Chapter Three

Rigid Body Kinematics

ATTITUDE coordinates (sometimes also referred to as attitude parameters) are sets of coordinates $\{x_1, x_2, \ldots, x_n\}$ that completely describe the orientation of a rigid body relative to some reference coordinate frame. There is an infinite number of attitude coordinates to choose from. Each set has strengths and weaknesses compared to other sets. This is analogous to choosing among the infinite sets of translational coordinates such as cartesian, polar or spherical coordinates to describe a spatial position of a point. However, describing the attitude of an object relative to some reference frame does differ in a fundamental way from describing the corresponding relative spatial position of a point. In cartesian space, the linear displacement between two spatial positions can grow arbitrarily large. On the other hand two rigid body (or coordinate frame) orientations can differ at most by a 180° rotation, a finite rotational displacement. If an object rotates past 180°, then its orientation actually starts to approach the starting angular position again. This concept of two orientations only being able to differ by finite rotations is important when designing control laws. A smart choice in attitude coordinates will be able to exploit this fact and produce a control law that is able to intelligently handle very large orientation errors.

The quest for "*the best rigid body orientation description*" is a very fundamental and important one. It has been studied by such great scholars as Euler, Jacobi, Hamilton, Cayley, Klein, Rodrigues and Gibbs and has led to a rich collection of elegant results. A good choice for attitude coordinates can greatly simplify the mathematics and avoid such pitfalls as mathematical and geometrical singularities or highly nonlinear kinematic differential equations. Among other things, a bad choice of attitude coordinates can artificially limit the operational range of a controlled system by requiring it to operate within the non-singular range of the chosen attitude parameters.

The following list contains four truths about rigid body attitude coordinates that are listed without proof.[1]

1. A minimum of three coordinates is required to describe the relative angular displacement between two reference frames $\mathcal{F}_1$ and $\mathcal{F}_2$.

2. Any minimal set of three attitude coordinates will contain at least one geometrical orientation where the coordinates are singular, namely at least two coordinates are undefined or not unique.

3. At or near such a geometric singularity, the corresponding kinematic differential equations are also singular.

4. The geometric singularities and associated numerical difficulties can be avoided altogether through a regularization.[2] Redundant sets of four or more coordinates exist which are universally determined and contain no geometric singularities.

3.1 Direction Cosine Matrix

Rigid body orientations are described using displacements of body-fixed referenced frames. The reference frame itself is usually defined using a set of three orthogonal, right-handed unit vectors. For notational purposes, a reference frame (or rigid body) is labeled through a script capital letter such as $\mathcal{F}$ and its associated unit base vectors are labeled with lower case letters such as $\hat{\boldsymbol{f}}_i$. There is always an infinity of ways to attach a reference frame to a rigid body. However, typically the reference frame base vectors are chosen such that they are aligned with the principal body axes.

Let the two reference frames $\mathcal{N}$ and $\mathcal{B}$ each be defined through sets of orthonormal right-handed sets of vectors $\{\hat{\boldsymbol{n}}\}$ and $\{\hat{\boldsymbol{b}}\}$ where we use the shorthand *vectrix* notation

$$\{\hat{\boldsymbol{n}}\} \equiv \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_2 \end{Bmatrix} \qquad \{\hat{\boldsymbol{b}}\} \equiv \begin{Bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_2 \end{Bmatrix} \tag{3.1}$$

The sets of unit vectors are shown in Figure 3.1. The reference frame $\mathcal{B}$ can be thought of being a generic rigid body and the reference frame $\mathcal{N}$ could be associated with some particular inertial coordinate system. Let the three angles α_{1i} be the angles formed between the first body vector $\hat{\boldsymbol{b}}_1$ and the three inertial axes. The cosines of these angles are called the direction cosines of $\hat{\boldsymbol{b}}_1$ relative to the $\mathcal{N}$ frame. The unit vector $\hat{\boldsymbol{b}}_1$ can be projected onto $\{\hat{\boldsymbol{n}}\}$ as

$$\hat{\boldsymbol{b}}_1 = \cos\alpha_{11}\hat{\boldsymbol{n}}_1 + \cos\alpha_{12}\hat{\boldsymbol{n}}_2 + \cos\alpha_{13}\hat{\boldsymbol{n}}_3 \tag{3.2}$$

Clearly the direction cosines $\cos\alpha_{1j}$ are the three orthogonal components of $\hat{\boldsymbol{b}}_j$. Analogously, the direction angles α_{2i} and α_{3i} between the unit vectors $\hat{\boldsymbol{b}}_2$ and $\hat{\boldsymbol{b}}_3$ and the reference frame $\mathcal{N}$ base vectors can be found. These vectors are then expressed as

$$\hat{\boldsymbol{b}}_2 = \cos\alpha_{21}\hat{\boldsymbol{n}}_1 + \cos\alpha_{22}\hat{\boldsymbol{n}}_2 + \cos\alpha_{23}\hat{\boldsymbol{n}}_3 \tag{3.3}$$

$$\hat{\boldsymbol{b}}_3 = \cos\alpha_{31}\hat{\boldsymbol{n}}_1 + \cos\alpha_{32}\hat{\boldsymbol{n}}_2 + \cos\alpha_{33}\hat{\boldsymbol{n}}_3 \tag{3.4}$$

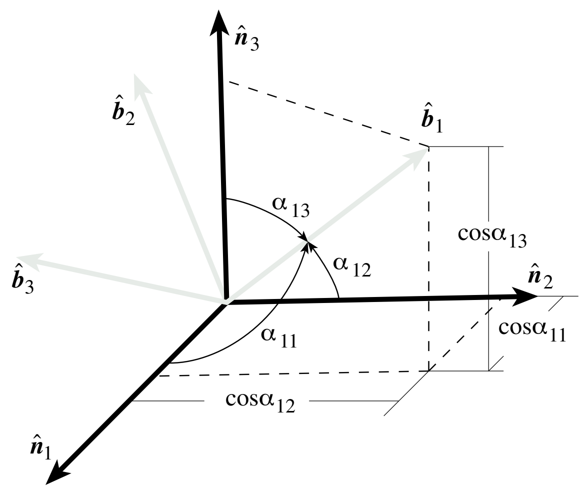

Figure 3.1: Direction Cosines

The set of orthonormal base vectors $\{\hat{\boldsymbol{b}}\}$ can be compactly expressed in terms of the base vectors $\{\hat{\boldsymbol{n}}\}$ as

$$\{\hat{\boldsymbol{b}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{n}}\} = [C]\{\hat{\boldsymbol{n}}\} \tag{3.5}$$

where the matrix $[C]$ is called the *direction cosine matrix*. Note that each entry of $[C]$ can be computed through

$$C_{ij} = \cos(\angle\, \hat{\boldsymbol{b}}_i, \hat{\boldsymbol{n}}_j) = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j \tag{3.6}$$

Analogously to Eq. (3.5), the set of $\{\hat{\boldsymbol{n}}\}$ vectors can be projected onto $\{\hat{\boldsymbol{b}}\}$ vectors as

$$\{\hat{\boldsymbol{n}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{21} & \cos\alpha_{31} \\ \cos\alpha_{12} & \cos\alpha_{22} & \cos\alpha_{32} \\ \cos\alpha_{13} & \cos\alpha_{23} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{b}}\} = [C]^T\{\hat{\boldsymbol{b}}\} \tag{3.7}$$

Substituting Eq. (3.7) into (3.5) yields

$$\{\hat{\boldsymbol{b}}\} = [C][C]^T\{\hat{\boldsymbol{b}}\} \tag{3.8}$$

which requires that

$$[C][C]^T = [I_{3\times 3}] \tag{3.9}$$

Similarly substituting Eq. (3.5) into (3.7) yields

$$[C]^T[C] = [I_{3\times 3}] \tag{3.10}$$

Eqs. (3.9) and (3.10) show that the direction cosine matrix $[C]$ is orthogonal.[1, 3–5] Therefore the inverse of $[C]$ is the transpose of $[C]$.

$$[C]^{-1} = [C]^T \tag{3.11}$$

Thanks to the orthogonality of the direction cosine matrix $[C]$, we will see below that the forward and inverse transformation (projection) of vectors between rotationally displaced reference frames can be accomplished without arithmetic.

Another important property of the direction cosine matrix is that its determinant is ± 1. This can be shown as follows.[5] From Eq. (3.9) it is evident that

$$\det\left(CC^T\right) = \det\left([I_{3\times 3}]\right) = 1 \tag{3.12}$$

Since $[C]$ is a square matrix this can be written as[6]

$$\det(C)\det(C^T) = 1 \tag{3.13}$$

Since $\det(C)$ is the same as $\det(C^T)$, this is further reduced to[7]

$$\left(\det(C)\right)^2 = 1 \Longleftrightarrow \det(C) = \pm 1 \tag{3.14}$$

As is shown by Goldstein in Ref. 8, if the reference frame base vectors $\{\hat{\boldsymbol{b}}\}$ and $\{\hat{\boldsymbol{n}}\}$ are right-handed, then $\det(C) = +1$. Goldstein also shows that the 3x3 direction cosine matrix $[C]$ will only have one real eigenvalue of ± 1. Again it will be $+1$ if the reference frame base vectors are right-handed.

In a standard coordinate transformation setting, the $[C]$ matrix is typically not restricted to projecting one set of base vectors from one reference frame onto another. Rather, the direction cosine's most powerful feature is the ability to directly project (or transform) an arbitrary vector, with components written in one reference frame, into a vector with components written in another reference frame. To show this let $\boldsymbol{v}$ be an arbitrary vector and let the reference frames $\mathcal{B}$ and $\mathcal{N}$ be defined as earlier. Let the scalars v_{b_i} be the vector components of $\boldsymbol{v}$ in the $\mathcal{B}$ reference frame.

$$\boldsymbol{v} = v_{b_1}\hat{\boldsymbol{b}}_1 + v_{b_2}\hat{\boldsymbol{b}}_2 + v_{b_3}\hat{\boldsymbol{b}}_3 = \{v_b\}^T\{\hat{\boldsymbol{b}}\} \tag{3.15}$$

Similarly $\boldsymbol{v}$ can be written in terms of $\mathcal{N}$ frame components v_{n_i} as

$$\boldsymbol{v} = v_{n_1}\hat{\boldsymbol{n}}_1 + v_{n_2}\hat{\boldsymbol{n}}_2 + v_{n_3}\hat{\boldsymbol{n}}_3 = \{v_n\}^T\{\hat{\boldsymbol{n}}\} \tag{3.16}$$

Substituting Eq. (3.7) into Eq. (3.16) the $\boldsymbol{v}$ vector components in the $\mathcal{N}$ frame can be directly projected into the $\mathcal{B}$ frame.

$$\boldsymbol{v}_b = [C]\boldsymbol{v}_n \tag{3.17}$$

Since the inverse of $[C]$ is simply $[C]^T$, the inverse transformation is

$$\boldsymbol{v}_n = [C]^T\boldsymbol{v}_b \tag{3.18}$$

The fact that Eqs. (3.17) and (3.18) are exactly analogous to Eqs. (3.5) and (3.7) is a fundamental property of Gibbsian vectors, and more generally, cartesian tensors.

Another common problem is that several cascading reference frames are present where each reference frame orientation is defined relative to the previous one, and it is desired to replace the sequence of projections by a single projection. Let $\{\hat{\boldsymbol{r}}\}$ contain the base vectors of the reference frame $\mathcal{R}$ whose relative orientation to the $\mathcal{B}$ frame is given through $[C']$.

$$\{\hat{\boldsymbol{r}}\} = [C']\{\hat{\boldsymbol{b}}\} \tag{3.19}$$

The basis vectors $\{\hat{\boldsymbol{n}}\}$ in the $\mathcal{N}$ frame can be projected directly into the $\mathcal{R}$ frame through

$$\{\hat{\boldsymbol{r}}\} = [C'][C]\{\hat{\boldsymbol{n}}\} = [C'']\{\hat{\boldsymbol{n}}\} \tag{3.20}$$

where the direction cosine matrix $[C''] = [C'][C]$ projects vectors in the $\mathcal{N}$ frame to vectors in the $\mathcal{R}$ frame. The direct transformation matrix from the first to the last cascading reference frame is clearly found by successive matrix-multiplications of each relative transformation matrix in reverse order as shown above. This property $[C''] = [C'][C]$ for composition of successive rotations is very important. When rotational coordinates are introduced to parameterize the $[C]$ matrix, the corresponding "composition" relationship among the three sets of coordinates is also of fundamental importance.

The direction cosine matrix is the most fundamental, but highly redundant, method of describing a relative orientation. As was mentioned earlier, the minimum number of parameter required to describe a reference frame orientation is three. The direction cosine matrix has nine entries. The six extra parameters in the matrix are made redundant through the orthogonality condition $[C][C]^T = [I_{3\times 3}]$. This is why in practice the elements of the direction cosine matrix are rarely used as coordinates to keep track of an orientation; instead less redundant attitude parameters are used. The biggest asset of the direction cosine matrix is the ability to easily transform vectors from one reference frame to another.

Example 3.1: Let the two reference frames $\mathcal{B}$ and $\mathcal{F}$ be defined relative to the inertial reference frame $\mathcal{N}$ by the orthonormal unit base vectors

$$\hat{\boldsymbol{b}}_1 = (0,1,0)^T \qquad \hat{\boldsymbol{b}}_2 = (1,0,0)^T \qquad \hat{\boldsymbol{b}}_3 = (0,0,-1)^T$$

$$\hat{\boldsymbol{f}}_1 = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right)^T \qquad \hat{\boldsymbol{f}}_2 = (0,0,1)^T \qquad \hat{\boldsymbol{f}}_3 = \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}, 0\right)^T$$

where the $\hat{\boldsymbol{b}}_i$ and $\hat{\boldsymbol{f}}_i$ vector components are written in the inertial $\mathcal{N}$ frame. Let us use the following notation to label the various direction cosine matrices. The matrix $[BN]$ maps vectors written in the $\mathcal{N}$ frame into vectors written in the $\mathcal{B}$ frame. Analogously, the matrix $[FB]$ maps vectors in the $\mathcal{B}$ frame into $\mathcal{F}$ frame vectors and so on. To find the entries of the various relative rotation matrices, note the following useful identity.

$$[FB]_{ij} = \cos\alpha_{ij} = \hat{\boldsymbol{f}}_i \cdot \hat{\boldsymbol{b}}_j$$

Given the base vectors of each frame, it is not necessary to find the angles between each set of vectors to find the appropriate direction cosine matrix. Since all base vectors have unit length, the inner product of the corresponding vectors will provide the needed direction cosines. The rotation matrices $[BN]_{ij} = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j$, $[FN]_{ij} = \hat{\boldsymbol{f}}_i \cdot \hat{\boldsymbol{n}}_j$ and $[FB]_{ij} = \hat{\boldsymbol{f}}_i \cdot \hat{\boldsymbol{b}}_j$ are

$$[BN] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad [FN] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

$$[FB] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

Instead of calculating the rotation matrix $[FB]$ from dot products of the respective base vectors, it could also be calculated using Eq. (3.20).

$$[FB] = [FN][BN]^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \quad \surd$$

To find the kinematic differential equation in terms of the direction cosine matrix $[C]$, let us write the instantaneous angular velocity vector $\boldsymbol{\omega}$ of the $\mathcal{B}$ frame relative to the $\mathcal{N}$ frame in $\mathcal{B}$ frame orthogonal components as

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3 \tag{3.21}$$

Let ${}^{\mathcal{N}}d/dt\{\hat{\boldsymbol{b}}\}$ be the derivative of the $\mathcal{B}$ frame base vectors taken in the $\mathcal{N}$ frame. Using the transport theorem we find[9]

$$\frac{{}^{\mathcal{N}}d}{dt}\{\hat{\boldsymbol{b}}_i\} = \frac{{}^{\mathcal{B}}d}{dt}\{\hat{\boldsymbol{b}}_i\} + \boldsymbol{\omega} \times \{\hat{\boldsymbol{b}}_i\} \tag{3.22}$$

Since the $\mathcal{B}$ frame base vectors are fixed within their frame the expression ${}^{\mathcal{B}}d/dt\{\hat{\boldsymbol{b}}\}$ is zero. After introducing the skew-symmetric tilde matrix operator

$$[\tilde{\boldsymbol{x}}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{3.23}$$

Eq. (3.22) leads to the vectrix equation

$$\frac{{}^{\mathcal{N}}d}{dt}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\} \tag{3.24}$$

Taking the time derivative of the right hand side of Eq. (3.5) we find

$$\frac{{}^{\mathcal{N}}d}{dt}\left([C]\{\hat{\boldsymbol{n}}\}\right) = \frac{d}{dt}\left([C]\right)\{\hat{\boldsymbol{n}}\} + [C]\frac{{}^{\mathcal{N}}d}{dt}\left(\{\hat{\boldsymbol{n}}\}\right) = [\dot{C}]\{\hat{\boldsymbol{n}}\} \tag{3.25}$$

where the short hand notation $d/dt([C]) = [\dot{C}]$ is used. Using Eq. (3.5), Eqs. (3.24) and (3.25) are combined to

$$\left([\dot{C}] + [\tilde{\boldsymbol{\omega}}][C]\right)\{\hat{\boldsymbol{n}}\} = 0 \tag{3.26}$$

Since Eq. (3.26) must hold for any set of $\{\hat{\boldsymbol{n}}\}$, the kinematic differential equation satisfied by the direction cosine matrix $[C]$ is found to be[1, 10]

$$[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C] \tag{3.27}$$

It can easily be verified that Eq. (3.9) is indeed an exact solution of above differential equation. Take the derivative of $[C][C]^T$

$$\frac{d}{dt}\left([C][C]^T\right) = [\dot{C}][C]^T + [C][\dot{C}]^T \tag{3.28}$$

and then substitute Eq. (3.27) to obtain

$$\frac{d}{dt}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}][C][C]^T - [C][C]^T[\tilde{\boldsymbol{\omega}}]^T \tag{3.29}$$

Making use of the orthogonality of $[C]$ and since $[\tilde{\boldsymbol{\omega}}] = -[\tilde{\boldsymbol{\omega}}]^T$ is skew-symmetric this simplifies to

$$\frac{d}{dt}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}] + [\tilde{\boldsymbol{\omega}}] = 0 \tag{3.30}$$

Since $[C][C]^T$ is a constant solution of the differential equation in Eq. (3.27), and Eq. (3.9) is satisfied initially, the solution of Eq. (3.27) will theoretically satisfy the orthogonality condition for all time. In practice, numerical solutions of Eq. (3.27) will slowly accumulate arithmetic errors so that the orthogonality condition $[C][C]^T - [I_{3\times3}] = 0$ is slightly in error. There are several ways to resolve this minor difficulty.

Given an arbitrary time history of $\boldsymbol{\omega}(t)$, Eq. (3.27) represents a rigorously linear differential equation which can be integrated to yield the instantaneous direction cosine matrix $[C]$. A major advantage of the kinematic differential equation for $[C]$ is that it is linear and universally applicable. There are no geometric singularities present in the attitude description or its kinematic differential equations. However, this advantage comes at the cost of having a highly redundant formulation. Several other attitude parameters will be presented in the following sections which include a minimal number (3) of attitude parameters. However, all minimal sets of attitude coordinates have kinematic differential equations which contain some degree of nonlinearity and also embody geometric and/or mathematical singularities. Only the once redundant Euler parameters (quaternions) will be found to retain a singularity free description and possess linear kinematic differential equations analogous to the direction cosine matrix.

3.2 Euler Angles

The most commonly used sets of attitude parameters are the Euler angles. They describe the attitude of a reference frame $\mathcal{B}$ relative to the frame $\mathcal{N}$ through three successive rotation angles $(\theta_1, \theta_2, \theta_3)$ about the sequentially displaced body fixed axes $\{\hat{\boldsymbol{b}}\}$. Note that the order of the axes about which the reference frame is rotated is important here. Performing three successive rotations about the 3rd, 2nd and 1st body axis, labeled (3-2-1) for short, does not yield the same orientation as if instead the rotation order is (1-2-3). Note these sequential rotations provide an instantaneous geometrical recipe for $\mathcal{N}$. Clearly, for $\mathcal{B}$ undergoing general motion, the $\theta_i(t)$ are time varying in a general way.

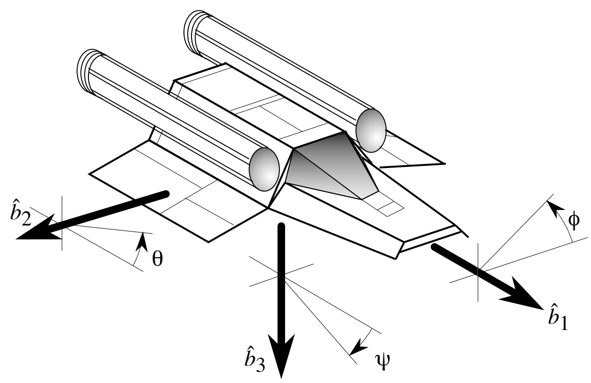

Figure 3.2: Yaw, Pitch and Roll Euler Angles

Aircraft and spacecraft orientations are commonly described through the Euler angles yaw, pitch and roll (ψ,θ,ϕ) as shown in Figure 3.2. They are usually measured relative to axes associated with a nominal flight path. The position of $\{\hat{\boldsymbol{b}}\}$ relative to $\{\hat{\boldsymbol{n}}\}$ is *described* by a sequence of three rigid rotations about prescribed body fixed axes. While the conceptual description is a sequence of rotations, we can consider the *instantaneous* values of these three angles and thereby establish a means for describing general, non-sequential rotations. The popularity of Euler angles stems from the fact that the relative attitude is easy to visualize for small angles. To transform components of a vector in the $\mathcal{N}$ frame into the $\mathcal{B}$ frame through a sequence of Euler angle rotations, the reference axes are first rotated about the $\hat{\boldsymbol{b}}_3$ axis by the yaw angle ψ, then about the $\hat{\boldsymbol{b}}_2$ axis by the pitch angle θ and finally about the $\hat{\boldsymbol{b}}_1$ axis by the roll angle ϕ as is shown in Figure 3.3. Thus the standard yaw-pitch-roll (ψ,θ,ϕ) angles are the (3-2-1) set of Euler angles.[11]

Another very popular set of Euler angles is the (3-1-3) set of Euler angles. These angles are commonly used by astronomers to define the orientation of orbit planes of the planets relative to the Earth's orbit plane.[1] While the (3-2-1) Euler angles are considered an *asymmetric* set, the (3-1-3) Euler angles are

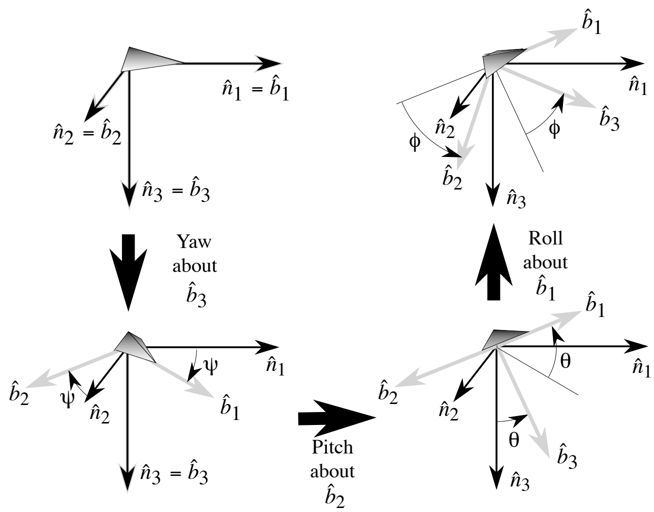

Figure 3.3: Successive Yaw, Pitch and Roll Rotations

a *symmetric* set since two rotations about the third body axis are performed. Instead of being called yaw, pitch and roll angles, the (3-1-3) Euler angles are called longitude of the ascending node Ω, inclination i and argument of the perihelion ω and are illustrated in Figure 3.4 below.[1, 12]

The direction cosine matrix introduced in section 3.1 can be parameterized in terms of the Euler angles. Since each Euler angle defines a successive rotation about the i-th body axis, let the three single-axis rotation matrices $[M_i(\theta)]$ be defined as

$$[M_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{3.31a}$$

$$[M_2(\theta)] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3.31b}$$

$$[M_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.31c}$$

Let the (α, β, γ) Euler angle sequence be $(\theta_1, \theta_2, \theta_3)$. Using Eq. (3.20) to combine successive rotations, the direction cosine matrix in terms of the (α, β, γ) Euler angles can be written as[1]

$$[C(\theta_1, \theta_2, \theta_3)] = [M_\gamma(\theta_3)][M_\beta(\theta_2)][M_\alpha(\theta_1)] \tag{3.32}$$

In particular, the direction cosine matrix in terms of the (3-2-1) Euler angles $(\theta_1, \theta_2, \theta_3) = (\psi, \theta, \phi)$ is[11]

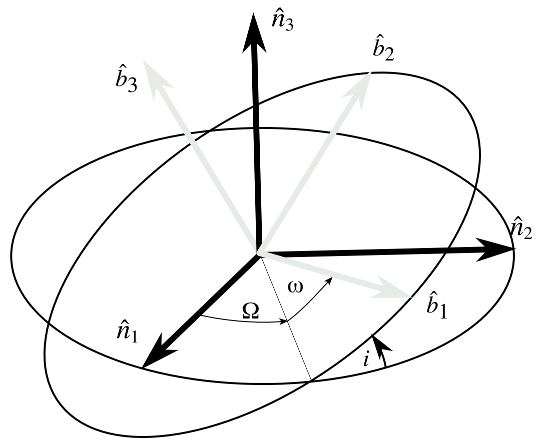

Figure 3.4: (3-1-3) Euler Angle Illustration

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$$[C] = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix} \tag{3.33}$$

where the short hand notation $c\xi = \cos\xi$ and $s\xi = \sin\xi$ is used. The inverse transformations from the direction cosine matrix $[C]$ to the (ψ, θ, ϕ) angles are

$$\psi = \theta_1 = \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right) \tag{3.34a}$$

$$\theta = \theta_2 = -\sin^{-1}(C_{13}) \tag{3.34b}$$

$$\phi = \theta_3 = \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right) \tag{3.34c}$$

In terms of the (3-1-3) Euler angles $(\theta_1, \theta_2, \theta_3) = (\Omega, i, \omega)$ the direction cosine matrix $[C]$ is written as[1]

$$[C] = \begin{bmatrix} c\theta_3 c\theta_1 - s\theta_3 c\theta_2 s\theta_1 & c\theta_3 s\theta_1 + s\theta_3 c\theta_2 c\theta_1 & s\theta_3 s\theta_2 \\ -s\theta_3 c\theta_1 - c\theta_3 c\theta_2 s\theta_1 & -s\theta_3 s\theta_1 + c\theta_3 c\theta_2 c\theta_1 & c\theta_3 s\theta_2 \\ s\theta_2 s\theta_1 & -s\theta_2 c\theta_1 & c\theta_2 \end{bmatrix} \tag{3.35}$$

The inverse transformations from the direction cosine matrix $[C]$ to the (3-1-3) Euler angles (Ω, i, ω) are

$$\Omega = \theta_1 = \tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right) \tag{3.36a}$$

$$i = \theta_2 = \cos^{-1}(C_{33}) \tag{3.36b}$$

$$\omega = \theta_3 = \tan^{-1}\left(\frac{C_{13}}{C_{23}}\right) \tag{3.36c}$$

The complete set of 12 transformations between the various Euler angle sets and the direction cosine matrix can be found in the Appendix C. We emphasize that while Eqs. (3.32)-(3.36) are easily established by sequential angular displacements, we consider the inverse situation; given a generally varying $[C]$ matrix, we can consider equations such as Eqs. (3.32)-(3.36) to hold at any instant in the motion, and thus $\{\psi(t), \theta(t), \phi(t)\}$ or $\{\Omega(t), i(t), \omega(t)\}$ can be considered as candidate coordinates for general rotational motion.

Note that each of the 12 possible sets of Euler angles has a geometric singularity where two angles are not uniquely defined. For the (3-2-1) Euler angles pitching up or down 90 degrees results in a geometric singularity. If the pitch angle is ± 90 degrees, then it does not matter if ψ=0 and ϕ=10 degrees or $\psi =$ 10 and $\phi = 0$ degrees. Only the sum $\psi + \phi$ is unique in this case. For the (3-1-3) Euler angles the geometric singularity occurs for an inclination angle of zero or 180 degrees. This geometric singularity also manifests itself in a mathematical singularity of the corresponding Euler angle kinematic differential equation.

Let $\boldsymbol{\theta} = \{\theta_1, \theta_2, \theta_3\}$ and $\boldsymbol{\phi} = \{\phi_1, \phi_2, \phi_3\}$ be two Euler angle vectors with identical rotation sequences. Often it is necessary to find the attitude that corresponds to performing two *successive* rotations, i.e. "adding" the two rotations. If a rigid body first performs the rotation $\boldsymbol{\theta}$ and then the rotation $\boldsymbol{\phi}$, then the final attitude is expressed relative to the original attitude through the vector $\boldsymbol{\varphi} = \{\varphi_1, \varphi_2, \varphi_3\}$ defined through

$$[FN(\boldsymbol{\varphi})] = [FB(\boldsymbol{\phi})][BN(\boldsymbol{\theta})] \tag{3.37}$$

Eq. (3.37) could be used to solve for $\boldsymbol{\varphi}$ in terms of the vector components of $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$. This process is very tedious and typically does not provide any simple, compact final expressions. However, for the case where $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ are vectors of symmetric Euler angles, then it is possible to obtain relatively compact transformations from the first two vectors into the overall vector using spherical geometry relationships.[5, 13]

A sample spherical triangle is shown in Figure 3.5. The following two spherical triangle laws are the only two required in deriving the symmetrical Euler angle successive rotation property. The *spherical law of sines* states that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \tag{3.38}$$

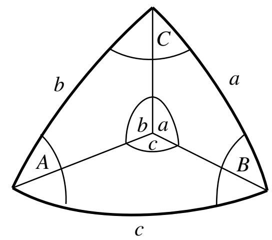
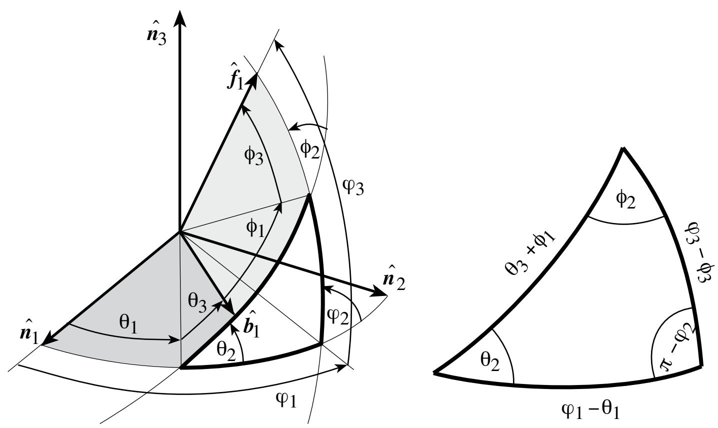


Figure 3.5: Spherical Triangle Labels



(i) Successive (3-1-3) Euler Angles **(ii)** Spherical Triangle

Figure 3.6: Illustration of Successive (3-1-3) Euler Angle Rotations

and the *spherical law of cosines* states that

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \tag{3.39a}$$

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b \tag{3.39b}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \tag{3.39c}$$

Figure 3.6(i) illustrates the orientation of the first body axis as it is first rotated from $\mathcal{N}$ to $\mathcal{B}$ with the (3-1-3) Euler angle vector $\boldsymbol{\theta}$ and then from $\mathcal{B}$ to $\mathcal{F}$ with the (3-1-3) vector $\boldsymbol{\phi}$. The (3-1-3) Euler angle description of the direct rotation from $\mathcal{N}$ to $\mathcal{F}$ is clearly given by the angles φ_1, φ_2 and φ_3. To obtain direct transformations from $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ to $\boldsymbol{\varphi}$, the bold spherical triangle in Figure 3.6(i) is used. The spherical arc lengths and angles of this triangle are labeled in Figure 3.6(ii). Using the spherical law of cosines we find that

$$\cos(\pi - \varphi_2) = -\cos\theta_2 \cos\phi_2 + \sin\theta_2 \sin\phi_2 \cos(\theta_3 + \phi_1) \tag{3.40}$$

This is trivially solved for the angle φ_2 as

$$\varphi_2 = \cos^{-1}\left(\cos\theta_2\cos\phi_2 - \sin\theta_2\sin\phi_2\cos(\theta_3+\phi_1)\right) \tag{3.41}$$

Using the spherical laws of sines, we are able to find the following expressions for φ_1 and φ_3:

$$\sin(\varphi_1-\theta_1) = \frac{\sin\phi_2}{\sin\varphi_2}\sin(\theta_3+\phi_1) \tag{3.42}$$

$$\sin(\varphi_3-\phi_3) = \frac{\sin\theta_2}{\sin\varphi_2}\sin(\theta_3+\phi_1) \tag{3.43}$$

To avoid quadrant problems, we prefer to find expressions of φ_1 and φ_3 that involve the tan function instead of the sin function. To accomplish this, using the spherical law of cosines we find the following two relationships:

$$\cos(\varphi_1-\theta_1) = \frac{\cos\phi_2 - \cos\theta_2\cos\varphi_2}{\sin\theta_2\sin\varphi_2} \tag{3.44}$$

$$\cos(\varphi_3-\phi_3) = \frac{\cos\theta_2 - \cos\phi_2\cos\varphi_2}{\sin\phi_2\sin\varphi_2} \tag{3.45}$$

Combining Eqs. (3.42) through (3.45), we are able to solve for φ_1 and φ_3 using the inverse tan function.

$$\varphi_1 = \theta_1 + \tan^{-1}\left(\frac{\sin\theta_2\sin\phi_2\sin(\theta_3+\phi_1)}{\cos\phi_2 - \cos\theta_2\cos\varphi_2}\right) \tag{3.46}$$

$$\varphi_3 = \phi_3 + \tan^{-1}\left(\frac{\sin\theta_2\sin\phi_2\sin(\theta_3+\phi_1)}{\cos\theta_2 - \cos\phi_2\cos\varphi_2}\right) \tag{3.47}$$

Using Eqs. (3.41), (3.46) and (3.47) to solve for $\boldsymbol{\phi}$ instead of back-solving $\boldsymbol{\varphi}$ out of the direction cosine matrix in Eq. (3.37) is numerically more efficient. While the Euler angle successive or composite rotation was developed for the (3-1-3) special case, the transformations in Eqs. (3.41), (3.46) and (3.47) actually hold for *any* symmetric rotation sequence.[5, 13] Asymmetric sets, however, will have to be composited using the corresponding direction cosine matrices.

On occasion it is required to find the *relative* attitude vector between two reference frame. For example, given the symmetric Euler angle vectors $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$, find the corresponding vector $\boldsymbol{\phi}$ which relates $\mathcal{B}$ to $\mathcal{F}$. Using the same spherical triangle in Figure 3.6(ii), we find the following closed form expressions for $\boldsymbol{\phi}$.

$$\phi_1 = -\theta_3 + \tan^{-1}\left(\frac{\sin\theta_2\sin\varphi_2\sin(\varphi_1-\theta_1)}{\cos\theta_2\cos\phi_2 - \cos\varphi_2}\right) \tag{3.48}$$

$$\phi_2 = \cos^{-1}\left(\cos\theta_2\cos\varphi_2 + \sin\theta_2\sin\varphi_2\cos(\varphi_1-\theta_1)\right) \tag{3.49}$$

$$\phi_3 = \varphi_3 - \tan^{-1}\left(\frac{\sin\theta_2\sin\varphi_2\sin(\varphi_1-\theta_1)}{\cos\theta_2 - \cos\phi_2\cos\varphi_2}\right) \tag{3.50}$$

Similar expressions can be found to express $\boldsymbol{\theta}$ in terms of $\boldsymbol{\phi}$ and $\boldsymbol{\varphi}$.

Example 3.2: Let the orientations of two spacecraft $\mathcal{B}$ and $\mathcal{F}$ relative to an inertial frame $\mathcal{N}$ be given through the asymmetric (3-2-1) Euler angles $\boldsymbol{\theta}_{\mathcal{B}} = (30, -45, 60)^T$ and $\boldsymbol{\theta}_{\mathcal{F}} = (10, 25, -15)^T$ degrees. What is the relative orientation of spacecraft $\mathcal{B}$ relative to $\mathcal{F}$ in terms of (3-2-1) Euler angles.

The orientation matrices $[BN]$ and $[FN]$ are found using Eq. (3.33).

$$[BN] = \begin{bmatrix} 0.612372 & 0.353553 & 0.707107 \\ -0.78033 & 0.126826 & 0.612372 \\ 0.126826 & -0.926777 & 0.353553 \end{bmatrix}$$

$$[FN] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

The direction cosine matrix $[BF]$ which describes the attitude of $\mathcal{B}$ relative to $\mathcal{F}$ is computed by using Eq. (3.20).

$$[BF] = [BN][FN]^T = \begin{bmatrix} 0.303372 & -0.0049418 & 0.952859 \\ -0.935315 & 0.1895340 & 0.298769 \\ -0.182075 & -0.9818620 & 0.052877 \end{bmatrix}$$

Using the transformations in Eq. (3.34) the relative (3-2-1) Euler angles are

$$\psi = \tan^{-1}\left(\frac{-0.0049418}{0.303372}\right) = 0.933242\ deg$$
$$\theta = -\sin^{-1}(0.952859) = -1.26252\ deg$$
$$\phi = \tan^{-1}\left(\frac{0.298769}{0.052877}\right) = -57.6097\ deg$$

Since ϕ is much larger than ψ and θ, the attitude of $\mathcal{B}$ could be described qualitatively to differ from $\mathcal{F}$ by a -57.6 degree roll. This result was not immediately obvious studying the original Euler angle vectors $\boldsymbol{\theta}_{\mathcal{B}}$ and $\boldsymbol{\theta}_{\mathcal{F}}$.

Let the vector $\boldsymbol{\omega}$ define the instantaneous rotational velocity of the $\mathcal{B}$ frame relative to the $\mathcal{N}$ frame. To avoid having to integrate the direction cosine matrix directly given an $\boldsymbol{\omega}$ time history, the Euler angle kinematic differential equations are needed. The (3-2-1) Euler kinematic differential equation is derived below. The methodology can be used for any set of Euler angles. The vector $\boldsymbol{\omega}$ is written in body frame components as

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3 \tag{3.51}$$

From Figure 3.3 it is evident that the $\mathcal{B}$ frame rotation can also be written in terms of the Euler angle rates $(\dot{\psi}, \dot{\theta}, \dot{\phi})$ as

$$\boldsymbol{\omega} = \dot{\psi}\hat{\boldsymbol{n}}_3 + \dot{\theta}\hat{\boldsymbol{b}}_2' + \dot{\phi}\hat{\boldsymbol{b}}_1 \tag{3.52}$$

The unit vector $\hat{\boldsymbol{b}}_2'$ is the direction of the body fixed axis $\hat{\boldsymbol{b}}_2$ before performing a roll ϕ about $\hat{\boldsymbol{b}}_1$ as is shown in Figure 3.3. It can be written in terms of $\{\hat{\boldsymbol{b}}\}$ as

$$\hat{\boldsymbol{b}}_2' = \cos\phi\hat{\boldsymbol{b}}_2 - \sin\phi\hat{\boldsymbol{b}}_3 \tag{3.53}$$

The direction cosine matrix in terms of the (3-2-1) Euler angles in Eq. (3.33) is used to express $\hat{\boldsymbol{n}}_3$ in terms of $\{\hat{\boldsymbol{b}}\}$.

$$\hat{\boldsymbol{n}}_3 = -\sin\theta\hat{\boldsymbol{b}}_1 + \sin\phi\cos\theta\hat{\boldsymbol{b}}_2 + \cos\phi\cos\theta\hat{\boldsymbol{b}}_3 \tag{3.54}$$

After substituting Eqs. (3.53) and (3.54) into Eq. (3.52) and then comparing terms with Eq. (3.51), the following kinematic equation is found.

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{bmatrix}-\sin\theta & 0 & 1\\ \sin\phi\cos\theta & \cos\phi & 0\\ \cos\phi\cos\theta & -\sin\phi & 0\end{bmatrix}\begin{pmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{pmatrix} \tag{3.55}$$

The kinematic differential equation of the (3-2-1) Euler angles is the inverse of Eq. (3.55).

$$\begin{pmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{pmatrix} = \frac{1}{\cos\theta}\begin{bmatrix}0 & \sin\phi & \cos\phi\\ 0 & \cos\phi\cos\theta & -\sin\phi\cos\theta\\ \cos\theta & \sin\phi\sin\theta & \cos\phi\sin\theta\end{bmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = [B(\psi,\theta,\phi)]\boldsymbol{\omega} \tag{3.56}$$

Similarly, the kinematic differential equations for the (3-1-3) Euler angles are found be

$$\boldsymbol{\omega} = \begin{bmatrix}\sin\theta_3\sin\theta_2 & \cos\theta_3 & 0\\ \cos\theta_3\sin\theta_2 & -\sin\theta_3 & 0\\ \cos\theta_2 & 0 & 1\end{bmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{pmatrix} \tag{3.57}$$

with the inverse relationship

$$\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{pmatrix} = \frac{1}{\sin\theta_2}\begin{bmatrix}\sin\theta_3 & \cos\theta_3 & 0\\ \cos\theta_3\sin\theta_2 & -\sin\theta_3\sin\theta_2 & 0\\ -\sin\theta_3\cos\theta_2 & -\cos\theta_3\cos\theta_2 & \sin\theta_2\end{bmatrix}\boldsymbol{\omega} = [B(\boldsymbol{\theta})]\boldsymbol{\omega} \tag{3.58}$$

The complete set of 12 transformations between the various Euler angle rates and the body angular velocity vector can be found in Appendix C. Note that the Euler angle kinematic differential equations encounter a singularity either at $\theta_2 = \pm 90$ degrees for the (3-2-1) set or at $\theta_2 = 0$ or 180 degrees for the (3-1-3) set. It turns out that all Euler angles sets encounter a singularity at specific second rotation angle θ_2 only. The first and third rotation angles θ_1 and θ_3 never lead to a singularity. In all cases, it can be verified that the singularity occurs for those θ_2 values that result in θ_1 and θ_3 being measured in the same plane. If the Euler angle set is symmetric, then the singular orientation is at $\theta_2 = 0$ or 180 degrees. If the Euler angle set is asymmetric, then the singular orientation is $\theta_2 = \pm 90$ degrees. Therefore asymmetric sets such as the (3-2-1) Euler reference frame. Symmetric sets as the (3-1-3) Euler angles would not be convenient to describe small departure rotations of $\{\hat{\boldsymbol{b}}\}$ from the $\{\hat{\boldsymbol{n}}\}$ axes since for small angles one would always operate very close to the singular attitude at $\theta_2 = 0$.

The Euler angles provide a compact, three parameter attitude description whose coordinates are easy to visualize. One main drawback of these angles is that a rigid body or reference frame is never further than a 90 degree rotation away from a singular orientation. Therefore their use in describing large, arbitrary and especially arbitrary rotations is limited. Also, their kinematic differential equations are fairly nonlinear, containing computationally intensive trigonometric functions. The linearized Euler angle kinematic differential equations are only valid for a relatively small domain of rotations.

3.3 Principal Rotation Vector

The following theorem has been very fundamental in the development of several types of attitude coordinates and is generally referenced to Euler.[14–16]

Theorem 3.1 (Euler's Principal Rotation) *A rigid body or coordinate reference frame can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rigid rotation through a principal angle Φ about the principal axis $\hat{\mathbf{e}}$; the principal axis being a judicious axis fixed in both the initial and final orientation.*

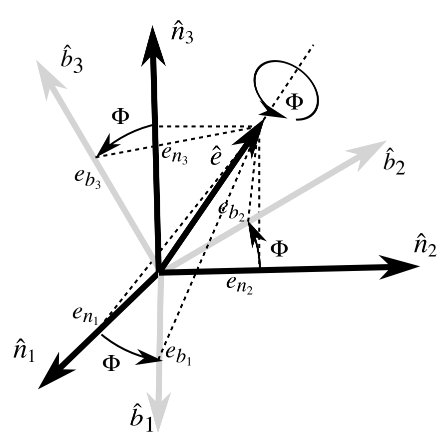


Figure 3.7: Illustration of Euler's Principal Rotation Theorem

This theorem can be visualized using Figure 3.7. Let the principal axis unit vector $\hat{\boldsymbol{e}}$ be written in $\mathcal{B}$ and $\mathcal{N}$ frame components as

$$\hat{\boldsymbol{e}} = e_{b_1}\hat{\boldsymbol{b}}_1 + e_{b_2}\hat{\boldsymbol{b}}_2 + e_{b_3}\hat{\boldsymbol{b}}_3 \tag{3.59a}$$

$$\hat{\boldsymbol{e}} = e_{n_1}\hat{\boldsymbol{n}}_1 + e_{n_2}\hat{\boldsymbol{n}}_2 + e_{n_3}\hat{\boldsymbol{n}}_3 \tag{3.59b}$$

Implicit in the theorem we see that $\hat{\boldsymbol{e}}$ will have the same vector components in the $\mathcal{B}$ as in the $\mathcal{N}$ reference frame; i.e. $e_{b_i} = e_{n_i} = e_i$. Eq. (3.5) shows that

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = [C] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{3.60}$$

must be true. Therefore the principal axis unit vector $\hat{\boldsymbol{e}}$ is the unit eigenvector of $[C]$ corresponding to the eigenvalue $+1$. Thus the proof of the Principal Rotation Theorem reduces to proving the $[C]$ has an eigenvalue of $+1$. This proof is given in Goldstein in Ref. 8. The eigenvalue $+1$ is unique and the corresponding eigenvector is unique to within a sign of Φ and $\hat{\boldsymbol{e}}$, except for the case of a zero rotation. In this case $[C] = [I_{3\times3}]$ and Φ would be zero, but there would be an infinity of unit axes $\hat{\boldsymbol{e}}$ such that $\hat{\boldsymbol{e}} = [I_{3\times3}]\hat{\boldsymbol{e}}$. For the general case, the lack of sign uniqueness of Φ and $\hat{\boldsymbol{e}}$ will not cause any practical problems. The sets $(\hat{\boldsymbol{e}}, \Phi)$ and $(-\hat{\boldsymbol{e}}, -\Phi)$ both describe the same orientation.

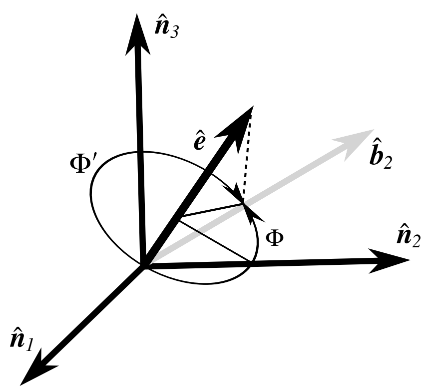


Figure 3.8: Illustration of Both Principal Rotation Angles

The principal rotation angle Φ is also not unique. Figure 3.7 shows the direction of the angle Φ labeled such that the shortest rotation about $\hat{\boldsymbol{e}}$ will be performed to move from $\mathcal{N}$ to $\mathcal{B}$. However, this is not necessary. If so desired, one can also rotate in the opposite direction by the angle Φ' and achieve the exact same orientation as shown in Figure 3.8. The difference between Φ and Φ' will always be 360 degrees. In most cases the magnitude of Φ is simply chosen to be less than or equal to 180 degrees.

To find the direction cosine matrix $[C]$ in terms of the principal rotation components $\hat{\boldsymbol{e}}$ and Φ, the fact is used that each reference frame base vector $\hat{\boldsymbol{n}}_i$

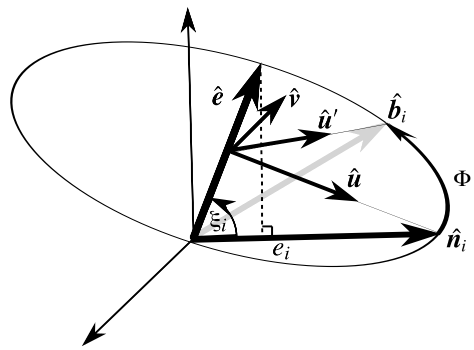


Figure 3.9: Mapping $\hat{\boldsymbol{n}}_i$ into $\hat{\boldsymbol{b}}_i$ Base Vectors

is related to $\hat{\boldsymbol{b}}_i$ through a single axis rotation about $\hat{\boldsymbol{e}}$. Let the unit principal axis vector be written as

$$\hat{\boldsymbol{e}} = e_1\hat{\boldsymbol{n}}_1 + e_2\hat{\boldsymbol{n}}_2 + e_3\hat{\boldsymbol{n}}_3 \tag{3.61}$$

and let ξ_i be the angle between $\hat{\boldsymbol{n}}_i$ and $\hat{\boldsymbol{e}}$ as shown in Figure 3.9. Let's note the following useful identity

$$\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{n}}_i = \cos\xi_i = e_i \tag{3.62}$$

Studying Figure 3.9 the base vector $\hat{\boldsymbol{b}}_i$ can be written as

$$\hat{\boldsymbol{b}}_i = \cos\xi_i\hat{\boldsymbol{e}} + \sin\xi_i\hat{\boldsymbol{u}}' = e_i\hat{\boldsymbol{e}} + \sin\xi_i\hat{\boldsymbol{u}}' \tag{3.63}$$

The unit vector $\hat{\boldsymbol{u}}'$ is given by

$$\hat{\boldsymbol{u}}' = \cos\Phi\hat{\boldsymbol{u}} + \sin\Phi\hat{\boldsymbol{v}} \tag{3.64}$$

It follows from the geometry of the single axis rotation that

$$\hat{\boldsymbol{v}} = \frac{\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i}{|\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i|} = \frac{1}{\sin\xi_i}\left(\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i\right) \tag{3.65}$$

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{v}} \times \hat{\boldsymbol{e}} = \frac{1}{\sin\xi_i}\left(\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i\right) \times \hat{\boldsymbol{e}} \tag{3.66}$$

The expression for $\hat{\boldsymbol{u}}$ can be further reduced by making use of the triple cross product identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \tag{3.67}$$

to the simpler form

$$\hat{\boldsymbol{u}} = \frac{1}{\sin\xi_i}\left(\hat{\boldsymbol{n}}_i - e_i\hat{\boldsymbol{e}}\right) \tag{3.68}$$

After substituting Eqs. (3.64), (3.65) and (3.68) into Eq. (3.63), each base vector $\hat{\boldsymbol{b}}_i$ is expressed in terms of reference frame N base vectors.

$$\hat{\boldsymbol{b}}_i = \cos\Phi \hat{\boldsymbol{n}}_i + (1-\cos\Phi)\,\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T \hat{\boldsymbol{n}}_i + \sin\Phi\,(\hat{\boldsymbol{e}}\times\hat{\boldsymbol{n}}_i) \tag{3.69}$$

where $\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T$ is the outer vector dot product of the vector $\hat{\boldsymbol{e}}$. Making use of the definition of $[\tilde{\boldsymbol{e}}]$ in Eq. (3.23) the set of base vectors $\{\hat{\boldsymbol{b}}\}$ can be expressed as

$$\{\hat{\boldsymbol{b}}\} = \left(\cos\Phi[I_{3\times3}] + (1-\cos\Phi)\,\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T - \sin\Phi[\tilde{\boldsymbol{e}}]\right)\{\hat{\boldsymbol{n}}\} \tag{3.70}$$

Using the relationship $\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}$, the direction cosine matrix can be directly extracted from Eq. (3.69) to be

$$[C] = \begin{bmatrix} e_1^2\Sigma + c\Phi & e_1e_2\Sigma + e_3 s\Phi & e_1e_3\Sigma - e_2 s\Phi \\ e_2e_1\Sigma - e_3 s\Phi & e_2^2\Sigma + c\Phi & e_2e_3\Sigma + e_1 s\Phi \\ e_3e_1\Sigma + e_2 s\Phi & e_3e_2\Sigma - e_1 s\Phi & e_3^2\Sigma + c\Phi \end{bmatrix} \tag{3.71}$$

where $\Sigma = 1 - c\Phi$. Again the short hand notation $c\Phi = \cos\Phi$ and $s\Phi = \sin\Phi$ was used here. The direction cosine matrix $[C]$ depends on four scalar quantities e_1, e_2, e_3 and Φ. However, only three degrees of freedom are present since the vector components e_i must abide by the unit constraint $\sum_i^3 e_i^2 = 1$.

By inspection of Eq. (3.71), the inverse transformation from the direction cosine matrix $[C]$ to the principal rotation elements is found to be

$$\cos\Phi = \frac{1}{2}\left(C_{11} + C_{22} + C_{33} - 1\right) \tag{3.72}$$

$$\hat{\boldsymbol{e}} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \frac{1}{2\sin\Phi}\begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix} \tag{3.73}$$

Note that Eq. (3.72) will yield a principal rotation angle within the range $0 \leq \Phi \leq 180$ degrees. The direction of $\hat{\boldsymbol{e}}$ in Eq. (3.73) will be such that the principal rotation parameterizing $[C]$ will be through a positive angle Φ about $\hat{\boldsymbol{e}}$. To find the second possible principal rotation angle Φ' one subtracts 360 degrees from Φ.

$$\Phi' = \Phi - 2\pi \tag{3.74}$$

The angle Φ' is equally valid as Φ and yields the same principal rotation axis $\hat{\boldsymbol{e}}$. The only difference being that a *longer* rotation (for $|\Phi| \leq \pi$) is being performed in the opposite direction. As with the sequential Euler angle rotations, the instantaneous principal rotation parameters $\{e_1(t), e_2(t), e_3(t), \Phi(t)\}$ can be considered coordinates associated with the instantaneous direction cosine matrix $[C(t)]$, and obviously does not restrict the body to actually execute the principal rotation.

Example 3.3: Let the B frame attitude relative the $\mathcal{N}$ frame be given by the

(3-2-1) Euler angles (10,25,-15) degrees. Find the corresponding principal rotation axis and angles.
Using Eq. (3.33) the direction cosine matrix $[BN]$ is

$$[BN] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

The first principal rotation angle Φ is found through Eq. (3.72).

$$\Phi = \cos^{-1}\left(\frac{1}{2}\left(0.892539 + 0.932257 + 0.875426 - 1\right)\right) = 31.7762°$$

The corresponding principal rotation axis is given though Eq. (3.73).

$$\hat{\boldsymbol{e}} = \frac{1}{2\sin\left(31.7762°\right)}\begin{pmatrix} -0.23457 - 0.325773 \\ 0.357073 - (-0.422618) \\ 0.157379 - (-0.275451) \end{pmatrix} = \begin{pmatrix} -0.532035 \\ 0.740302 \\ 0.410964 \end{pmatrix}$$

The second principal rotation angle Φ' calculated using Eq. (3.74).

$$\Phi' = 31.7762° - 360° = -328.2238°$$

Either principal rotation element sets $(\hat{\boldsymbol{e}}, \Phi)$ or $(\hat{\boldsymbol{e}}, \Phi')$ describes the identical attitude as the original (3-2-1) Euler angles.

Many important attitude parameters that are derived from Euler's principal rotation axis $\hat{\boldsymbol{e}}$ and angle Φ can be written in the general form

$$\boldsymbol{p} = f(\Phi)\hat{\boldsymbol{e}} \tag{3.75}$$

where $f(\Phi)$ could be any scalar function of Φ. All these attitude coordinate vectors have the same direction and differ only by their magnitude $|\boldsymbol{p}| = f(\Phi)$.

The principal rotation vector $\boldsymbol{\gamma}$ is simply defined as

$$\boldsymbol{\gamma} = \Phi\hat{\boldsymbol{e}} \tag{3.76}$$

Therefore the magnitude of $\boldsymbol{\gamma}$ is $f(\Phi) = \Phi$. This attitude vector has a very interesting relationship to the direction cosine matrix that can be verified to also hold for higher dimensional orthogonal projections as shown in Ref. 17. To gain more insight, consider the special case of a pure single-axis rotation about a fixed $\hat{\boldsymbol{e}}$ with the rotation angle being Φ. The angular velocity vector for this case is

$$\boldsymbol{\omega} = \dot{\Phi}\hat{\boldsymbol{e}} \tag{3.77}$$

or in matrix form:

$$[\tilde{\boldsymbol{\omega}}] = \dot{\Phi}[\tilde{\boldsymbol{e}}] \tag{3.78}$$

Substituting Eq. (3.78) into Eq. (3.27) leads to the following development:

$$\begin{aligned} \frac{d[C]}{dt} &= -\frac{d\Phi}{dt}[\tilde{\boldsymbol{e}}][C] \\ \frac{d[C]}{d\Phi} &= -[\tilde{\boldsymbol{e}}][C] \\ [C] &= e^{-\Phi[\tilde{\boldsymbol{e}}]} \end{aligned} \tag{3.79}$$

The last step holds true for $[\tilde{\boldsymbol{e}}]$ being a constant matrix for a rotation about a fixed axis. Due to Euler's principal rotation theorem, however, any arbitrary rotation can be instantaneously described by the equivalent single-axis rotation. Euler's theorem means that Eq. (3.79) holds at any instant for an arbitrary time varying direction cosine matrix $[C]$. Note for time-varying $[C]$, however, that $\hat{\boldsymbol{e}}$ and Φ must be considered time-varying. Using Eq. (3.76) the rotation matrix $[C]$ is related to $\boldsymbol{\gamma}$ through

$$[C] = e^{-[\tilde{\boldsymbol{\gamma}}]} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-[\tilde{\boldsymbol{\gamma}}]\right)^n \tag{3.80}$$

It turns out that this mapping also holds for higher dimensional proper orthogonal matrices $[C]$. For the case of three-dimensional rotations, the infinite power series in Eq. (3.80) can more conveniently be written as a finite, closed form solution.[5, 17]

$$[C] = e^{-\Phi[\tilde{\boldsymbol{e}}]} = [I_{3\times 3}]\cos\Phi - \sin\Phi[\tilde{\boldsymbol{e}}] + (1-\cos\Phi)\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T \tag{3.81}$$

To find the inverse transformation from $[C]$ to $\boldsymbol{\gamma}$, the inverse matrix logarithm is taken.

$$[\tilde{\boldsymbol{\gamma}}] = -\ln[C] = \sum_{n=0}^{\infty} \frac{1}{n}(1-[C])^n \tag{3.82}$$

This inverse mapping is defined everywhere except for $\Phi = 0$ and $\Phi = \pm 180$ degree rotations. For these rotations, the non-uniqueness of the $\boldsymbol{\gamma}$ vector that leads to mathematical difficulties. Otherwise a vector $\boldsymbol{\gamma}$ is reliably returned corresponding to a principal rotation of less than or equal to 180 degrees.

Example 3.4: In Example 3.3 it was shown that the direction cosine matrix

$$[BN] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.23457 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

represents the equivalent orientation as the principal rotation vector

$$\boldsymbol{\gamma} = 0.55460rad \begin{pmatrix} -0.532035 \\ 0.740302 \\ 0.410964 \end{pmatrix} = \begin{pmatrix} -0.295067 \\ 0.410571 \\ 0.227921 \end{pmatrix}$$

To verify the mapping in Eq. (3.80) let's write $[\tilde{\gamma}]$ using the definition of tilde matrix operator in Eq. (3.23).

$$[\tilde{\gamma}] = \begin{bmatrix} 0 & -0.227921 & 0.410571 \\ 0.227921 & 0 & 0.295067 \\ -0.410571 & -0.295067 & 0 \end{bmatrix}$$

Using software packages such as Mathematica or MATLAB, the matrix exponential mapping in Eq. (3.80) can be solved numerically for the corresponding direction cosine matrix $[BN]$.

$$[BN] = e^{-[\tilde{\gamma}]} = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix} \surd$$

Let $(\Phi_1, \hat{\boldsymbol{e}}_1)$ be the principal rotation elements that relate the $\mathcal{B}$ frame relative to the $\mathcal{N}$ frame, while $(\Phi_2, \hat{\boldsymbol{e}}_2)$ orients the $\mathcal{F}$ frame relative to the $\mathcal{B}$ frame. The $\mathcal{F}$ frame is related directly to the $\mathcal{N}$ frame by the elements $(\Phi, \hat{\boldsymbol{e}})$ through the relationship

$$[FN(\Phi, \hat{\boldsymbol{e}})] = [FB(\Phi_2, \hat{\boldsymbol{e}}_2)][BN(\Phi_1, \hat{\boldsymbol{e}}_1)] \tag{3.83}$$

Instead of solving for the overall principal rotation elements through the corresponding direction cosine matrix, it is possible to express $(\Phi, \hat{\boldsymbol{e}})$ directly in terms of $(\Phi_1, \hat{\boldsymbol{e}}_1)$ and $(\Phi_2, \hat{\boldsymbol{e}}_2)$ through[5]

$$\Phi = 2\cos^{-1}\left(\cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2} - \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2\right) \tag{3.84}$$

$$\hat{\boldsymbol{e}} = \frac{\cos\frac{\Phi_2}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}_1 + \cos\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_2 + \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2}{\sin\frac{\Phi}{2}} \tag{3.85}$$

This composite rotation property is easily derived from the Euler parameter composite rotation property shown in the next section. Given the two principal rotation element sets $(\Phi_1, \hat{\boldsymbol{e}}_1)$ and $(\Phi, \hat{\boldsymbol{e}})$, the relative orientation set $(\Phi_2, \hat{\boldsymbol{e}}_2)$ is expressed similarly through

$$\Phi_2 = 2\cos^{-1}\left(\cos\frac{\Phi}{2}\cos\frac{\Phi_1}{2} + \sin\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}}_1\right) \tag{3.86}$$

$$\hat{\boldsymbol{e}}_2 = \frac{\cos\frac{\Phi_1}{2}\sin\frac{\Phi}{2}\hat{\boldsymbol{e}} - \cos\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}_1 + \sin\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}} \times \hat{\boldsymbol{e}}_1}{\sin\frac{\Phi_2}{2}} \tag{3.87}$$

The kinematic differential equation of the principal rotation vector $\boldsymbol{\gamma}$ is given by[5, 18–20]

$$\dot{\boldsymbol{\gamma}} = \left[[I_{3\times 3}] + \frac{1}{2}[\tilde{\gamma}] + \frac{1}{\Phi^2}\left(1 - \frac{\Phi}{2}\cot\left(\frac{\Phi}{2}\right)\right)[\tilde{\gamma}]^2\right]\boldsymbol{\omega} \tag{3.88}$$

where $\Phi = \| \boldsymbol{\gamma} \|$. The inverse transformation of Eq. (3.88) is

$$\boldsymbol{\omega} = \left[[I_{3\times3}] - \left(\frac{1-\cos\Phi}{\Phi^2} \right) [\tilde{\gamma}] + \left(\frac{\Phi - \sin\Phi}{\Phi^3} \right) [\tilde{\gamma}]^2 \right] \dot{\boldsymbol{\gamma}} \tag{3.89}$$

As expected, the kinematic differential equation in Eq. (3.88) contains a 0/0 type mathematical singularity for zero rotations where $\Phi = 0$ degrees. Therefore, the principal rotation vector is not well suited for use in small motion feedback control type applications where the reference state is the zero rotation. Further, the mathematical expression in Eq. (3.88) is rather complex, containing polynomial fractions of degrees up to three in addition to trigonometric functions. This makes $\boldsymbol{\gamma}$ less attractive to describe large arbitrary rotations as compared to some other, closely related, attitude parameters that will be presented in the next few sections.

Example 3.5: Given the prescribed body angular velocity vector $\boldsymbol{\omega} = \omega(t)\hat{\boldsymbol{e}}$ for a single axis rotation, Eq. (3.88) yields the following kinematic differential equation for the principal rotation vector $\boldsymbol{\gamma} = \Phi\hat{\boldsymbol{e}}$.

$$\dot{\boldsymbol{\gamma}} = \left[[I_{3\times3}] - \frac{\Phi}{2}[\tilde{\gamma}] + \frac{1}{\Phi^2}\left(1 - \frac{\Phi}{2}\cot\left(\frac{\Phi}{2}\right)\right)\Phi^2[\tilde{\gamma}]^2 \right] \omega(t)\hat{\boldsymbol{e}}$$

Noting that $[\tilde{\gamma}]\hat{\boldsymbol{e}} = \Phi[\tilde{e}]\hat{\boldsymbol{e}} = 0$, this is simplified to

$$\dot{\boldsymbol{\gamma}} = \omega(t)\hat{\boldsymbol{e}}$$

Therefore the general expression in Eq. (3.88) simplifies to the single axis result in Eq. (3.77).

The principal rotation elements $\hat{\boldsymbol{e}}$ and Φ have had a fundamental influence on the derivation of many sets of attitude coordinates. All of the following attitude parameters will be directly derived from these principal rotation elements.

3.4 Euler Parameters

Another popular set of attitude coordinates are the four Euler parameters (quaternions). They provide a redundant, nonsingular attitude description and are well suited to describe arbitrary, large rotations. The Euler parameter vector $\boldsymbol{\beta}$ is defined in terms of the principal rotation elements as

$$\beta_0 = \cos(\Phi/2) \tag{3.90a}$$
$$\beta_1 = e_1 \sin(\Phi/2) \tag{3.90b}$$
$$\beta_2 = e_2 \sin(\Phi/2) \tag{3.90c}$$
$$\beta_3 = e_3 \sin(\Phi/2) \tag{3.90d}$$

It is evident since $e_1^2+e_2^2+e_3^2=1$, that the β_i's satisfy the holonomic constraint

$$\beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2=1 \tag{3.91}$$

Note that this constraint geometrically describes a four-dimensional unit sphere. Any rotation described through the Euler parameters has a trajectory on the surface of this constraint sphere. Given a certain attitude, there are actually two sets of Euler parameters that will describe the same orientation. This is due to the non-uniqueness of the principal rotation elements themselves. Switching between the sets $(\hat{\boldsymbol{e}},\Phi)$ and $(-\hat{\boldsymbol{e}},-\Phi)$ will yield the same Euler parameter vector $\boldsymbol{\beta}$. However, if the second principal rotation angle Φ' is used, another Euler parameter vector $\boldsymbol{\beta}'$ is found. Using Eq. (3.74) one can show that

$$\begin{aligned}\beta_0' &= \cos\left(\frac{\Phi'}{2}\right)=\cos\left(\frac{\Phi}{2}-\pi\right)=-\cos\left(\frac{\Phi}{2}\right)=-\beta_0\\ \beta_i' &= e_i\sin\left(\frac{\Phi'}{2}\right)=e_i\sin\left(\frac{\Phi'}{2}-\pi\right)=-e_i\sin\left(\frac{\Phi}{2}\right)=-\beta_i\end{aligned}$$

Therefore the vector $\boldsymbol{\beta}'=-\boldsymbol{\beta}$ describes the same orientation as the vector $\boldsymbol{\beta}$. This results in the following interesting observation. Since any point on the unit constraint sphere surface represent a specific orientation, the anti-pole to that point represents the exact same orientation. The difference between the two attitude descriptions is that one specifies the orientation through the shortest single axis rotation, the other through the longest. From Eq.(3.90a) it is clear that in order to choose the Euler parameter vector corresponding to the shortest rotation (i.e. $|\Phi|\leq 180$ degrees), the coordinate β_0 must be chosen to be non-negative.

Using the trigonometric identities

$$\begin{aligned}\sin\Phi &= 2\sin(\Phi/2)\cos(\Phi/2)\\ \cos\Phi &= 2\cos^2(\Phi/2)-1\end{aligned}$$

in Eq. (3.71), the direction cosine matrix can be written in terms of the Euler parameters as

$$[C]=\begin{bmatrix}\beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2)\\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1)\\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2\end{bmatrix} \tag{3.92}$$

The fact that $\boldsymbol{\beta}$ and $-\boldsymbol{\beta}$ produces the same direction cosine matrix $[C]$ can be easily verified in Eq. (3.92). All Euler parameters appear in quadratic product pairs, thus changing the signs of all β_i components has no effect on the resulting $[C]$ matrix. It is evident that the most general angular motion of a reference frame generates two arcs on the four dimensional unit sphere (the geodesic arcs generated by $\boldsymbol{\beta}(t)$ and $-\boldsymbol{\beta}(t)$). This elegant description is universally nonsingular and is unique to within the sign $\pm\boldsymbol{\beta}(t)$. The inverse transformations from

$[C]$ to the Euler parameters can be found through inspection of Eq. (3.92) to be

$$\beta_0 = \pm\frac{1}{2}\sqrt{C_{11} + C_{22} + C_{33} + 1} \tag{3.93a}$$

$$\beta_1 = \frac{C_{23} - C_{32}}{4\beta_0} \tag{3.93b}$$

$$\beta_2 = \frac{C_{31} - C_{13}}{4\beta_0} \tag{3.93c}$$

$$\beta_3 = \frac{C_{12} - C_{21}}{4\beta_0} \tag{3.93d}$$

Note that the non-uniqueness of the Euler parameters is evident again in this inverse transformation. By keeping the + sign in Eq. (3.93a) one restricts the corresponding principal rotation angle Φ to be less than or equal to 180 degrees. From a practical point of few this non-uniqueness does not pose any difficulties. Initially one simply picks an initial condition on one Euler parameter trajectory and then remains with it either through solving an associated kinematic differential developed below, or using elementary continuity logic.

Clearly Eq. (3.93) has a 0/0 type mathematical singularity whenever $\beta_0 \rightarrow 0$. This corresponds to the $\boldsymbol{\beta}$ vector describing any 180 degree principal rotation. A computationally superior algorithm has been developed by Stanley in Ref. 21. First the the four β_i^2 terms are computed.

$$\beta_0^2 = \frac{1}{4}\left(1 + Trace[C]\right) \tag{3.94a}$$

$$\beta_1^2 = \frac{1}{4}\left(1 + 2C_{11} - Trace[C]\right) \tag{3.94b}$$

$$\beta_2^2 = \frac{1}{4}\left(1 + 2C_{22} - Trace[C]\right) \tag{3.94c}$$

$$\beta_3^2 = \frac{1}{4}\left(1 + 2C_{33} - Trace[C]\right) \tag{3.94d}$$

Then Stanley takes the square root of the largest β_i^2 found in Eq. (3.94) where the sign of β_i is arbitrarily chosen to be positive. The other β_j's are found by dividing the appropriate three of the following six in Eq. (3.95) by the chosen largest β_i coordinate.

$$\beta_0\beta_1 = (C_{23} - C_{32})/4 \tag{3.95a}$$

$$\beta_0\beta_2 = (C_{31} - C_{13})/4 \tag{3.95b}$$

$$\beta_0\beta_3 = (C_{12} - C_{21})/4 \tag{3.95c}$$

$$\beta_2\beta_3 = (C_{23} + C_{32})/4 \tag{3.95d}$$

$$\beta_3\beta_1 = (C_{31} + C_{13})/4 \tag{3.95e}$$

$$\beta_1\beta_2 = (C_{12} + C_{21})/4 \tag{3.95f}$$

To find the alternate set of Euler parameter, the sign of the chosen β_i would simply be set negative.

Example 3.6: Let's use Stanley's method to find the Euler parameters of the direction cosine matrix $[C]$.

$$[C] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

Using the expressions in Eq. (3.94) the absolute values of the four Euler parameter are found.

$$\beta_0^2 = 0.925055 \qquad \beta_1^2 = 0.021214$$

$$\beta_2^2 = 0.041073 \qquad \beta_3^2 = 0.012657$$

The β_0 term is selected as the largest element and used in Eqs. (3.95a) through (3.95c) to find the Euler parameter vector.

$$\boldsymbol{\beta} = (0.961798, -0.14565, 0.202665, 0.112505)^T$$

The alternate Euler parameter vector would be found be simply reversing the sign of each element in $\boldsymbol{\beta}$.

A very important *composite rotation* property of the Euler parameters is the manner in which they allow two sequential rotations to be combined into one overall composite rotation. Let the Euler parameter vector $\boldsymbol{\beta}'$ describe the first, $\boldsymbol{\beta}''$ the second and $\boldsymbol{\beta}$ the composite rotation. From Eq. (3.20) it is clear that

$$[FN(\boldsymbol{\beta})] = [FB(\boldsymbol{\beta}'')][BN(\boldsymbol{\beta}')] \tag{3.96}$$

Using Eq. (3.92) in Eq. (3.96) and equating corresponding elements leads to following elegant transformation that bi-linearly combines $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}''$ into $\boldsymbol{\beta}$.

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta_0'' & -\beta_1'' & -\beta_2'' & -\beta_3'' \\ \beta_1'' & \beta_0'' & \beta_3'' & -\beta_2'' \\ \beta_2'' & -\beta_3'' & \beta_0'' & \beta_1'' \\ \beta_3'' & \beta_2'' & -\beta_1'' & \beta_0'' \end{bmatrix} \begin{pmatrix} \beta_0' \\ \beta_1' \\ \beta_2' \\ \beta_3' \end{pmatrix} \tag{3.97}$$

By *transmutation* of Eq.(3.97) an alternate expression $\boldsymbol{\beta} = [G(\boldsymbol{\beta}')]\boldsymbol{\beta}''$ is found

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta_0' & -\beta_1' & -\beta_2' & -\beta_3' \\ \beta_1' & \beta_0' & -\beta_3' & \beta_2' \\ \beta_2' & \beta_3' & \beta_0' & -\beta_1' \\ \beta_3' & -\beta_2' & \beta_1' & \beta_0' \end{bmatrix} \begin{pmatrix} \beta_0'' \\ \beta_1'' \\ \beta_2'' \\ \beta_3'' \end{pmatrix} \tag{3.98}$$

where the components of the matrix $[G(\boldsymbol{\beta}')]$ are given in Eq. (3.98). Note the useful identity

$$[G(\boldsymbol{\beta})]^T \boldsymbol{\beta} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.99}$$

By inspection, it is evident that the *4*x*4* matrices in Eqs. (3.97) and (3.98) are orthogonal. These transformations provide a simple, nonsingular and bilinear method to combine two successive rotations described through Euler parameters. For other attitude parameters such as the Euler angles, this same composite transformation would yield a very complicated, transcendental expression.

Example 3.7: Using Stanley's method, the direction cosine matrices $[BN]$ and $[FB]$ defined in Example 3.1 can be parameterized through the Euler parameter vectors $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}''$ respectively as

$$[BN] \Rightarrow \quad \boldsymbol{\beta}' = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T$$

$$[FB] \Rightarrow \quad \boldsymbol{\beta}'' = \left(\frac{1}{2}\sqrt{\frac{\sqrt{3}}{2}+1}, -\frac{1}{2}\sqrt{\frac{\sqrt{3}}{2}+1}, \frac{-\sqrt{2}}{4\sqrt{2+\sqrt{3}}}, \frac{\sqrt{2}}{4\sqrt{2+\sqrt{3}}}\right)^T$$

Note that the vector $\boldsymbol{\beta}'$ describes the attitude of the $\mathcal{B}$ frame relative to the $\mathcal{N}$ frame, while the vector $\boldsymbol{\beta}''$ describes the $\mathcal{F}$ frame attitude relative to the $\mathcal{B}$ frame. Eq. (3.97) can be used to combine the two successive attitude vectors into one vector $\boldsymbol{\beta}$ which directly describes the $\mathcal{F}$ frame orientation relative to the $\mathcal{N}$ frame.

$$\boldsymbol{\beta} = \frac{1}{2\sqrt{2}}\left(\sqrt{3}, \sqrt{3}, 1, 1\right)^T$$

To verify that $\boldsymbol{\beta}$ does indeed parameterize the direction cosine matrix $[FN]$ given in Example 3.1, it can be back substituted into Eq. (3.92) to yield

$$[FN] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \quad \checkmark$$

The kinematic differential equation for the Euler parameters can be derived by differentiating the β_i's in Eq. (3.93). The following development will establish the kinematic equation for $\dot{\beta}_0$ only, the remaining $\dot{\beta}_i$ equations can be developed in an analogous manner. After taking the derivative of Eq. (3.93a), $\dot{\beta}_0$ is expressed as

$$\dot{\beta}_0 = \frac{\dot{C}_{11} + \dot{C}_{22} + \dot{C}_{33}}{8\beta_0} \tag{3.100}$$

After using the expressions for $\dot{C}_{ii}$ given in Eq. (3.27), the term $\dot{\beta}_0$ is rewritten as

$$\dot{\beta}_0 = \frac{1}{2}\left(-\frac{C_{23} - C_{32}}{4\beta_0}\omega_1 - \frac{C_{31} - C_{13}}{4\beta_0}\omega_2 - \frac{C_{12} - C_{21}}{4\beta_0}\omega_3\right) \tag{3.101}$$

Using Eqs. (3.93b) through (3.93d), the $\dot{\beta}_0$ differential equation is simplified to

$$\dot{\beta}_0 = \frac{1}{2}\left(-\beta_1\omega_1 - \beta_2\omega_2 - \beta_3\omega_3\right) \tag{3.102}$$

After performing a similar derivation for the $\dot{\beta}_1$, $\dot{\beta}_2$ and $\dot{\beta}_3$ terms, the four coupled kinematic differential equations for the Euler parameters are found to be the exceptionally elegant matrix form

$$\begin{pmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \tag{3.103}$$

or by transmutation of Eq. (3.103), the kinematic differential equation has the elegant form

$$\begin{pmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{3.104}$$

Note that the transformation matrix relating $\dot{\boldsymbol{\beta}}$ and $\boldsymbol{\omega}$ is orthogonal and singularity free. The inverse transformation from $\boldsymbol{\omega}$ to $d(\boldsymbol{\beta})/dt$ is always defined. Further, the Euler parameter kinematic differential equation of Eq. (3.103) is rigorously linear if $\omega_i(t)$ are known functions of time only. If $\omega_i(t)$ are themselves coordinates, then Eqs. (3.103) and (3.104) are more generally considered bi-linear. This makes the Euler parameters very attractive attitude coordinates for attitude estimation problems where the kinematic differential equation is linearized. All three parameter sets of attitude coordinates always have kinematic differential equations which are nonlinear and contain 0/0 type mathematical singularities. In attitude estimation problems their linearization is only locally valid. Whereas the linear (or bi-linear) property of the Euler parameter kinematic differential equation is globally valid. The Euler parameter kinematic differential equation in Eq. (3.104) can be written compactly as

$$\dot{\boldsymbol{\beta}} = \frac{1}{2}[B(\boldsymbol{\beta})]\boldsymbol{\omega} \tag{3.105}$$

where the *4*x*3* matrix $[B(\boldsymbol{\beta})]$ is defined as

$$[B(\boldsymbol{\beta})] = \begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \tag{3.106}$$

By carrying out the matrix algebra, the following useful identities can easily be verified.

$$[B(\boldsymbol{\beta})]^T \boldsymbol{\beta} = \mathbf{0} \tag{3.107}$$

$$[B(\boldsymbol{\beta})]^T \boldsymbol{\beta}' = -[B(\boldsymbol{\beta}')]^T \boldsymbol{\beta} \tag{3.108}$$

It is easily verified that the normalization condition $\boldsymbol{\beta}^T\boldsymbol{\beta} = 1$ is a rigorous analytical integral of Eqs. (3.103), (3.104). However, in practice the norm of $\boldsymbol{\beta}$ may slightly differ form 1 when numerically integrating Eq. (3.103). It is therefore necessary to take care to reimpose this condition differentially after each numerical integration step, if the solution is to remain valid over long time intervals. However, in contrast to the re-normalization of $[C(t)]$ to satisfy $[C]^T[C] = [I_{3\times3}]$ when solving Eq. (3.27), only one scalar condition needs to be considered when integrating $\boldsymbol{\beta}(t)$.

In control applications, often the four Euler parameters are broken up into two groups. The parameter β_0 is single out since it contains no information regarding the corresponding principal rotation axis of the orientation being represented. In effect, if is a scalar measure of the three dimensional rigid body attitude measure whose value is +1 or -1 if the attitude is zero. The remaining three Euler parameters are grouped together into a three-dimensional vector as

$$\boldsymbol{\epsilon} \equiv (\beta_1, \beta_2, \beta_3)^T \tag{3.109}$$

If the attitude goes to zero, then so will this vector. From Euler parameter differential equation in Eq. (3.104), it is evident that the differential equations for $\dot{\beta}_0$ and $\dot{\boldsymbol{\epsilon}}$ are of the form

$$\dot{\beta}_0 = -\frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{\epsilon} \tag{3.110}$$

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega} \tag{3.111}$$

The 3×3 matrix $[T]$ is defined as

$$[T(\beta_0, \boldsymbol{\epsilon})] = \beta_0[I_{3\times3}] + [\tilde{\boldsymbol{\epsilon}}] \tag{3.112}$$

3.5 Classical Rodrigues Parameters

The origin of the classical Rodrigues parameter vector $\boldsymbol{q}$ (or Gibbs vector) dates back over a hundred years to the French mathematician O. M. Rodrigues. This rigid body attitude coordinate set reduces the redundant Euler parameters to a minimal three parameter set through the transformation

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3 \tag{3.113}$$

The inverse transformation from classical Rodrigues parameters to Euler parameters is given by

$$\beta_0 = \frac{1}{\sqrt{1 + \boldsymbol{q}^T\boldsymbol{q}}} \tag{3.114a}$$

$$\beta_i = \frac{q_i}{\sqrt{1 + \boldsymbol{q}^T\boldsymbol{q}}} \qquad i = 1, 2, 3 \tag{3.114b}$$

Using the definitions in Eq. (3.90) the vector $\boldsymbol{q}$ is expressed directly in terms of the principal rotation elements as the elegant transformation

$$\boldsymbol{q} = \tan\frac{\Phi}{2}\hat{\boldsymbol{e}} \tag{3.115}$$

From Eqs. (3.113) and (3.115) it is evident that the classical Rodrigues parameters go singular whenever $\Phi \to \pm 180$ degrees. Very large rotations can be described with these parameters without ever approaching a geometric singularity. For rotations with $|\Phi| \leq 90^{\text{o}}$, it is evident that $\boldsymbol{q}(t)$ locates points near the origin bounded by the unit sphere. Compare this $\pm 180^{\text{o}}$ nonsingular range to the Euler angles where any orientation is never more than 90 degrees away from a singularity.

The small angle behavior of the classical Rodrigues parameters is also more linear than compared to the small angle behavior of any Euler angle set. Linearizing Eq. (3.115) it is evident that

$$\boldsymbol{q} \approx \frac{\Phi}{2}\hat{\boldsymbol{e}} \tag{3.116}$$

This means that classical Rodrigues parameters will linearize roughly to an "angle over 2" type quantity, whereas the Euler angles linearize as an angle type quantity well removed from singular points.

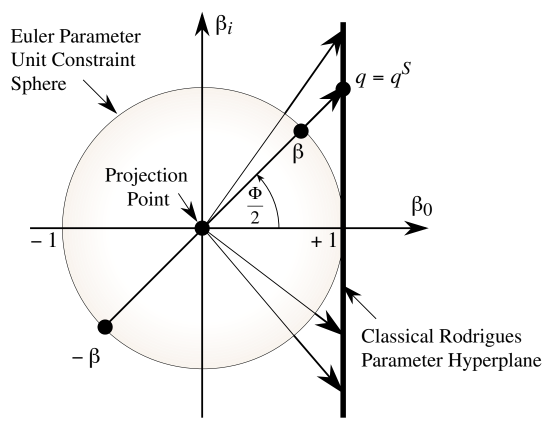


Figure 3.10: Stereographic Projection of Euler Parameters to Classical Rodrigues Parameters

As discussed in Ref. 22, the classical Rodrigues parameters can be viewed as a special set of stereographic orientation parameters. Stereographic projections are used to map a higher-dimensioned spherical surface onto a lower-dimensioned hyperplane. In this case, the surface of the four-dimensional Euler parameter unit constraint sphere in Eq. (3.91) is mapped (projected) onto a three-dimensional hyperplane though Eq. (3.113). Figure 3.10 illustrates how such a projection would yield the classical Rodrigues parameters. The projection point is chosen to be the origin $\boldsymbol{\beta} = 0$ and the hyperplane upon which all Euler parameter coordinates are projected is the tangent surface at $\beta_0 = 1$. Note that on the constraint sphere surface $\beta_0 = 1$ corresponds to a $\Phi = 0$ degrees, $\beta_0 = 0$ corresponds to $\Phi = \pm 180$ degrees and $\beta_0 =$ -1 represents $\Phi = \pm 360$ degrees. The transformation in Eq. (3.113) maps any Euler parameter set on the unit constraint sphere surface onto a corresponding point located on the classical Rodrigues parameter hyperplane.

All stereographic orientation parameters can be viewed as a projection of the constraint sphere onto some hyperplane. Since the Euler parameters themselves are not unique, the corresponding stereographic orientation parameters are also generally not unique. The set corresponding to the projection of the Euler parameter set $-\boldsymbol{\beta}$ is referred to as the *shadow* set and is differentiated from the original set by a superscript S.[22] However, it turns out that the shadow set of the classical Rodrigues parameters are indeed identical to the original classical Rodrigues parameters as is easily verified by reversing the β_i signs in Eq. (3.113) or by inspection of Figure 3.10.

$$q_i^S = \frac{-\beta_i}{-\beta_0} = q_i \tag{3.117}$$

The direction cosine matrix in terms of the classical Rodrigues parameters can be found by using their definition in Eq. (3.113) in the direction cosine matrix formulation in Eq. (3.92). The resulting parameterization is in matrix form[4, 22]

$$[C] = \frac{1}{1 + \boldsymbol{q}^T\boldsymbol{q}} \begin{bmatrix} 1+q_1^2-q_2^2-q_3^2 & 2\left(q_1q_2 + q_3\right) & 2\left(q_1q_3 - q_2\right) \\ 2\left(q_2q_1 - q_3\right) & 1-q_1^2+q_2^2-q_3^2 & 2\left(q_2q_3 + q_1\right) \\ 2\left(q_3q_1 + q_2\right) & 2\left(q_3q_2 - q_1\right) & 1-q_1^2-q_2^2+q_3^2 \end{bmatrix} \tag{3.118}$$

and in vector form[5, 22]

$$[C] = \frac{1}{1 + \boldsymbol{q}^T\boldsymbol{q}} \left(\left(1 - \boldsymbol{q}^T\boldsymbol{q}\right)[I_{3\times3}] + 2\boldsymbol{q}\boldsymbol{q}^T - 2[\tilde{\boldsymbol{q}}]\right) \tag{3.119}$$

The simplest way to extract the classical Rodrigues parameters from a given direction cosine matrix is to determine the Euler parameters first and then use Eq. (3.113) to find the corresponding Rodrigues parameters. Note the following useful identity.

$$[C(\boldsymbol{q})]^T = [C(-\boldsymbol{q})] \tag{3.120}$$

Since $\boldsymbol{q}$ defines the relative orientation of a second frame to a first frame, the relative orientation of the second frame relative to the first corresponds simply to reversing the sign of $\boldsymbol{q}$ as in

$$\{\hat{\boldsymbol{n}}\} = [C(\boldsymbol{q})]^T\{\hat{\boldsymbol{b}}\} = [C(-\boldsymbol{q})]\{\hat{\boldsymbol{b}}\} \tag{3.121}$$

This elegant property doesn't exist with Euler angles.

Similar to the direction cosine matrices and Euler parameters, the classical Rodrigues parameter vectors have a composite rotation property. Given two attitude vectors $\boldsymbol{q}'$ and $\boldsymbol{q}''$, let the overall composite attitude vector $\boldsymbol{q}$ be defined through the quadratically nonlinear condition

$$[FN(\boldsymbol{q})] = [FB(\boldsymbol{q}'')][BN(\boldsymbol{q}')] \tag{3.122}$$

However, solving for an overall transformation from $\boldsymbol{q}'$ and $\boldsymbol{q}''$ to $\boldsymbol{q}$ using Eq. (3.122) is very cumbersome. Using the successive rotation property of the Euler parameters and the definition of the classical Rodrigues parameters in Eq. (3.113), the composite attitude vector $\boldsymbol{q}$ is expressed directly in terms of $\boldsymbol{q}'$ and $\boldsymbol{q}''$ through[5, 23]

$$\boldsymbol{q} = \frac{\boldsymbol{q}'' + \boldsymbol{q}' - \boldsymbol{q}'' \times \boldsymbol{q}'}{1 - \boldsymbol{q}'' \cdot \boldsymbol{q}'} \tag{3.123}$$

Assume that the attitude vectors $\boldsymbol{q}$ and $\boldsymbol{q}'$ are given and the relative attitude vector $\boldsymbol{q}''$ is to be found. With direction cosine matrices and Euler parameters the two attitude descriptions were related through an orthogonal matrix which made finding the relative attitude description trivial. This is no longer the case with the classical Rodrigues parameter composite rotation property. However, we can use Eq. (3.122) to solve for $[FB(\boldsymbol{q}'')]$ first using the orthogonality of the direction cosine matrices.

$$[FB(\boldsymbol{q}'')] = [FN(\boldsymbol{q})][BN(\boldsymbol{q}')]^T \tag{3.124}$$

Using the identity in Eq. (3.120), this is rewritten as

$$[FB(\boldsymbol{q}'')] = [FN(\boldsymbol{q})][BN(-\boldsymbol{q}')] \tag{3.125}$$

which then leads to the desired direct transformation from $\boldsymbol{q}$ and $\boldsymbol{q}'$ to the relative orientation vector $\boldsymbol{q}''$.

$$\boldsymbol{q}'' = \frac{\boldsymbol{q} - \boldsymbol{q}' + \boldsymbol{q} \times \boldsymbol{q}'}{1 + \boldsymbol{q} \cdot \boldsymbol{q}'} \tag{3.126}$$

A similar transformation could be found to express $\boldsymbol{q}'$ in terms of $\boldsymbol{q}$ and $\boldsymbol{q}''$.

The kinematic differential equation of the classical Rodrigues parameters is found by taking the derivative of Eq. (3.113) and then substituting the corresponding expressions for $\dot{\beta}_i$ given in Eq. (3.104). The resulting matrix formulation is[4]

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} 1 + q_1^2 & q_1 q_2 - q_3 & q_1 q_3 + q_2 \\ q_2 q_1 + q_3 & 1 + q_2^2 & q_2 q_3 - q_1 \\ q_3 q_1 - q_2 & q_3 q_2 + q_1 & 1 + q_3^2 \end{bmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{3.127}$$

and the compact vector matrix form is

$$\dot{\boldsymbol{q}} = \frac{1}{2} \left[[I_{3\times3}] + [\tilde{\boldsymbol{q}}] + \boldsymbol{q}\boldsymbol{q}^T \right] \boldsymbol{\omega} \tag{3.128}$$

Note that the above kinematic differential equation contains no trigonometric functions and only has a quadratic nonlinearity. It is defined for any rotation except for $\Phi = \pm 180$ degrees. As $\boldsymbol{q}(t)$ approaches $\Phi = \pm 180^{\text{o}}$, both $\boldsymbol{q}(t)$ and $\dot{\boldsymbol{q}}(t)$ diverge to infinity. The inverse transformation of Eq. (3.128) is given by[5]

$$\boldsymbol{\omega} = \frac{2}{1 + \boldsymbol{q}^T\boldsymbol{q}}\left([I_{3\times3}] - [\tilde{\boldsymbol{q}}]\right)\dot{\boldsymbol{q}} \tag{3.129}$$

As is evident, for $(\boldsymbol{q}, \dot{\boldsymbol{q}}) \to \infty$, the transformation of Eq. (3.129) exhibits an ∞/∞ type singular behavior near $|\Phi| \to \pm 180$ degrees.

There exists a very elegant, analytically exact transformation between the orthogonal direction cosine matrix $[C]$ and the classical Rodrigues parameter vector $\boldsymbol{q}$ called the Cayley Transform.[4, 5, 10, 17, 24] What is remarkable is that this transformation holds for proper orthogonal matrices of dimensions higher than three. A proper orthogonal matrix is an orthogonal matrix with a determinant of $+1$. Thus it is possible to parameterize any proper orthogonal $[C]$ matrix by a minimal set of higher-dimensional classical Rodrigues parameters.

The Cayley Transform parameterizes a proper orthogonal matrix $[C]$ as a function of a skew-symmetric matrix $[Q]$:

$$[C] = ([I] - [Q])\left([I] + [Q]\right)^{-1} = \left([I] + [Q]\right)^{-1}([I] - [Q]) \tag{3.130}$$

The matrix product order is irrelevant in this transformation. Another surprising property of this transformation is that the inverse transformation from the skew-symmetric matrix Q back to the $[C]$ matrix *has exactly the same form as the forward transformation* in Eq. (3.130):

$$[Q] = ([I] - [C])\left([I] + [C]\right)^{-1} = \left([I] + [C]\right)^{-1}([I] - [C]) \tag{3.131}$$

For the case where $[C]$ is a 3x3 rotation matrix, the transformation in Eq. (3.130) yields the standard three-dimensional Rodrigues parameters. This can be verified by setting $[Q] = [\tilde{\boldsymbol{q}}]$ in Eq. (3.130), carrying out the 3×3 special case algebra implicit in Eq. (3.130) and comparing the result to Eq. (3.118). The kinematic differential equation of the $[C]$ is given in Eq. (3.27). This expression also holds for matrix dimensions higher than three.[4, 10] Since the Cayley Transform parameterizes a proper orthogonal matrix in terms of an "orientation coordinate" type quantity $[Q]$, the matrix $[\tilde{\boldsymbol{\omega}}]$ represents an analogous "angular velocity" cross product matrix which can be defined as[4, 10, 17]

$$[\tilde{\boldsymbol{\omega}}] = 2\left([I] + [Q]\right)^{-1}[\dot{Q}]\left([I] - [Q]\right)^{-1} \tag{3.132}$$

The kinematic differential equation of the higher dimensional Rodrigues parameters is obtained by differentiation of Eq. (3.131), and substituting Eqs. (3.27) and (3.130) as

$$[\dot{Q}] = \frac{1}{2}([I] + [Q])[\tilde{\boldsymbol{\omega}}]([I] - [Q]) \tag{3.133}$$

It can readily be verified that the 3×3 special case of Eq. (3.133) is equivalent to Eq. (3.127); so again the general $n \times n$ case contains the classical 3×3 results.

Example 3.8: Given the orthogonal 4x4 matrix $[C]$,

$$[C] = \begin{bmatrix} 0.505111 & -0.503201 & -0.215658 & 0.667191 \\ 0.563106 & -0.034033 & -0.538395 & -0.626006 \\ 0.560111 & 0.748062 & 0.272979 & 0.228387 \\ -0.337714 & 0.431315 & -0.767532 & 0.332884 \end{bmatrix}$$

it is easy to verify that $[C]$ can be parameterized in terms of higher dimensional classical Rodrigues parameters. Using MATLAB to solve Eq. (3.131), the skew-symmetric 4x4 matrix $[Q]$ is found to be

$$[Q] = \begin{bmatrix} 0 & 0.5 & 0.2 & -0.3 \\ -0.5 & 0 & 0.7 & 0.6 \\ -0.2 & -0.7 & 0 & -0.4 \\ 0.3 & -0.6 & 0.4 & 0 \end{bmatrix}$$

where the six upper diagonal elements of $[Q]$ are the higher dimensional classical Rodrigues elements.

3.6 Modified Rodrigues Parameters

The Modified Rodrigues Parameters (MRPs) are an elegant recent addition to the family of attitude parameters.[5, 22, 25–27] The MRP vector $\boldsymbol{\sigma}$ is defined in terms of the Euler parameters as the transformation

$$\sigma_i = \frac{\beta_i}{1+\beta_0} \qquad i = 1, 2, 3 \tag{3.134}$$

The inverse transformation is given by

$$\beta_0 = \frac{1-\sigma^2}{1+\sigma^2} \quad \beta_i = \frac{2\sigma_i}{1+\sigma^2} \quad i = 1, 2, 3 \tag{3.135}$$

where the notation $\sigma^{2n} = \left(\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)^n$ is introduced. Substituting Eq. (3.90) into Eq. (3.134) the MRP can be expressed in terms of the principal rotation elements as

$$\boldsymbol{\sigma} = \tan\frac{\Phi}{4}\hat{\boldsymbol{e}} \tag{3.136}$$

Studying Eq. (3.136), it is evident that the MRP have a geometric singularity at $\Phi = \pm 360$ degrees. Any rotation can be described except a complete revolution back to the original orientation. This gives $\boldsymbol{\sigma}$ twice the rotational range of the classical Rodrigues parameters. Also note that for small rotations the MRPs linearize as $\boldsymbol{\sigma} \approx (\Phi/4)\,\hat{\boldsymbol{e}}$.

Observing Eq. (3.134) it is evident that these equations are well-behaved except near the singularity at $\beta_0 = -1$, where $\Phi \to \pm 360^{\circ}$. Also, the inverse

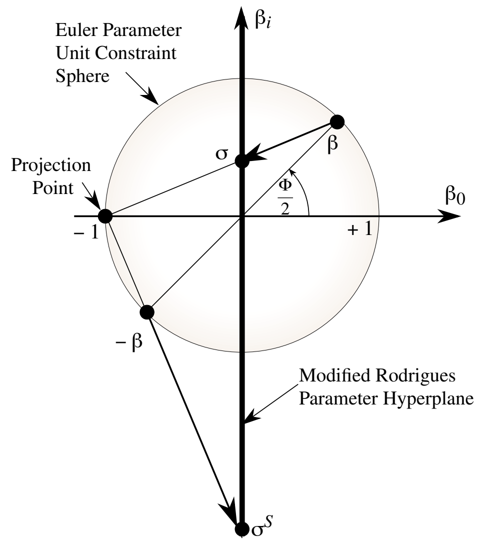


Figure 3.11: Stereographic Projection of Euler Parameters to Modified Rodrigues Parameters

transformation of Eq. (3.135) is well-behaved everywhere except at $|\boldsymbol{\sigma}| \to \infty$; we see from Eq. (3.136) that this again occurs at $\Phi \to 360^{\circ}$.

The MRP vector $\boldsymbol{\sigma}$ can be transformed directly into the classical Rodrigues parameter vector $\boldsymbol{q}$ through

$$\boldsymbol{q} = \frac{2\boldsymbol{\sigma}}{1-\sigma^2} \tag{3.137}$$

with the inverse transformation being

$$\boldsymbol{\sigma} = \frac{\boldsymbol{q}}{1+\sqrt{1+\boldsymbol{q}^T\boldsymbol{q}}} \tag{3.138}$$

Naturally, these transformations are singular at $\Phi = \pm 180$ degrees since the classical Rodrigues parameters are singular at this orientation.

As are the classical Rodrigues parameters, the MRPs are also a particular set of stereographic orientation parameters. Equation (3.134) describes a stereographic projection of the Euler parameter unit sphere onto the MRP hyperplane normal to the β_0 axis at $\beta_0 = 0$, where the projection point is at $\boldsymbol{\beta} = (-1, 0, 0, 0)$. This is illustrated in Figure 3.11. As a ± 360 degree principal rotation is approached (i.e. $\beta_0 \to -1$), the projection of the corresponding point on the constraint sphere goes to infinity. This illustrates the singular behavior of the MRPs as they describe a complete revolution.

However, contrary to the classical Rodrigues parameters, the projection of the alternate Euler parameter vector $-\boldsymbol{\beta}$ results in a distinct set of shadow (or "image") MRPs as can be seen in Figure 3.11. Each MRP vector is an equally valid attitude description satisfying the same kinematic differential equation. Therefore one can arbitrarily switch between the two vectors through the mapping[22, 26]

$$\sigma_i^S = \frac{-\beta_i}{1-\beta_0} = \frac{-\sigma_i}{\sigma^2} \quad i = 1,2,3 \tag{3.139}$$

where the choice as to which vector is the original and which the shadow vector is arbitrary. We usually let $\boldsymbol{\sigma}$ denote the mapping point interior to the unit sphere and $\boldsymbol{\sigma}^S$ the point point exterior to the unit sphere. As with the non-uniqueness of the principal rotation vector $\boldsymbol{\gamma}$ and the Euler parameter vector $\boldsymbol{\beta}$, one set of MRPs always corresponds to a principal rotation $\Phi \leq 180$ degrees and the other to $\Phi \geq 180$ degrees. From Eq. (3.136) it is clear that

$$\begin{aligned} |\boldsymbol{\sigma}| &\leq 1 \quad if \quad \Phi \leq 180^{\circ} \\ |\boldsymbol{\sigma}| &\geq 1 \quad if \quad \Phi \geq 180^{\circ} \\ |\boldsymbol{\sigma}| &= 1 \quad if \quad \Phi = 180^{\circ} \end{aligned} \tag{3.140}$$

The behavior is seen in Figure 3.11. The unit sphere $|\boldsymbol{\sigma}| = 1$, corresponding to all principal rotations of 180° from the origin, is of particular importance. As one set of MRPs exits the unit sphere, the other (shadow) set enters. The mapping in Eq. (3.139) can be written in terms of the principal rotation elements using the definitions of β_i in Eq. (3.90) as

$$\boldsymbol{\sigma}^S = \tan\left(\frac{\Phi - 2\pi}{4}\right)\hat{\boldsymbol{e}} \tag{3.141}$$

Using Eq. (3.74) this can be written directly in terms of the alternate principal rotation angle Φ'.

$$\boldsymbol{\sigma}^S = \tan\left(\frac{\Phi'}{4}\right)\hat{\boldsymbol{e}} \tag{3.142}$$

Eq. (3.142) clearly shows that the shadow MRP vector is a direct result of the alternate principal rotation vector.

The shadow MRPs have a singular orientation at $\Phi = 0$ degrees as compared to the original MRPs, which are singular at $\Phi = \pm 360$ degrees. This allows one to avoid MRP singularities all together by switching between original and shadow MRP sets as one MRP vector approaches a singular orientation. On which surface $\boldsymbol{\sigma}^T\boldsymbol{\sigma} = c$ one switches is arbitrary. However, switching between the two MRPs whenever the vector $\boldsymbol{\sigma}$ penetrates the surface $\boldsymbol{\sigma}^T\boldsymbol{\sigma} = 1$ has many positive aspects. For one, the map between the two MRP vectors simplifies on this surface to $\boldsymbol{\sigma}^S = -\boldsymbol{\sigma}$. Further, the magnitude of $\boldsymbol{\sigma}$ will remain bounded above by 1. Having a bounded norm of an attitude description is useful since it reflects the fundamental fact that two orientations can only differ by a finite rotation. Also, the current MRP attitude description will always describe the shortest principal rotation because of Eq. (3.140). Therefore the combined set of original and shadow MRPs with the switching surface $\boldsymbol{\sigma}^T\boldsymbol{\sigma} = 1$ provides for a nonsingular, bounded, minimal attitude description. It is ideally suited to describe large, arbitrary motions. The combined set is also useful in a feedback control type setting. For example, it linearizes well for small angles and has a bounded maximum norm of 1 which makes the selection of feedback gains easier.

The direction cosine matrix in terms of the MRP is found by substituting Eq. (3.135) into Eq. (3.92) and is given as[5, 22, 26, 27]

$$[C] = \frac{1}{(1+\sigma^2)^2}\begin{bmatrix} 4\left(\sigma_1^2-\sigma_2^2-\sigma_3^2\right)+(1-\sigma^2)^2 & 8\sigma_1\sigma_2+4\sigma_3(1-\sigma^2) & 8\sigma_1\sigma_3-4\sigma_2(1-\sigma^2) \\ 8\sigma_2\sigma_1-4\sigma_3(1-\sigma^2) & 4\left(-\sigma_1^2+\sigma_2^2-\sigma_3^2\right)+(1-\sigma^2)^2 & 8\sigma_2\sigma_3+4\sigma_1(1-\sigma^2) \\ 8\sigma_3\sigma_1+4\sigma_2(1-\sigma^2) & 8\sigma_3\sigma_2-4\sigma_1(1-\sigma^2) & 4\left(-\sigma_1^2-\sigma_2^2+\sigma_3^2\right)+(1-\sigma^2)^2 \end{bmatrix} \tag{3.143}$$

In compact vector form $[C]$ is parameterized in terms of the MRP as[5, 22]

$$[C] = [I_{3\times3}] + \frac{8[\tilde{\boldsymbol{\sigma}}]^2 - 4\left(1-\sigma^2\right)[\tilde{\boldsymbol{\sigma}}]}{\left(1+\sigma^2\right)^2} \tag{3.144}$$

As is the case with the classical Rodrigues parameters, the simplest method to extract the MRP from a given direction cosine matrix is the first extract the Euler parameters and then use Eq. (3.134) to find the MRP vector $\boldsymbol{\sigma}$. If $\beta_0 \geq 0$ is chosen when extracting the Euler parameters, then $|\boldsymbol{\sigma}| \leq 1$. If β_0 is chosen to be negative, then the alternate MRP vector corresponding to a larger principal rotation angle is found.

The MRPs enjoy the same relative rotation identity as did the classical Rodrigues parameters.

$$[C(\boldsymbol{\sigma})]^T = [C(-\boldsymbol{\sigma})] \tag{3.145}$$

Given two MRP vectors $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$, let the overall MRP vector $\boldsymbol{\sigma}$ be defined through

$$[FN(\boldsymbol{\sigma})] = [FB(\boldsymbol{\sigma}'')][BN(\boldsymbol{\sigma}')] \tag{3.146}$$

Starting with the Euler parameter successive rotation property and using the MRP definitions in Eq. (3.134), the MRP successive rotation property is expressed as[5]

$$\boldsymbol{\sigma} = \frac{(1-|\boldsymbol{\sigma}'|^2)\boldsymbol{\sigma}'' + (1-|\boldsymbol{\sigma}''|^2)\boldsymbol{\sigma}' - 2\boldsymbol{\sigma}'' \times \boldsymbol{\sigma}'}{1 + |\boldsymbol{\sigma}'|^2|\boldsymbol{\sigma}''|^2 - 2\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}''} \tag{3.147}$$

Using Eq. (3.145), we are able to express the relative attitude vector $\boldsymbol{\sigma}''$ in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ as

$$\boldsymbol{\sigma}'' = \frac{(1-|\boldsymbol{\sigma}'|^2)\boldsymbol{\sigma} - (1-|\boldsymbol{\sigma}|^2)\boldsymbol{\sigma}' + 2\boldsymbol{\sigma} \times \boldsymbol{\sigma}'}{1 + |\boldsymbol{\sigma}'|^2|\boldsymbol{\sigma}|^2 + 2\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}} \tag{3.148}$$

While these expressions are more complicated than their Euler parameter or classical Rodrigues parameter counterparts, they do provide a numerically efficient method to compute the composition of two MRP vectors or find the relative MRP attitude vector.

Example 3.9: Given the Euler parameter vector $\boldsymbol{\beta}$

$$\boldsymbol{\beta} = (0.961798, -0.14565, 0.202665, 0.112505)^T$$

the MRP vector $\boldsymbol{\sigma}$ is found using Eq. (3.134)

$$\sigma_1 = \frac{-0.14565}{1+0.961798} = -0.0742431$$
$$\sigma_2 = \frac{0.202665}{1+0.961798} = 0.103306$$
$$\sigma_3 = \frac{0.112505}{1+0.961798} = 0.0573479$$

The alternate shadow MRP vector $\boldsymbol{\sigma}^S$ can be found using $-\boldsymbol{\beta}$ instead of $\boldsymbol{\beta}$ in Eq. (3.134).

$$\sigma_1^S = \frac{0.14565}{1-0.961798} = 3.81263$$
$$\sigma_2^S = \frac{-0.202665}{1-0.961798} = -5.30509$$
$$\sigma_3^S = \frac{-0.112505}{1-0.961798} = -2.945$$

Note that if the direct mapping in Eq. (3.139) is used the same vector $\boldsymbol{\sigma}^S$ is obtained. Since the vector $|\boldsymbol{\sigma}| = 0.139546 \leq 1$, it represents the shorter principal rotation angle of $\Phi = 7.94$ degrees. The vector $|\boldsymbol{\sigma}^S| = 7.16611 \geq 1$ represents the longer principal rotation angle $\Phi' = \Phi - 360° = -328.224°$.

The kinematic differential equation of the MRPs is found in a similar manner as the one for the classical Rodrigues parameters. The resulting matrix formulation is[22, 27]

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\begin{bmatrix} 1-\sigma^2+2\sigma_1^2 & 2\left(\sigma_1\sigma_2-\sigma_3\right) & 2\left(\sigma_1\sigma_3+\sigma_2\right) \\ 2\left(\sigma_2\sigma_1+\sigma_3\right) & 1-\sigma^2+2\sigma_2^2 & 2\left(\sigma_2\sigma_3-\sigma_1\right) \\ 2\left(\sigma_3\sigma_1-\sigma_2\right) & 2\left(\sigma_3\sigma_2+\sigma_1\right) & 1-\sigma^2+2\sigma_3^2 \end{bmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \tag{3.149}$$

The MRP kinematic differential equation in vector form is[5, 22]

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[\left(1-\sigma^2\right)[I_{3\times 3}] + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega} = \frac{1}{4}[B(\boldsymbol{\sigma})]\boldsymbol{\omega} \tag{3.150}$$

Note that the MRPs retain a kinematic differential equation very similar to the classical Rodrigues parameters with only quadratic nonlinearity present. This equation holds for either set of MRPs. However, the resulting vector $\dot{\boldsymbol{\sigma}}$ will depend on which set of MRPs is being used. Just as a mapping exists between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^S$, a direct mapping between $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\sigma}}^S$ is given by[28]

$$\dot{\boldsymbol{\sigma}}^S = -\frac{\dot{\boldsymbol{\sigma}}}{\sigma^2} + \frac{1}{2}\left(\frac{1+\sigma^2}{\sigma^4}\right)\boldsymbol{\sigma}\boldsymbol{\sigma}^T\boldsymbol{\omega} \tag{3.151}$$

Let the matrix $[B]$ transform $\boldsymbol{\omega}$ in Eqs. (3.149) and (3.150) into $\dot{\boldsymbol{\sigma}}$. Turns out that this $[B]$ matrix is almost orthogonal except for a generally non-unit scaling factor. The inverse of $[B]$ can be written as

$$[B]^{-1} = \frac{1}{\left(1+\sigma^2\right)^2}[B]^T \tag{3.152}$$

To prove Eq. (3.152) let's study the expression $[B]^T[B]$. Using Eq. (3.150) this is written as

$$[B]^T[B] = \left(\left(1-\sigma^2\right)[I_{3\times3}] - 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)\left(\left(1-\sigma^2\right)[I_{3\times3}] + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)$$

After carrying out all the matrix multiplications the $[B]^T[B]$ expression is reduced to

$$[B]^T[B] = \left(1-\sigma^2\right)^2[I_{3\times3}] - 4[\tilde{\boldsymbol{\sigma}}]^2 + 4\boldsymbol{\sigma}\boldsymbol{\sigma}^T$$

which can be further simplified using the identity $[\tilde{\boldsymbol{\sigma}}]^2 = \boldsymbol{\sigma}\boldsymbol{\sigma}^T - \sigma^2[I_{3\times3}]$ to

$$[B]^T[B] = \left(1+\sigma^2\right)^2[I_{3\times3}]$$

At this point it is trivial to verify that Eq. (3.152) must hold. The inverse transformation of Eqs. (3.149) and (3.150) then is in matrix notation

$$\boldsymbol{\omega} = \frac{4}{\left(1+\sigma^2\right)^2}[B]^T\dot{\boldsymbol{\sigma}} \tag{3.153}$$

and in vector form[5]

$$\boldsymbol{\omega} = \frac{4}{\left(1+\sigma^2\right)^2}\left[\left(1-\sigma^2\right)[I_{3\times3}] - 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\dot{\boldsymbol{\sigma}} \tag{3.154}$$

Like the classical Rodrigues parameters, the MRPs can also be used to minimally parameterize higher-dimensional proper orthogonal matrix $[C]$. Let the $[S]$ be a skew-symmetric matrix. The extended Cayley transform of $[C]$ in terms of $[S]$ is[17, 29]

$$[C] = \left([I_{3\times3}] - [S]\right)^2\left(1+[S]\right)^{-2} = \left(1+[S]\right)^{-2}\left([I_{3\times3}] - [S]\right)^2 \tag{3.155}$$

where the order of the matrix products is again irrelevant. For the case where $[C]$ is a 3x3 matrix, then $[S]$ is the same as $[\tilde{\boldsymbol{\sigma}}]$. Therefore Eq. (3.155) transforms a higher dimensional proper orthogonal $[C]$ into higher dimensional MRPs.

Unfortunately no direct inverse transformation exists like Eq. (3.131) for the higher order Cayley transforms.[17] The transformation is achieved indirectly through the matrix $[W]$, where it is defined as the matrix square root of $[C]$.

$$[C] = [W][W] \tag{3.156}$$

Since $[C]$ is orthogonal, it can be spectrally decomposed as

$$[C] = [V][D][V]^* \tag{3.157}$$

where $[V]$ is the orthogonal eigenvector matrix and $[D]$ is the diagonal eigenvalue matrix with entries of unit magnitude. The "$*$" operator stands for the adjoint operator which performs the complex conjugate transpose of a matrix. The matrix $[W]$ can be computed as

$$[W] = [V]\begin{bmatrix} \ddots & & 0 \\ & \sqrt{[D]_{ii}} & \\ 0 & & \ddots \end{bmatrix}[V]^T \tag{3.158}$$

The eigenvalues of $[C]$ are typically complex conjugate pairs. If the dimension of $[C]$ is odd, then the extra eigenvalue is real. For proper orthogonal matrices it is $+1$ and its square root is also chosen to be $+1$. The resulting $[W]$ matrix will then itself also be an proper orthogonal matrix. As Ref. 17 shows, the geometric interpretation of $[W]$ is that it represents the same "higher-dimensional" orientation as $[C]$ except that the corresponding principal rotation angles are halved.

The standard Cayley transforms in Eqs. (3.130) and (3.131) can be applied to map $[W]$ into $[S]$ and back.

$$[W] = ([I] - [S])([I] + [S])^{-1} = ([I] + [S])^{-1}([I] - [S]) \tag{3.159}$$

$$[S] = ([I] - [W])([I] + [W])^{-1} = ([I] + [W])^{-1}([I] - [W]) \tag{3.160}$$

Therefore, to obtain a higher-dimensional MRP representation of $[C]$, the matrix $[W]$ must be found first and then substituted into Eq. (3.160). Note that substituting Eq. (3.159) into Eq. (3.156) a direct forward transformation from $[S]$ to $[C]$ is found.

$$[C] = ([I] - [S])^2([I] + [S])^{-2} = ([I] + [S])^{-2}([I] - [S])^2 \tag{3.161}$$

The kinematic differential equations for $[S]$ are not written directly in terms of $[C]$ as they were for the classical Cayley transform. Instead the $[W]$ matrix is used. Being an orthogonal matrix, its kinematic differential equation is of the same form as Eq. (3.27)

$$[\dot{W}] = -[\tilde{\mathbf{\Omega}}][W] \tag{3.162}$$

where $[\tilde{\boldsymbol{\Omega}}]$ is the corresponding angular velocity matrix. It is related to the $[\tilde{\boldsymbol{\omega}}]$ matrix in Eq. (3.27) through

$$[\tilde{\boldsymbol{\omega}}] = [\tilde{\boldsymbol{\Omega}}] + [W][\tilde{\boldsymbol{\Omega}}][W]^T \tag{3.163}$$

Analogously to Eq. (3.133), the kinematic differential equation of the $[S]$ matrix is given by

$$[\dot{S}] = \frac{1}{2}\left([I] + [S]\right)[\tilde{\boldsymbol{\Omega}}]\left([I] - [S]\right) \tag{3.164}$$

Example 3.10: Consider the same orthogonal 4x4 matrix $[C]$ as is defined in Example 3.8. Using MATLAB, its matrix square root $[W]$ is found to be

$$[W] = \begin{bmatrix} 0.86416 & -0.35312 & -0.14580 & 0.32754 \\ 0.37209 & 0.69343 & -0.44177 & -0.43076 \\ 0.25488 & 0.50816 & 0.79065 & 0.22734 \\ -0.22320 & 0.36911 & -0.39807 & 0.80962 \end{bmatrix}$$

Using Eq. (3.160) the higher dimensional, skew-symmetric MRP matrix $[S]$ representing $[C]$ is found.

$$[S] = \begin{bmatrix} 0 & 0.20952 & 0.10114 & -0.14383 \\ -0.20952 & 0 & 0.28309 & 0.24040 \\ -0.10114 & -0.28309 & 0 & -0.17471 \\ 0.14383 & -0.24040 & 0.17471 & 0 \end{bmatrix}$$

By back substitution of this $[S]$ into Eq. (3.155) it can be verified that it does indeed parameterize $[C]$.

3.7 Other Attitude Parameters

There exists a multitude of other attitude parameters sets in addition to those discussed so far. This section will briefly outline a selected few.

3.7.1 Stereographic Orientation Parameters

The Stereographic Orientation Parameters (SOPs) are introduced in Ref. 22. They are formed by projecting the Euler parameter constraint surface, a four-dimensional unit hypersphere, onto a three-dimensional hyperplane. The projection point can be anywhere on or within the constraint hypersphere, while the mapping hyperplane is chosen to be a unit distance away from the projection point.

There are two types of SOPs, the symmetric and asymmetric sets. The symmetric sets have a mapping hyperplane that is perpendicular to the β_0 axis. Since $\beta_0 = \cos\Phi/2$ only contains information about the principal rotation angle, the resulting sets will all have a geometric singularity at a specific principal rotation angle Φ only, regardless of the corresponding principal rotation axis $\hat{\boldsymbol{e}}$. The classical and modified Rodrigues parameters are examples of symmetric SOPs.

Asymmetric SOPs have a mapping hyperplane which is not perpendicular to the β_0 axis. The condition for a geometric singularity will now depend on both the principal rotation axis $\hat{\boldsymbol{e}}$ and the angle Φ. As an example, consider the asymmetric SOP vector $\boldsymbol{\eta}$. It is formed by having a projection point at $\beta_1 = -1$ and having a mapping hyperplane at $\beta_1 = 0$. In terms of the Euler parameters it is defined as

$$\eta_1 = \frac{\beta_0}{1+\beta_1} \quad \eta_2 = \frac{\beta_2}{1+\beta_1} \quad \eta_3 = \frac{\beta_3}{1+\beta_1} \tag{3.165}$$

with the inverse transformation being

$$\beta_0 = \frac{2\eta_1}{1+\eta^2} \quad \beta_1 = \frac{1-\eta^2}{1+\eta^2} \quad \beta_2 = \frac{2\eta_2}{1+\eta^2} \quad \beta_3 = \frac{2\eta_3}{1+\eta^2} \tag{3.166}$$

where $\eta^2 = \boldsymbol{\eta}^T\boldsymbol{\eta}$. From Eq. (3.165) it is evident that $\boldsymbol{\eta}$ has a geometric singularity whenever $\beta_1 \to -1$. This means that $\boldsymbol{\eta}$ goes singular whenever it represents a pure single-axis rotation about the first body axis by the principal angles Φ_1 = -180 degrees or Φ_2 = +540 degrees. This type of asymmetric principal angle rotation range is typical for all asymmetric SOPs. However, since the $\boldsymbol{\eta}$ vector has a distinct shadow counter part, any geometric singularities can be avoided by switching between the two sets through the mapping

$$\boldsymbol{\eta}^S = -\frac{\boldsymbol{\eta}}{\eta^2} \tag{3.167}$$

The direction cosine matrix is written in terms of the $\boldsymbol{\eta}$ vector components as

$$[C] = \frac{1}{(1+\eta^2)^2}\begin{bmatrix} 4\left(\eta_1^2-\eta_2^2-\eta_3^2\right)+(1-\eta^2)^2 & 8\eta_1\eta_3+4\eta_2(1-\eta^2) & -8\eta_1\eta_2+4\eta_3(1-\eta^2) \\ -8\eta_1\eta_3+4\eta_2(1-\eta^2) & 4\left(\eta_1^2+\eta_2^2-\eta_3^2\right)-(1-\eta^2)^2 & 8\eta_2\eta_3+4\eta_1(1-\eta^2) \\ 8\eta_1\eta_2+4\eta_3(1-\eta^2) & 8\eta_2\eta_3-4\eta_1(1-\eta^2) & 4\left(\eta_1^2-\eta_2^2+\eta_3^2\right)-(1-\eta^2)^2 \end{bmatrix} \tag{3.168}$$

The kinematic differential equation of the $\boldsymbol{\eta}$ vector is

$$\dot{\boldsymbol{\eta}} = \frac{1}{4}\begin{bmatrix} -1-2\eta_1^2+\eta^2 & 2\left(\eta_1\eta_3-\eta_2\right) & -2\left(\eta_1\eta_2+\eta_3\right) \\ 2\left(\eta_3-\eta_1\eta_2\right) & 2\left(\eta_2\eta_3+\eta_1\right) & -1-2\eta_2^2+\eta^2 \\ -2\left(\eta_1\eta_3+\eta_2\right) & 1+2\eta_3^2-\eta^2 & 2\left(\eta_1-\eta_2\eta_3\right) \end{bmatrix}\boldsymbol{\omega} \tag{3.169}$$

Having a projection point on the constraint surface provides for the largest possible range of singularity free rotations. This is evident when comparing the classical and the modified Rodrigues parameters. The classical Rodrigues parameters have a projection point within the constraint hypersphere at $\beta_0 = 0$. Their principal rotation range is half of that of the MRPs whose projection is on the constraint surface at $\beta_0 = -1$.

3.7.2 Higher Order Rodrigues Parameters

The Higher Order Rodrigues Parameters (HORP) are introduced in Ref. 29. The classical Cayley transform in Eq. (3.130) is expanded such that it parameterized nxn orthogonal matrices through a skew-symmetric, higher order Rodrigues parameter matrix X.

$$[C] = ([I_{3\times3}] - X)^m \, ([I_{3\times3}] + X)^{-m} \tag{3.170}$$

The corresponding attitude vector $\boldsymbol{x}$ is given by

$$\boldsymbol{x} = \tan\left(\frac{\Phi}{2m}\right)\hat{\boldsymbol{e}} \tag{3.171}$$

For $m = 1$ the vector $\boldsymbol{x}$ is the classical Rodrigues vector and for $m = 2$ it is the MRP vector. Note that the domain of validity of the $\boldsymbol{x}$ vector is $|\Phi| < m\pi$. The HORP sets are generally also not unique as is the case with the MRPs. Corresponding "shadow" sets can be used here too to avoid any geometric singularities. Note that for a given m there are typically m sets of possible HORPs.

A particular set of HORP is the $\boldsymbol{\tau}$ vector where $m = 4$. In terms of the Euler parameters, the first two HORP vectors $\boldsymbol{\tau}$ are defined through

$$\tau_i = \frac{\beta_i}{1 + \beta_0 \pm \sqrt{2\,(1 + \beta_0)}} \qquad i = 1, 2, 3 \tag{3.172}$$

with the inverse transformation being

$$\beta_0 = 2\left(\frac{1 - \tau^2}{1 + \tau^2}\right)^2 - 1 \quad \beta_i = \frac{4\tau_i\left(1 - \tau^2\right)}{\left(1 + \tau^2\right)^2} \quad i = 1, 2, 3 \tag{3.173}$$

where $\tau^{2n} = \left(\boldsymbol{\tau}^T\boldsymbol{\tau}\right)^n$. Each vector τ defined in Eq. (3.172) can be mapped to the corresponding shadow vector τ^S through

$$\boldsymbol{\tau}^S = -\boldsymbol{\tau}\left(\frac{1 - \tau^2}{2\tau^2 + (1 + \tau^2)\,\tau}\right) \tag{3.174}$$

where $\tau = \sqrt{\tau^2}$. Combined Eqs. (3.172) and (3.174) yield the four possible HORP vectors for $m = 4$. In terms of the principal rotation elements, the four sets can be expressed as

$$\boldsymbol{\tau} = \tan\left(\frac{\Phi - 2k\pi}{8}\right) \qquad k = 0, 1, 2, 3 \tag{3.175}$$

Therefore it will always be possible to switch from one $\boldsymbol{\tau}$ vector to another in order to avoid geometric singularities.

The kinematic differential equations of the $\boldsymbol{\tau}$ vector are

$$\dot{\boldsymbol{\tau}} = \frac{1}{8\,(1-\tau^2)}\left[2\left(3-\tau^2\right)\boldsymbol{\tau}\boldsymbol{\tau}^T + 4\left(1-\tau^2\right)[\tilde{\boldsymbol{\tau}}] + (1-6\tau^2+\tau^4)[I_{3\times3}]\right]\omega \tag{3.176}$$

Note that the kinematic differential equations of the HORP lose the simple second order polynomial form that is present for the classical and modified Rodrigues parameters. Also, while the $\boldsymbol{\tau}$ vector itself is defined for rotations up to $\Phi = m\pi$, the kinematic differential equations encounter mathematical singularities of the type $0/0$ whenever $\tau^2 \to 0$. This corresponds to $\Phi \to \pm 360$ degrees. By using the mapping in Eq. (3.174) to transform a τ vector to an alternate set whenever $|\boldsymbol{\tau}| \geq \tan(\Phi/8)$ any geometrical and mathematical singularities are avoided all together.

3.7.3 The (w, z) Coordinates

The (w, z) attitude coordinates were introduced by Tsiotras and Longuski in Ref. 30. They are a minimal coordinate set and lend themselves well to be used in control problems of under actuated axially-symmetric spacecraft.[31] The complex coordinate w describes the heading of the one of the body axes, typically the spin axis. The coordinate z is the relative rotation angle about this axis defined by w. Let the heading of the chosen body axis be given by the vector $\hat{\boldsymbol{b}}_i = (a, b, c)^T$. Since the vector $\hat{\boldsymbol{b}}_i$ is a unit vector, the three components a, b and c are not independent. They must satisfy the constraint sphere equation

$$a^2 + b^2 + c^2 = 1 \tag{3.177}$$

By performing a stereographic projection of the constraint sphere from the projection point $(0, 0, -1$ onto the complex (w_1, w_2) plane, the three redundant axis heading coordinates (a, b, c) are reduced to the complex variable w.

$$w = w_1 + iw_2 = \frac{b - ia}{1 + c} \tag{3.178}$$

The inverse transformation from w to (a, b, c) is given by

$$a = \frac{i(w - \bar{w})}{1 + |w|^2} \quad b = \frac{w + \bar{w}}{1 + |w|^2} \quad c = \frac{1 - |w|^2}{1 + |w|^2} \tag{3.179}$$

Let's assume that the spin axis is the third body axis, then the direction cosine matrix in terms of (w, z) is given by

$$[C] = \frac{1}{1 + |w|^2} \begin{bmatrix} Re\left[\left(1 + w^2\right)e^{iz}\right] & Im\left[\left(1 + w^2\right)e^{iz}\right] & -2Im(w) \\ Im\left[\left(1 - \bar{w}^2\right)e^{-i\bar{z}}\right] & Re\left[\left(1 - \bar{w}^2\right)e^{-iz}\right] & 2Re(w) \\ 2Im(we^{iz}) & -2Re(we^{iz}) & 1 - |w|^2 \end{bmatrix} \tag{3.180}$$

The kinematic differential equations of the (w, z) coordinates are given by

$$\dot{w}_1 = \omega_3 w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2}\left(1 + w_1^2 - w_2^2\right) \tag{3.181a}$$

$$\dot{w}_2 = -\omega_3 w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2}\left(1 + w_2^2 - w_1^2\right) \tag{3.181b}$$

$$\dot{z} = \omega_3 - \omega_1 w_2 \omega_2 w_1 \tag{3.181c}$$

3.7.4 Cayley-Klein Parameters

The Cayley-Klein parameters are a set of four complex parameters which are closely related to the Euler parameter vector $\boldsymbol{\beta}$. They form a once-redundant, non-singular set of attitude parameters. Let $i = \sqrt{-1}$, then they are defined in terms of $\boldsymbol{\beta}$ as[16]

$$\begin{aligned} \alpha &= \beta_0 + i\beta_3 & \beta &= -\beta_2 + i\beta_1 \\ \gamma &= \beta_2 + i\beta_1 & \delta &= \beta_0 - i\beta_3 \end{aligned} \tag{3.182}$$

The inverse transformation from the Euler parameters to the Cayley-Klein parameters is

$$\begin{aligned} \beta_0 &= (\alpha + \delta)/2 & \beta_1 &= -i(\beta + \gamma)/2 \\ \beta_2 &= -(\beta - \gamma)/2 & \beta_3 &= -i(\alpha - \delta)/2 \end{aligned} \tag{3.183}$$

The direction cosine matrix is parameterized by the Cayley-Klein parameters as

$$[C] = \begin{bmatrix} \left(\alpha^2-\beta^2-\gamma^2+\delta^2\right)/2 & i\left(-\alpha^2+\beta^2-\gamma^2+\delta^2\right)/2 & (\beta\delta - \alpha\gamma) \\ i\left(\alpha^2+\beta^2-\gamma^2-\delta^2\right)/2 & \left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)/2 & -i(\alpha\gamma + \beta\delta) \\ (\gamma\delta - \alpha\beta) & i(\alpha\beta + \gamma\delta) & (\alpha\delta + \beta\gamma) \end{bmatrix} \tag{3.184}$$

3.8 Homogeneous Transformations

All previous sections in this chapter deal with methods to describe the relative orientation of one coordinate frame to another. In particular, the direction cosine matrix is a convenient tool to map a vector with components taken in one reference frame to a vector with components taken in another. However, one underlying assumption here is that both reference frames have the same origin. In other words, any translational differences between the two frames in questions is not taken into account when the vector components are mapped from one frame to another.

Figure 3.12 shows an illustration where two coordinates frames differ both in orientation and in their origins. Let us define the following two reference fame $\mathcal{N}$ and $\mathcal{B}$.

$$\mathcal{N} : \{\mathcal{O}_{\mathcal{N}}, \hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$$
$$\mathcal{B} : \{\mathcal{O}_{\mathcal{B}}, \hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$$

Let the position vector from the $\mathcal{N}$ frame origin to the $\mathcal{B}$ frame origin be given by $\boldsymbol{r}_{\mathcal{B}/\mathcal{N}}$. The position vector of point P is expressed in $\mathcal{B}$ frame components as ${}^{\mathcal{B}}\boldsymbol{r}_p$. These vector components are mapped into $\mathcal{N}$ frame components by pre-multiplying by the direction cosine matrix $[NB]$. While this provides the correct $\mathcal{N}$ frame components of the vector $\boldsymbol{r}_P$, it does not provide the correct

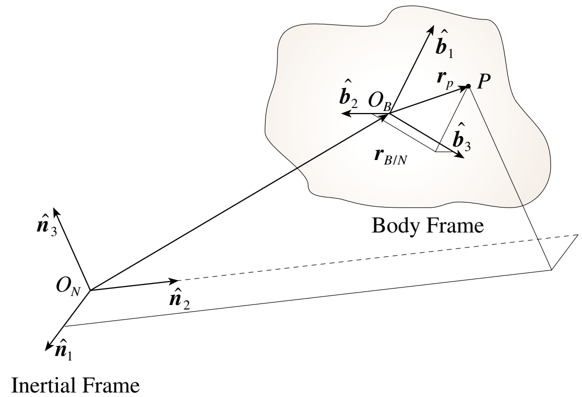


Figure 3.12: Illustration of two Coordinate Frames with Different Origins and Orientations

position vector of point P as seen by the $\mathcal{N}$ frame since the two frames have different origins. To obtain these vector components, we compute

$$^{\mathcal{N}}\boldsymbol{r}_p = {}^{\mathcal{N}}\boldsymbol{r}_{\mathcal{B}/\mathcal{N}} + [NB]^{\mathcal{B}}\boldsymbol{r}_p \tag{3.185}$$

By defining the 4×4 *homogeneous transformation*[32]

$$[\mathcal{N}\mathcal{B}] = \begin{bmatrix} NB & {}^{\mathcal{N}}\boldsymbol{r}_{\mathcal{B}/\mathcal{N}} \\ 0_{1\times 3} & 1 \end{bmatrix} \tag{3.186}$$

it is possible to transform the position vector taken in $\mathcal{B}$ frame components directly into the corresponding position vector in $\mathcal{N}$ frame components. In robotics literature, this transformation is typically referred to as ${}^{\mathcal{N}}_{\mathcal{B}}T$. To accomplish this, we define the 4×1 position vector

$$^{\mathcal{B}}\boldsymbol{p} = \begin{bmatrix} {}^{\mathcal{B}}\boldsymbol{r}_p \\ 1 \end{bmatrix} \tag{3.187}$$

Observing Eq. (3.185), it is clear that

$$^{\mathcal{N}}\boldsymbol{p} = [\mathcal{N}\mathcal{B}]^{B}\boldsymbol{p} \tag{3.188}$$

This formula is very convenient when computing the position coordinate of a chain of bodies such as are typically found in robotics applications. However, care must be taken when considering the order of the translational and rotational differences between the two frames. The homogenous transformation, as shown in Eq. (3.186), performs the translation first and the rotation second. This order is important. Assume a rotational joint has a telescoping member attached to it. To compute the homogeneous transformation from the joint to the telescoping member tip, a rotation must be performed first and a translation second.

Note that this homogenous transformation matrix abides by the same successive transformation property as the direction cosine matrix does. Consider the two vectors

$$^{\mathcal{A}}\boldsymbol{p} = [\mathcal{AB}]^{\mathcal{B}}\boldsymbol{p} \tag{3.189}$$

$$^{\mathcal{N}}\boldsymbol{p} = [\mathcal{NA}]^{\mathcal{A}}\boldsymbol{p} \tag{3.190}$$

Substituting Eq. (3.189) into (3.190), we find that

$$^{\mathcal{N}}\boldsymbol{p} = [\mathcal{NA}][\mathcal{AB}]^{\mathcal{B}}\boldsymbol{p} = [\mathcal{NB}]^{\mathcal{B}}\boldsymbol{p} \tag{3.191}$$

Thus, two successive transformations are combined through

$$[\mathcal{NB}] = [\mathcal{NA}][\mathcal{AB}] \tag{3.192}$$

However, the inverse matrix formula for the homogeneous transformation is not quite as elegant as the matrix inverse of the orthogonal direction cosine matrix. The following partitioned matrix inverse is convenient to compute the inverse of the $[\mathcal{NB}]$. Let $[M]$ be defined as[4]

$$[M] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{3.193}$$

Then the inverse is given by

$$[M]^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\Delta^{-1}CA^{-1} & -A^{-1}B\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{bmatrix} \tag{3.194}$$

with the Schur complement being defined as

$$\Delta = D - CA^{-1}B \tag{3.195}$$

Substituting

$$[A] = [NB] \qquad [B] = [^{\mathcal{N}}\boldsymbol{r}_{\mathcal{B}/\mathcal{N}}]$$
$$[C] = [0_{1\times 3}] \qquad [D] = [1]$$

the Schur complement is given by

$$\Delta = [1] \tag{3.196}$$

and the inverse of the homogeneous transformation is the remarkable simply formula:

$$[\mathcal{NB}] = \begin{bmatrix} [NB]^T & -[NB]^T\, {}^{\mathcal{N}}\boldsymbol{r}_{\mathcal{B}/\mathcal{N}} \\ 0_{1\times 3} & 1 \end{bmatrix} \tag{3.197}$$

Here the fact was used that $[NB]$ is orthogonal and that $[NB]^{-1} = [NB]^T$.

Problems

3.1 Given three reference frames $\mathcal{N}$, $\mathcal{B}$ and $\mathcal{F}$, let the unit base vectors of the reference frames $\mathcal{B}$ and $\mathcal{F}$ be

$$\hat{\boldsymbol{b}}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \hat{\boldsymbol{b}}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \hat{\boldsymbol{b}}_3 = \frac{1}{3\sqrt{2}}\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

and

$$\hat{\boldsymbol{f}}_1 = \frac{1}{4}\begin{pmatrix} 3 \\ -2 \\ \sqrt{3} \end{pmatrix} \quad \hat{\boldsymbol{f}}_2 = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} \quad \hat{\boldsymbol{f}}_3 = \frac{-1}{4}\begin{pmatrix} \sqrt{3} \\ 2\sqrt{3} \\ 1 \end{pmatrix}$$

where the base vector components are written in the $\mathcal{N}$ frame. Find the direction cosine matrices $[BF]$ that describes the orientation of the $\mathcal{B}$ frame relative to the $\mathcal{F}$ frame, along with the direction cosine matrices $[BN]$ and $[FN]$ that map vectors in the $\mathcal{N}$ frame into respective $\mathcal{B}$ or $\mathcal{F}$ frame vectors.

3.2 Let the vector $\boldsymbol{v}$ be written in $\mathcal{B}$ frame components as

$${}^{\mathcal{B}}\boldsymbol{v} = 1\hat{\boldsymbol{b}}_1 + 2\hat{\boldsymbol{b}}_2 - 3\hat{\boldsymbol{b}}_3$$

The orientation of the $\mathcal{B}$ frame relative to the $\mathcal{N}$ frame is given through the direction cosine matrix

$$[BN] = \begin{bmatrix} -0.87097 & 0.45161 & 0.19355 \\ -0.19355 & -0.67742 & 0.70968 \\ 0.45161 & 0.58065 & 0.67742 \end{bmatrix}$$

a) Find the direction cosine matrix $[NB]$ that maps vectors with components in the $\mathcal{B}$ frame into a vector with $\mathcal{N}$ frame components.

b) Find the $\mathcal{N}$ frame components of the vector $\boldsymbol{v}$.

3.3 Using the direction cosine matrix $[BN]$ in Problem 3.2, find its *real* eigenvalue and corresponding eigenvector.

3.4 ♣ The angular velocity vectors of a spacecraft $\mathcal{B}$ and a reference frame motion $\mathcal{R}$ relative to the inertial frame $\mathcal{N}$ are given by $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ and $\boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}}$. The vector $\boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}}$ is given in $\mathcal{R}$ frame components, while $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ is given in $\mathcal{B}$ frame components. The error angular velocity vector of the spacecraft relative to the reference motion is then given by $\delta\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}}$. Find the relative error angular acceleration vector $\delta\dot{\boldsymbol{\omega}}$ with components expressed in the $\mathcal{B}$ frame.

a) Find $\delta\dot{\boldsymbol{\omega}}$ by only assigning vector frames at the last step.

b) Find $\delta\dot{\boldsymbol{\omega}}$ by first expressing $\delta\boldsymbol{\omega}$ in $\mathcal{B}$ frame components as ${}^{\mathcal{B}}\delta\boldsymbol{\omega} = {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [BR]{}^{\mathcal{R}}\boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}}$ and then performing an inertial derivative.

3.5 The reference frames $\mathcal{N} : \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ and $\mathcal{B} : \{\hat{\boldsymbol{b}}_L, \hat{\boldsymbol{b}}_\theta, \hat{\boldsymbol{b}}_r\}$ are shown in Figure 3.13.

a) Find the direction cosine matrix $[BN]$ in terms of the angle ϕ.

b) Given the vector ${}^{\mathcal{B}}\boldsymbol{v} = 1\hat{\boldsymbol{b}}_r + 1\hat{\boldsymbol{b}}_\theta + 2\hat{\boldsymbol{b}}_L$, find the vector ${}^{\mathcal{N}}\boldsymbol{v}$.

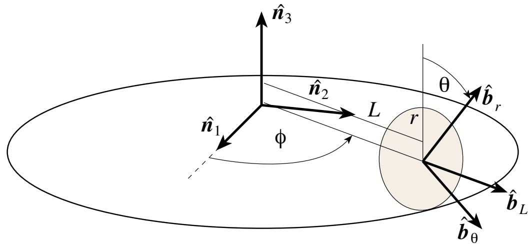


Figure 3.13: Disk Rolling on Circular Ring

3.6 Starting with Eqs. (3.21) and (3.22), verify Eq. (3.24).

3.7 Parameterize the direction cosine matrix $[C]$ in terms of (2-3-2) Euler angles. Also, find appropriate inverse transformations from $[C]$ back to the (2-3-2) Euler angles.

3.8 Find the kinematic differential equations of the (2-3-2) Euler angles. What is the geometric condition for which these equations will encounter a mathematical singularity.

3.9 Given the (3-2-1) Euler angles $\psi = 10°$, $\theta = -15°$ and $\phi = 20°$ and their rates $\dot{\psi} = 2°/s$, $\dot{\theta} = 1°/s$ and $\dot{\phi} = 0°/s$, find the vectors ${}^{\mathcal{B}}\boldsymbol{\omega}$ and ${}^{\mathcal{N}}\boldsymbol{\omega}$.

3.10 The orientation of an object is given in terms of the (3-1-3) Euler angles $(-30°, 40°, 20°)$,

a) find the corresponding principal rotation axis $\hat{\boldsymbol{e}}$

b) find the two principal rotation angles Φ and Φ'

3.11 A spacecraft performs a $45°$ single axis rotation about $\hat{\boldsymbol{e}} = \frac{1}{\sqrt{3}}(1,1,1)^T$. Find the corresponding (3-2-1) yaw, pitch and roll angles that relate the final attitude to the original attitude.

3.12 Verify that the exponential matrix mapping $[C] = e^{-\Phi[\tilde{\boldsymbol{e}}]}$ does have the finite form given in Eq. (3.81).

3.13 Verify that Eq. (3.89) is indeed the inverse mapping of the differential kinematic equation of $\dot{\boldsymbol{\gamma}}$ given in Eq. (3.88).

3.14 ♣ Starting from the direction cosine matrix $[C]$ in Eq. (3.71) written in terms of the principal rotation elements, derive the parameterization of $[C]$ in terms of the Euler parameters.

3.15 ♣ Derive the composite rotation property of the Euler parameter vector given in Eqs. (3.97) and (3.98).

3.16 Derive the kinematic differential equations for the second, third and fourth Euler parameter.

3.17 Verify the transformation in Eq. (3.114) which maps a classical Rodrigues parameter vector into an Euler parameter vector.

3.18 Show the details of transforming the classical Rodrigues parameter definition in terms of the Euler parameters $q_i = \beta_i/\beta_0$ into the expression $q_i = \tan\frac{\Phi}{2}\hat{e}_i$ which is in terms of the principal rotation elements.

3.19 ♣ Show that the classical Rodrigues parameters are indeed a stereographic projection of the Euler parameter constraint surface (a four-dimensional unit hypersphere) onto the three-dimensional hyperplane tangent to $\beta_0 = 1$ with the projection point being $\boldsymbol{\beta} = (0,0,0,0)^T$.

3.20 Given the classical Rodrigues parameter vector $\boldsymbol{q} = (0.5, -0.2, 0.8)^T$. Use the Cayley transform in Eq. (3.130) to find the corresponding direction cosine matrix $[C]$. Also, verify that this $[C]$ is the same as is obtained through the mapping in Eq. (3.118) or (3.119).

3.21 Verify the transformation in Eq. (3.135) which maps a MRP vector into an Euler parameter vector.

3.22 Show the details of transforming the MRP definition in terms of the Euler parameters $\sigma_i = \beta_i/(1+\beta_0)$ into the expression $\sigma_i = \tan\frac{\Phi}{4}\hat{e}_i$ which is in terms of the principal rotation elements.

3.23 ♣ Show that the MRPs are a stereographic projection of the Euler parameter constraint surface (a four-dimensional unit hypersphere) onto the three-dimensional hyperplane tangent to $\beta_0 = 0$ with the projection point being $\boldsymbol{\beta} = (-1,0,0,0)^T$.

3.24 Derive the MRP parameterization of the direction cosine matrix $[C]$ given in Eq. (3.143).

3.25 Let the initial attitude vector be given through the MRP vector $\boldsymbol{\sigma}(t_0) = (0,0,0)^T$. The body angular velocity vector $\boldsymbol{\omega}(t)$ is given as $(1, 0.5, -0.7)^T$ rad/second. Integrate the resulting rotation for 5 seconds and use the mapping between "original" and "shadow" MRPs in Eq. (3.139) to enforce $|\boldsymbol{\sigma}| \leq 1$.

3.26 ♣ Derive the mapping between $\dot{\boldsymbol{\sigma}}$ and its shadow counter part $\dot{\boldsymbol{\sigma}}^S$ in Eq. (3.151) starting with the kinematic differential equation of the MRP in Eq. (3.150) and Eq. (3.139).

3.27 Given the MRP vector $\boldsymbol{\sigma} = (-0.25, -0.4, 0.3)^T$. Use the Cayley transform in Eq. (3.161) to find the corresponding direction cosine matrix $[C]$. Also, verify that this $[C]$ is the same as is obtained through the mapping in Eq. (3.143) or (3.144).

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