Euler's Totient Function

A Comprehensive Guide with Visual Explanations

Mathematical Foundations Series

February 5, 2026

Abstract

This document provides a comprehensive exploration of Euler's Totient Function, denoted as $\phi(n)$. We examine its definition, properties, calculation methods, and practical applications in modern cryptography, particularly in RSA encryption. Through detailed examples, visual diagrams, step-by-step explanations, pseudocode, flowcharts, and complete Java implementations, we demonstrate how this fundamental number-theoretic function plays a crucial role in contemporary information security.

Contents

1 Introduction

Euler's Totient Function, also known as Euler's phi function, is one of the most important functions in number theory. Named after the Swiss mathematician Leonhard Euler, this function counts the number of integers from 1 to n that are relatively prime to n (i.e., integers that share no common factor with n other than 1).

Key Point

Two numbers are **relatively prime** (or coprime) if their greatest common divisor (GCD) is 1.

2 Mathematical Definition

Definition 2.1: Euler's Totient Function

For a positive integer n, Euler's totient function $\phi(n)$ is defined as:

$$\phi(n) = |\{k \in \mathbb{Z} : 1 \leq k \leq n \text{ and } \gcd(k, n) = 1\}|$$

where $|\cdot|$ denotes the cardinality (count) of the set.

In simpler terms, $\phi(n)$ counts how many numbers from 1 to n are coprime with n.

3 The Core Concept: A Detailed Example

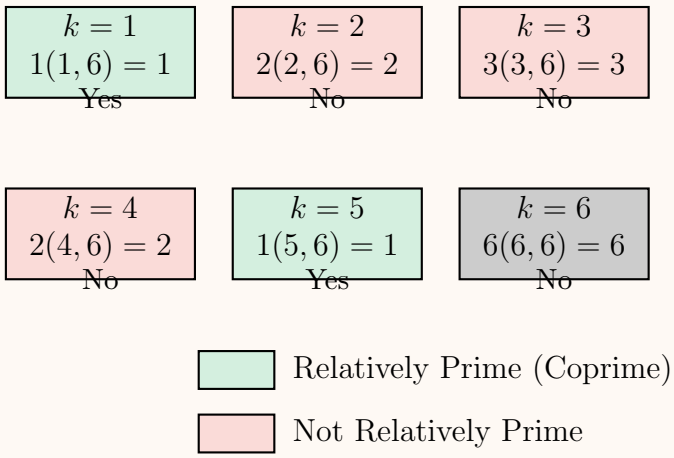
Let's explore the calculation of $\phi(6)$ in detail to understand the fundamental concept.

3.1 Step-by-Step Calculation for $n = 6$

Example 3.1: Computing $\phi(6)$

We need to check each integer from 1 to 6 and determine if it is relatively prime to 6.

Checking GCD for $n = 6$



Detailed Analysis:

1. **Check $k = 1$:** $\gcd(1, 6) = 1 \quad \Rightarrow$ Coprime
   The number 1 is coprime to every integer.

2. **Check $k = 2$:** $\gcd(2, 6) = 2 \quad \Rightarrow$ Not Coprime
   Both 2 and 6 are even, sharing the common factor 2.

3. **Check $k = 3$:** $\gcd(3, 6) = 3 \quad \Rightarrow$ Not Coprime
   Both 3 and 6 are divisible by 3.

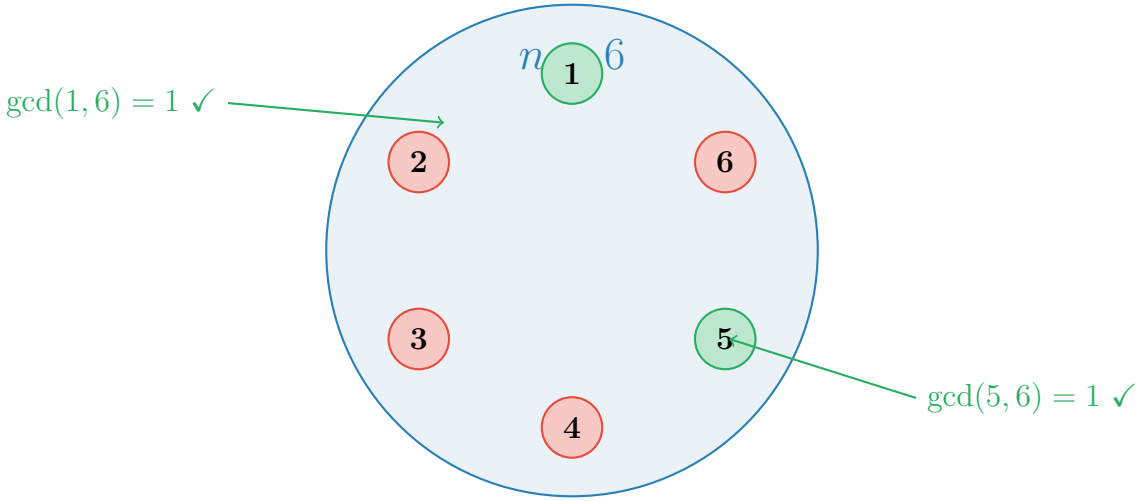
4. **Check $k = 4$:** $\gcd(4, 6) = 2 \quad \Rightarrow$ Not Coprime
   Both share the common factor 2.

5. **Check $k = 5$:** $\gcd(5, 6) = 1 \quad \Rightarrow$ Coprime
   5 is prime and does not divide 6.

6. **Check $k = 6$:** $\gcd(6, 6) = 6 \quad \Rightarrow$ Not Coprime
   A number always shares all its factors with itself.

Conclusion: There are exactly 2 numbers (1 and 5) that are relatively prime to 6.
Therefore, $\boxed{\phi(6) = 2}$

3.2 Visual Representation of Coprimality

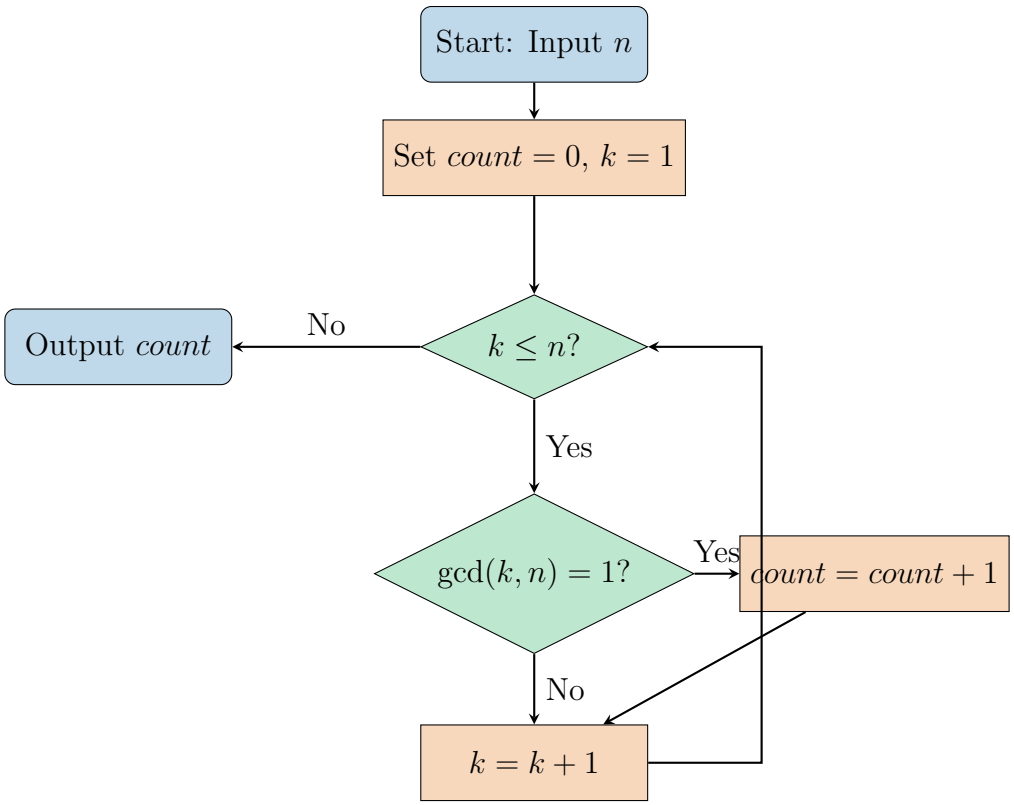


$\phi(6) = 2$ coprime numbers

4 Calculation Methods

4.1 Direct Method (Brute Force)

The most straightforward approach is to check each number from 1 to n:



Complexity: $O(n \log n)$ due to GCD calculations.

4.2 Formula-Based Method

For more efficient calculation, we use the multiplicative property of $\phi(n)$.

Theorem 4.1: Euler's Product Formula

If n has the prime factorization $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$, then:

$$\phi(n) = n \prod_{p|n, p \text{ prime}} \left(1 - \frac{1}{p}\right) = n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

Example 4.1: Using the Formula for $n = 6$

First, find the prime factorization of 6:

$$6 = 2^1 \cdot 3^1$$

The prime factors are $p_1 = 2$ and $p_2 = 3$.
Apply the formula:

$$\begin{aligned} \phi(6) &= 6 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \\ &= 6 \cdot \frac{1}{2} \cdot \frac{2}{3} \\ &= 6 \cdot \frac{2}{6} \\ &= \boxed{2} \end{aligned}$$

This matches our brute-force result!

5 Additional Examples

5.1 Example: $\phi(12)$

Example 5.1: Computing $\phi(12)$

Method 1: Direct Counting
Check each number from 1 to 12:

k	gcd(k,12)	Coprime?
1	1	Yes ✓
2	2	No
3	3	No
4	4	No
5	1	Yes ✓
6	6	No
7	1	Yes ✓
8	4	No
9	3	No
10	2	No
11	1	Yes ✓
12	12	No

Numbers coprime to 12: $\{1, 5, 7, 11\}$
Therefore, $\phi(12) = 4$
Method 2: Using the Formula
Prime factorization: $12 = 2^2 \cdot 3^1$

$$\begin{aligned}\phi(12) &= 12 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \\ &= 12 \cdot \frac{1}{2} \cdot \frac{2}{3} \\ &= 12 \cdot \frac{1}{3} \\ &= \boxed{4}\end{aligned}$$

5.2 Example: Prime Numbers

Theorem 5.1: Totient of a Prime

If p is prime, then:

$$\phi(p) = p - 1$$

Proof: Every number from 1 to $p-1$ is coprime to a prime p, since p has no divisors other than 1 and itself.

Example 5.2: $\phi(11)$

Since 11 is prime:

$$\phi(11) = 11 - 1 = \boxed{10}$$

6 Properties of Euler's Totient Function

1. **Multiplicative Property:** If $\gcd(m, n) = 1$, then:

$$\phi(m \cdot n) = \phi(m) \cdot \phi(n)$$

2. **Power of Prime:** For a prime p and positive integer k:

$$\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$$

3. **Sum Property:** For any positive integer $n > 1$:

$$\sum_{d|n} \phi(d) = n$$

where the sum is over all divisors d of n.

7 Why Do We Use It? Applications

7.1 RSA Encryption - The Most Famous Application

RSA (Rivest–Shamir–Adleman) is the most widely used public-key cryptosystem, securing almost all modern internet communications (HTTPS, SSH, email encryption, etc.). Euler's totient function is **central** to its operation.

RSA Encryption System

Key Generation

Step 1: Choose two large prime numbers p and q

Step 2: Compute $n = p \times q$

Step 3: Compute $\phi(n) = (p-1)(q-1)$ ← Uses Totient!

Step 4: Choose e such that $1 < e < \phi(n)$ and $\gcd(e, \phi(n)) = 1$

Step 5: Find d such that $d \cdot e \equiv 1 \pmod{\phi(n)}$

Public Key: (n, e) **Private Key:** (n, d)

Encryption
Message: M
Ciphertext: C
$C \equiv M^e \pmod{n}$

Send C →

Decryption
Ciphertext: C
Message: M
$M \equiv C^d \pmod{n}$

7.1.1 Concrete RSA Example

Example 7.1: Simple RSA Encryption

Key Generation:

1. Choose primes: $p = 5$, $q = 11$
2. Compute: $n = 5 \times 11 = 55$
3. Compute totient: $\phi(55) = \phi(5) \times \phi(11) = 4 \times 10 = 40$
4. Choose: $e = 3$ (since $\gcd(3, 40) = 1$)
5. Find d: We need $3d \equiv 1 \pmod{40}$
   Testing: $3 \times 27 = 81 = 2 \times 40 + 1 \equiv 1 \pmod{40}$
   So $d = 27$

Public Key: $(n = 55, e = 3)$
Private Key: $(n = 55, d = 27)$
Encryption: Message $M = 13$

$$\begin{aligned} C &\equiv 13^3 \pmod{55} \\ &\equiv 2197 \pmod{55} \\ &\equiv 52 \pmod{55} \end{aligned}$$

Ciphertext: $C = 52$
Decryption:

$$M \equiv 52^{27} \pmod{55}$$

Using modular exponentiation techniques:

$$M \equiv 13 \pmod{55}$$

We successfully recovered the original message!

7.2 Why Totient Function Makes RSA Secure

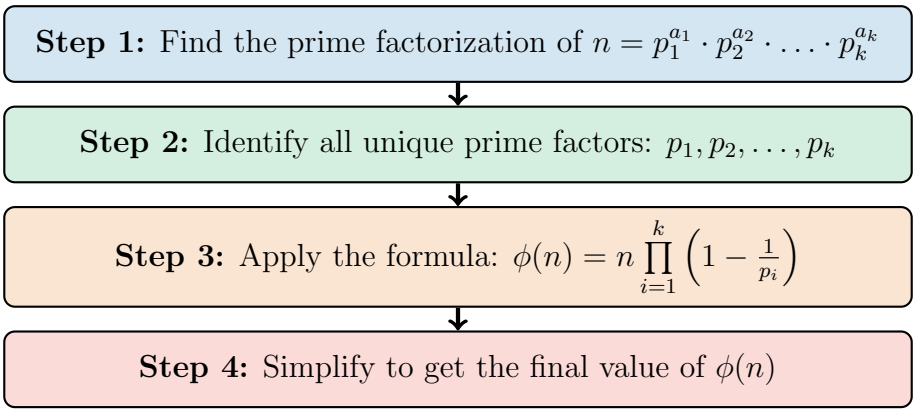
Key Point

The security of RSA relies on the fact that:

- Computing $\phi(n)$ is **easy** if you know the prime factors p and q
- Computing $\phi(n)$ is **extremely hard** if you only know n but not its factorization

This asymmetry creates the "trapdoor" that makes public-key cryptography possible.

8 Step-by-Step Algorithm Summary



9 Practice Problems

9.1 Problem 1

Calculate $\phi(20)$.

Solution:

- Prime factorization: $20 = 2^2 \times 5$
- Prime factors: 2 and 5
- Apply formula: $\phi(20) = 20 \times \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{5}\right)$
- $= 20 \times \frac{1}{2} \times \frac{4}{5} = 20 \times \frac{4}{10} = 8$

Answer: $\phi(20) = 8$

9.2 Problem 2

Calculate $\phi(100)$.

Solution:

- Prime factorization: $100 = 2^2 \times 5^2$
- Prime factors: 2 and 5
- Apply formula: $\phi(100) = 100 \times \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{5}\right)$
- $= 100 \times \frac{1}{2} \times \frac{4}{5} = 100 \times \frac{4}{10} = 40$

Answer: $\phi(100) = 40$

10 Conclusion

Euler's Totient Function is a cornerstone of number theory with profound applications in modern cryptography. Its elegant mathematical properties enable secure communication across the internet through RSA encryption. Understanding $\phi(n)$ provides insight into:

- The distribution of coprime numbers
- Multiplicative number theory
- The foundations of public-key cryptography
- Computational complexity and security

The function's dual nature—easy to compute with factorization, hard without—exemplifies the beautiful connection between pure mathematics and practical applications in computer science.

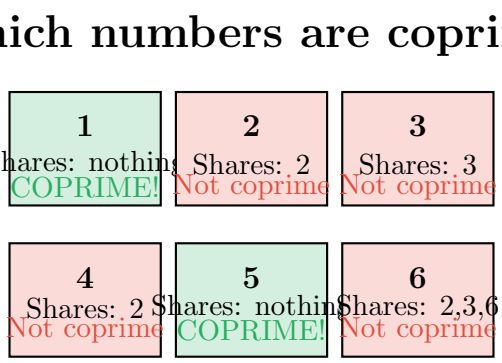
11 Simple Explanation: How the Algorithms Work

This section explains the algorithms in the simplest way possible with visual examples.

11.1 Understanding the Problem

Question: How many numbers from 1 to n don't share any factors with n?

Example with $n = 6$:



Answer: Only 1 and 5 are coprime to 6, so $\phi(6) = 2$

11.2 Method 1: Simple Counting (Brute Force)

11.2.1 The Idea in Plain English

1. Start with number 1
2. Check: Does it share factors with n? (Use GCD)
3. If GCD = 1, count it!
4. Move to next number
5. Repeat until you reach n

11.2.2 Pseudocode (Super Simple)

Simple Pseudocode

```
Start counting from 0

For each number i from 1 to n:
    Find GCD of i and n

    If GCD equals 1:
        Add 1 to count

Return count
```

11.2.3 Example Walkthrough: $\phi(6)$

Step	Number i	GCD(i,6)	Count
Start	-	-	0
1	1	1	0+1=1
2	2	2	1 (no change)
3	3	3	1 (no change)
4	4	2	1 (no change)
5	5	1	1+1=2
6	6	6	2 (no change)
Final	-	-	**2**

11.2.4 Java Code Explained Line-by-Line

Simple Java Code with Explanations

```
// Step 1: Create a counter variable
int count = 0;

// Step 2: Check every number from 1 to n
for (int i = 1; i <= n; i++) {

    // Step 3: Find GCD of current number and n
    int gcdValue = gcd(i, n);

    // Step 4: If GCD is 1, they're coprime!
    if (gcdValue == 1) {
        count++;  // Add 1 to counter
    }
}

// Step 5: Return how many we counted
return count;
```

What Each Line Does:

- `int count = 0;` - Start counting from zero
- `for (int i = 1; i <= n; i++)` - Check numbers 1,2,3,...,n
- `gcd(i, n)` - Find greatest common divisor
- `if (gcdValue == 1)` - Is it coprime?
- `count++;` - Yes! Count it
- `return count;` - Give back the answer

11.3 Method 2: Smart Formula (Optimized)

11.3.1 The Idea in Plain English

Instead of checking every number, we use a mathematical trick:

1. Find which prime numbers divide n
2. Use the formula: $\phi(n) = n \times (1 - \frac{1}{p})$ for each prime p
3. Example: If $n = 6$, primes are 2 and 3
4. $\phi(6) = 6 \times (1 - \frac{1}{2}) \times (1 - \frac{1}{3}) = 6 \times \frac{1}{2} \times \frac{2}{3} = 2$

11.3.2 Pseudocode (Super Simple)

Smart Formula Pseudocode

```
Start with result = n

For each number p from 2 to sqrt(n):

    If p divides n:
        Remove all p's from n
        Apply formula: result = result - result/p

If n is still > 1:
    Apply formula one more time

Return result
```

11.3.3 Visual Example: $\phi(12)$

Step 1: Find $12 = 2 \times 2 \times 3$

Start: $n = 12$, $result = 12$

Check $p = 2$: Does 2 divide 12? **YES!**
Remove all 2's: $12 \div 2 \div 2 = 3$

Apply formula: $result = 12 - 12/2 = 12 - 6 = 6$

Check $p = 3$: Too big (stop loop)
But n=3 is still ¿ 1, so it's a prime!

Apply formula: $result = 6 - 6/3 = 6 - 2 = 4$

Answer: $\phi(12) = 4$

11.3.4 Java Code Explained Line-by-Line

Optimized Code with Explanations

```
// Step 1: Start with result = n
int result = n;

// Step 2: Check small numbers (only up to sqrt)
for (int p = 2; p * p <= n; p++) {

    // Step 3: Does p divide n?
    if (n % p == 0) {

        // Step 4: Remove ALL occurrences of p
        while (n % p == 0) {
            n = n / p;  // Keep dividing
        }

        // Step 5: Apply the magic formula
        result = result - result / p;
    }
}

// Step 6: If n > 1, it's a big prime factor
if (n > 1) {
    result = result - result / n;
}

// Step 7: Return answer
return result;
```

What Each Part Does:

1. `result = n` - Start with the number itself

2. `p * p <= n` - Only check up to square root (saves time!)
3. `n % p == 0` - Is p a factor?
4. `while (n % p == 0)` - Remove ALL copies of p
5. `result - result/p` - Math trick instead of multiplication
6. `if (n > 1)` - Check for one large prime at the end

11.4 Why the Formula Works: Simple Explanation

Math: $\phi(n) = n \times (1 - \frac{1}{p})$

Algebra: $n \times (1 - \frac{1}{p}) = n \times \frac{p-1}{p} = n - \frac{n}{p}$
In Code: `result - result/p`
Example:

- If $n = 12$ and $p = 2$
- Formula says: $12 \times (1 - \frac{1}{2}) = 12 \times \frac{1}{2} = 6$
- Code does: `12 - 12/2 = 12 - 6 = 6`
- Same answer!

11.5 Complete Example: $\phi(20)$ Both Methods

11.5.1 Method 1: Count Every Number

Number	GCD(i,20)	Coprime?
1	1	YES
2	2	No
3	1	YES
4	4	No
5	5	No
...	...	...
19	1	YES
20	20	No

Count = 8 numbers: {1, 3, 7, 9, 11, 13, 17, 19}

11.5.2 Method 2: Use Formula

Step-by-step:

1. $20 = 2 \times 2 \times 5$
2. Primes: 2 and 5
3. Start: $result = 20$
4. For prime 2: $result = 20 - 20/2 = 20 - 10 = 10$

5. For prime 5: $result = 10 - 10/5 = 10 - 2 = 8$
6. Answer: $\phi(20) = 8$

Both methods give the same answer!

11.6 Quick Reference: Which Method to Use?

Situation	Method 1 (Counting)	Method 2 (Formula)
Learning/Understanding	BEST	Harder
Small numbers (n ¡ 100)	Good	Good
Large numbers (n ¿ 1000)	Too slow	BEST
Code simplicity	Simple	Medium
Speed	Slow	Fast

12 Algorithm Implementation

This section provides practical implementations of Euler's Totient Function, including pseudocode, flowcharts, and Java code examples.

12.1 Pseudocode Algorithms

12.1.1 Method 1: Brute Force Approach

Algorithm 1: Brute Force Totient

```
Algorithm: EulerTotientBruteForce(n)
Input: A positive integer n
Output: phi(n) - count of numbers coprime to n

1. count <- 0
2. for i <- 1 to n do
3.     if gcd(i, n) = 1 then
4.         count <- count + 1
5.     end if
6. end for
7. return count

Time Complexity: O(n log n)
Space Complexity: O(1)
```

12.1.2 Method 2: Optimized Using Prime Factorization

Algorithm 2: Optimized Totient (Formula-Based)

```
Algorithm: EulerTotientOptimized(n)
Input: A positive integer n
Output: phi(n) using Euler's product formula

1. result <- n
2. temp <- n
3. p <- 2
4. while p * p <= temp do
5.     if temp mod p = 0 then
6.         // Remove all occurrences of p
7.         while temp mod p = 0 do
8.             temp <- temp / p
9.         end while
10.        // Apply formula: result *= (1 - 1/p)
11.        result <- result - result / p
12.    end if
13.    p <- p + 1
14. end while
15. // If temp > 1, then it's a prime factor
16. if temp > 1 then
17.     result <- result - result / temp
18. end if
19. return result

Time Complexity: O(sqrt(n))
Space Complexity: O(1)
```

12.2 Java Implementation

12.2.1 Method 1: Brute Force Implementation

Java Code: Brute Force Method

```
public class EulerTotient {

    /**
     * Computes GCD using Euclidean Algorithm
     * @param a First number
     * @param b Second number
     * @return GCD of a and b
     */
    public static int gcd(int a, int b) {
        if (b == 0) {
            return a;
```

```
        }
        return gcd(b, a % b);
    }

    /**
     * Computes Euler's Totient using brute force
     * Time Complexity: O(n log n)
     * @param n Input number
     * @return phi(n)
     */
    public static int phiBruteForce(int n) {
        int count = 0;

        // Check each number from 1 to n
        for (int i = 1; i <= n; i++) {
            if (gcd(i, n) == 1) {
                count++;
            }
        }

        return count;
    }

    // Example usage
    public static void main(String[] args) {
        int n = 6;
        System.out.println("phi(" + n + ") = "
                           + phiBruteForce(n));
        // Output: phi(6) = 2

        n = 12;
        System.out.println("phi(" + n + ") = "
                           + phiBruteForce(n));
        // Output: phi(12) = 4
    }
}
```

12.2.2 Method 2: Optimized Implementation

Java Code: Optimized Method (Formula-Based)

```
public class EulerTotientOptimized {

    /**
     * Computes Euler's Totient using prime factorization
     * Time Complexity: O(sqrt(n))
     * @param n Input number
```

```
     * @return phi(n)
     */
    public static int phiOptimized(int n) {
        int result = n;  // Initialize result as n

        // Consider all prime factors of n
        for (int p = 2; p * p <= n; p++) {

            // Check if p is a prime factor
            if (n % p == 0) {

                // Remove all occurrences of p
                while (n % p == 0) {
                    n = n / p;
                }

                // Apply formula: phi(n) *= (1 - 1/p)
                // Equivalent to: result = result - result/p
                result = result - result / p;
            }
        }

        // If n is still greater than 1,
        // then it's a prime factor
        if (n > 1) {
            result = result - result / n;
        }

        return result;
    }

    // Example usage with multiple test cases
    public static void main(String[] args) {
        int[] testCases = {6, 12, 20, 100, 11};

        System.out.println("Euler's Totient Function");
        System.out.println("========================");

        for (int n : testCases) {
            System.out.printf("phi(%3d) = %3d%n",
                              n, phiOptimized(n));
        }

        /* Output:
         * Euler's Totient Function
         * ========================
         * phi(  6) =   2
```

```
         * phi( 12) =   4
         * phi( 20) =   8
         * phi(100) =  40
         * phi( 11) =  10
         */
    }
}
```

12.2.3 Complete Program with Both Methods

Complete Java Program

```
public class TotientCalculator {

    // GCD using Euclidean algorithm
    public static int gcd(int a, int b) {
        return (b == 0) ? a : gcd(b, a % b);
    }

    // Brute force method
    public static int phiBruteForce(int n) {
        int count = 0;
        for (int i = 1; i <= n; i++) {
            if (gcd(i, n) == 1) count++;
        }
        return count;
    }

    // Optimized method
    public static int phiOptimized(int n) {
        int result = n;
        for (int p = 2; p * p <= n; p++) {
            if (n % p == 0) {
                while (n % p == 0) n /= p;
                result -= result / p;
            }
        }
        if (n > 1) result -= result / n;
        return result;
    }

    // Main method with comparison
    public static void main(String[] args) {
        int n = 100;

        long startTime, endTime;
```

```
        // Test brute force
        startTime = System.nanoTime();
        int result1 = phiBruteForce(n);
        endTime = System.nanoTime();
        System.out.println("Brute Force:");
        System.out.println("  Result: phi(" + n
                           + ") = " + result1);
        System.out.println("  Time: "
                           + (endTime - startTime) + " ns");

        // Test optimized
        startTime = System.nanoTime();
        int result2 = phiOptimized(n);
        endTime = System.nanoTime();
        System.out.println("\nOptimized:");
        System.out.println("  Result: phi(" + n
                           + ") = " + result2);
        System.out.println("  Time: "
                           + (endTime - startTime) + " ns");
    }
}
```

12.3 Understanding the Code

12.3.1 Key Points

1. **GCD Function:** Uses Euclidean algorithm with recursion. The base case is when $b = 0$.

2. **Brute Force:** Simple and intuitive. Checks every number from 1 to n. Good for understanding but slow for large n.

3. **Optimized Method:** Uses Euler's product formula. Only checks up to $\sqrt{n}$, making it much faster.

4. **Formula Application:** Instead of $(1 - \frac{1}{p})$, we use `result - result/p` to avoid floating-point arithmetic.

12.3.2 Time Complexity Comparison

Method	Time Complexity	Best For
Brute Force	$O(n \log n)$	Small values, learning
Optimized	$O(\sqrt{n})$	Large values, production

13 References

1. Euler, L. (1763). "Theoremata arithmetica nova methodo demonstrata"

2. Rivest, R., Shamir, A., & Adleman, L. (1978). "A Method for Obtaining Digital Signatures and Public-Key Cryptosystems"

3. Hardy, G. H., & Wright, E. M. (2008). "An Introduction to the Theory of Numbers"

4. Apostol, T. M. (1976). "Introduction to Analytic Number Theory"