + \frac{u_\infty^2}{L} = [\nu]\frac{u_\infty}{L^2} + [\nu]\frac{u_\infty}{\epsilon^2L^2} \tag{12}$$

Given that the term $[v]\frac{u_\infty}{L^2}$ is divided by L^2 we neglet it to due it being extremely small. We end up with:

$$\epsilon^2 \sim \frac{\nu}{u_\infty L} \tag{13}$$

So: $\epsilon \sim Re^{-\frac{1}{2}}$

And for y-momentum:

$$u\frac{dv}{dx} + v\frac{dv}{dy} = -\frac{1}{\rho}\frac{dP}{dy} + \nu\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}\Big) \tag{14}$$

We can rewrite this as:

$$\frac{\epsilon u_\infty^2}{L} + \frac{\epsilon^{\cancel{2}}u_\infty^2}{\cancel{\epsilon}L} + \frac{u_\infty^2}{\epsilon L} = \epsilon^2u_\infty L\Big(\frac{\epsilon u_\infty}{L^2} + \frac{\epsilon u_\infty}{\epsilon^2L^2}\Big) \tag{15}$$

And each term is

$$\mathcal{O}(\epsilon)\ \mathcal{O}(\epsilon)\ \mathcal{O}\Big(\frac{1}{\epsilon}\Big)\ \mathcal{O}(\epsilon^3)\ \mathcal{O}(\epsilon) \tag{16}$$

The wall normal pressure gradient is, by $\mathcal{O}(\epsilon^2)$, the leading order term. In other words, the pressure gradient has a magnitude $\mathcal{O}(\epsilon^2)$ greater than any other term, so it has to be 0. So $\frac{dP}{dy} = 0$, which means that there is no variation in pressure in the normal direction to the surface. So the surface in the BL is the same as the pressure in the rest of the flow along the direction normal to the surface.
Combined the governing equations of the BL are:

Governing equations of the BL

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \tag{17}$$

$$\frac{dP}{dy} = 0 \tag{18}$$

$$u\frac{du}{dx} + v\frac{du}{dy} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{d^2u}{dy^2} \tag{19}$$

There are called **Prandtl's Boundary-Layers Equations**.
We have to remember the boundary conditions:

Boundary Conditions

$$u = v = 0 \ \ @ \ \ y = 0 \ \ (No - Slip) \tag{20}$$

$$u_\infty \Rightarrow u_e \ \ as \ \ y \Rightarrow \infty \tag{21}$$

Where u_e is the "edge velocity" from the potential flow. Or in other words the velocity of the flow outside of the BL.

[1] $\mathcal{O}(x)$ is used to indicate the magntidue of a function as a term of x.

[2] D'Alembert's paradox states that there are no drag forces on an object moving in an incompressible and inviscid flow. This is a consequence of the assumption that a high Reynold's number results in the lack of a boundary layer. However, even at high Reynold's numbers, a thin boundary layer exists.

[3] v is not a function of $\delta or L$. These are used just to show the magnitude. In fact $\frac{\delta}{L}(or \epsilon) << 1$.