Complex Potential and Complex Velocity

Mauro Patimo

December 10, 2023

Since the potential and the streamline have a very specific relation

$$u = \frac{d\phi}{dx} = \frac{d\psi}{dy}, \quad v = \frac{d\phi}{dy} = -\frac{d\psi}{dx} \tag{1}$$

These two equations fullfill the Cauchy-Riemann conditions, so we can write the potential as a complex function:

$$F(z) = \phi(x, y) + i\psi(x, y) \tag{2}$$

where $z = x + iy$, and $i = \sqrt{-1}$.
This complex analytical function is called **complex potential**, and it allows us to describe the behaviour of two independent variables into a single complex variable.

$$\frac{dF}{dz} = \frac{dF}{dx} = \frac{1}{i}\frac{dF}{dy} = \frac{d\phi}{dx} + i\frac{d\psi}{dx} = -i\frac{d\phi}{dy} + \frac{d\psi}{dy} \tag{3}$$

We can then write the complex potential as:

$$\frac{dF}{dz} = u - iv = \bar{\omega} \tag{4}$$

Where $\bar{\omega}$ is the complex conjugate of $\omega = u + iv$.
Complex quantities can be expressed as x and y components in the complez plane (Argand diagram): Remember Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

For example:

$$\frac{1}{z} = \frac{1}{x+iy}\frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$
$$\frac{1}{z} = z^{-1} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) = \underbrace{\frac{rcos\theta}{r^2}}_{\frac{x}{x^2+y^2}} - i\underbrace{\frac{rsin\theta}{r^2}}_{\frac{y}{x^2+y^2}}$$

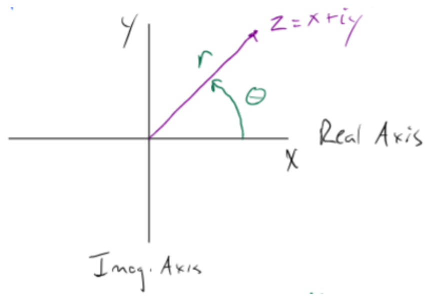

Figure 1: Argand Diagram

So a basic Potential Flow solution in complex form looks like:

$$F(z) = \underbrace{U_\infty(xcos\alpha + ysin\alpha)}_{\phi} + i\underbrace{U_\infty(ycos\alpha - xsin\alpha)}_{\psi} \tag{5}$$

$$F(z) = U_\infty e^{i\alpha} z$$
$$\bar{\omega} = \frac{dF}{dz} = U_\infty e^{i\alpha} = U_\infty cos\alpha - iU_\infty sin\alpha = u - iv$$

A source/sink in complex form is:

$$F(z) = \frac{\sigma}{2\pi} lnr + i\frac{\sigma}{2\pi}\theta = \frac{\sigma}{2\pi}\ln z \tag{6}$$

and

$$\frac{dF}{dz} = \frac{\sigma}{2\pi z} = \frac{\sigma}{2\pi r}[cos\theta - i\sin\theta]$$

For a vortex:

$$F(z) = \frac{\Gamma}{2\pi} lnz = \frac{\Gamma}{2\pi} lnr - i\frac{\Gamma}{2\pi}\theta \tag{7}$$

and

$$\frac{dF}{dz} = \frac{\Gamma}{2\pi i z} = -\frac{\Gamma}{2\pi r}[sin\theta + i\cos\theta]$$

So, a doublet in complex form is:

$$F(z) = \frac{\mu}{2\pi z} = \frac{\mu}{2\pi r}[cos\theta - i\sin\theta] \tag{8}$$
$$\frac{dF}{dz} = -\frac{\mu}{2\pi z^2} = -\frac{\mu}{2\pi r^2}[cos2\theta - isin2\theta] \tag{9}$$

We can use $e^{i\theta}$ to rotate the velocity fields. Since $e^{i\theta}$ (where θ is the angle of rotation) has a unit magnitude, this won't change the mangitude of the function

we are multiplying it by.
For example, if we want to rotate a doublet by θ:

$$F(z) = \frac{\mu}{2\pi z} = \frac{\mu}{2\pi r}[cos\theta - i\sin\theta]$$
$$F(z)e^{i\theta} = \frac{\mu}{2\pi r}[cos\theta - i\sin\theta][cos\theta + i\sin\theta] = \frac{\mu}{2\pi r}$$
$$\frac{dF}{dz}e^{i\theta} = -\frac{\mu}{2\pi r^2}[cos2\theta - isin2\theta][cos\theta + i\sin\theta] = -\frac{\mu}{2\pi r^2}e^{2i\theta}$$

If we want to rotate the free stream by $90°$, $\theta = 90°$. Thus, $e^{-i\frac{\pi}{2}} = \frac{i}{2}$, which allows the vortex to be orthogonal to the source/sink.
I higly suggest using this website to understand complex potentials and velocity with intercative exmaples: Complex Analysis