Educational Material: Math Application Problem Set

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Abstract

This material is developed to provide interesting and useful application of mathematics and physics for high school and undergraduate students. The problems are related to other professional fields, such as physics, biology, chemistry, and everyday life. The author decided to archive the material such that it is easy to be shared and used.

Move to the next page for the problems and solutions.

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1 Calculus + Biology Problems

1. PCR Optimization Problem

   You are running a biology lab business and you need to find the right temperature for an enzyme (polymerase) to work.

   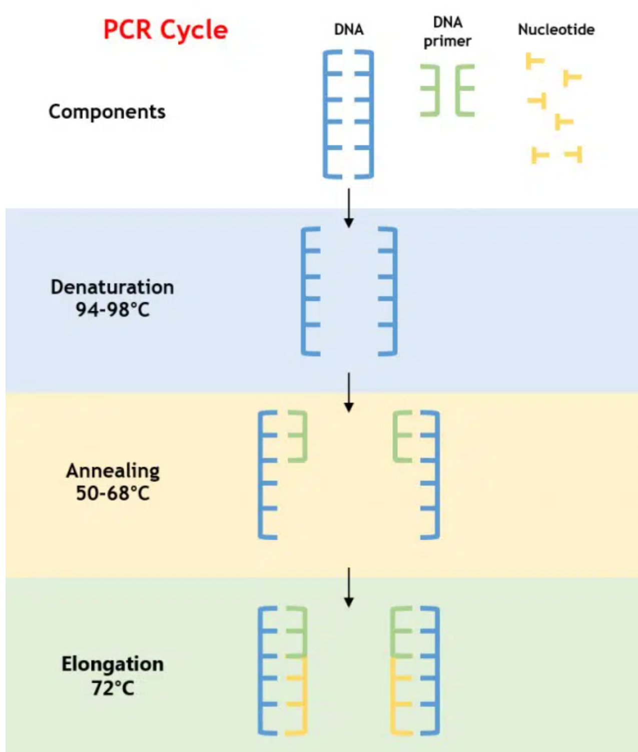

   Figure 1: PCR process diagram. Source: clinisciences

   Polymerase Chain Reaction (PCR) is a widely used technique in biology and bioengineering to duplicate DNA copies and amplify the quantity of DNA. It requires multiple components, but one key ingredient is polymerase. We will focus on polymerase to simplify the problem.

   The most widely-used polymerase for PCR is Taq polymerase. This protein is special in the sense that it doesn't get denatured in high temperature. This makes it the best polymerase for PCR.

Bio Quiz: Do you know why it is good for the polymerase to be heat-resistent for PCR? If you know, try explaining 😀. If you don't, you can look up online too.

(a) We want to find the best temperature for the "annealing" process, where primers attach to single-stranded DNA helix. Let's say the efficiency of annealing process A is given as a function of temperature T. The functional form is as follows:

$$A(T) = e^{-a(T-50)^2}$$

where a is a positive number. Find the temperature value T_A that maximizes the annealing efficiency $A(T)$. What is the best efficiency in this case?

(b) Now, we want to find the best temperature for the "elongation" process, where polymerase extends DNA strands by putting nucleotides. The efficiency of elongation E is also given as a function of temperature T. The functional form is as follows:

$$E(T) = e^{-b(T-80)^2}$$

where b is a positive number. Find the best temperature value T_E that maximizes the efficiency of elongation $E(T)$. What is the best efficiency in this case?

(c) If we perform the annealing process at one temperature T_1, and the elongation process at another temperature T_2, the overall efficiency is defined as the product of the annealing efficiency at T_1 and the elongation efficiency at T_2 as follows:

$$\begin{aligned} A(T_1) \cdot E(T_2) &= e^{-a(T_1-50)^2} \cdot e^{-b(T_2-80)^2} \\ &= e^{-a(T_1-50)^2-b(T_2-80)^2} \end{aligned}$$

Confirm that you get the efficiency of 1 when you plug in the best temperatures you calculated: $T_A \to T_1$ and $T_E \to T_2$.

Confirm that when $T_1 \neq T_A$ or $T_2 \neq T_E$, the overall efficiency drops below 1.

(d) Emergency! Our PCR machine is broken and we cannot set two different temperatures for annealing and elongation separately. We can only set one single temperature for annealing and elongation. In this case, we would like to maximize the overall efficiency $O(T)$, defined as the product of $A(T)$ and $E(T)$:

$$\begin{aligned} O(T) = A(T) \cdot E(T) &= e^{-a(T-50)^2} \cdot e^{-b(T-80)^2} \\ &= e^{-a(T-50)^2-b(T-80)^2} \end{aligned}$$

Find the best temperature value T_O that maximizes the overall emergency efficiency $O(T)$. The result should be expressed with a and b.

(e) What is the T_O value if $a = b$?

(f) What is the T_O value if $a = 4b$?

(g) Show that T_O always falls between T_A and T_E regardless of the value of a and b, given that $a > 0$ and $b > 0$. Does it make intuitive sense that T_O must be inbetween T_A and T_E?

(h) What is the best emergency efficiency $O(T)$ when $a = b = \frac{1}{1000}$? Use a calculator to evaluate. Compared to the normal situation, what is the efficiency loss in the emergency situation?

The solution of the problems can be found in: page 6 of the following document:

maverick-oh.github.io/assets/pdf/Educational_Material.pdf

2 Calculus + Biology Solutions

1. PCR Optimization Solution

   (a) From $A(T) = e^{-a(T-50)^2}$,

   $$\frac{\mathrm{d}A}{\mathrm{d}T} = e^{-a(T-50)^2} \cdot \big(-2a(T-50) \big).$$

   Because the exponential part is always positive, the sign of $\frac{\mathrm{d}A}{\mathrm{d}T}$ is the same as the sign of $-2a(T-50)$.

   $$\operatorname{sign}\left(\frac{\mathrm{d}A}{\mathrm{d}T}\right) = \operatorname{sign}\big(-2a(T-50) \big).$$

   When $T = 50$, $\operatorname{sign}\left(\frac{\mathrm{d}A}{\mathrm{d}T}\right) = 0$ and we can verify that the sign $\left(\frac{\mathrm{d}A}{\mathrm{d}T}\right)$ changes from positive to negative around $T = 50$, which means it is a local maximum.
   $T_A = 50$ and best efficiency is given as $A(T_A) = e^0 = 1$.

   (b) Similarly to the previous problem, from $E(T) = e^{-b(T-80)^2}$,

   $$\operatorname{sign}\left(\frac{\mathrm{d}E}{\mathrm{d}T}\right) = \operatorname{sign}\big(-2b(T-80) \big).$$

   At $T = 80$, $E(T)$ has a local maximum.
   $T_E = 80$ and best efficiency is given as $E(T_E) = e^0 = 1$.

   (c) $A(T_1) \cdot E(T_2) = e^{-a(T_1-50)^2-b(T_2-80)^2}$. Plugging $T_A = 50 \to T_1$ and $T_E = 80 \to T_2$ gives the efficiency 1:

   $$A(50) \cdot E(80) = e^{-a(50-50)^2-b(80-80)^2} = e^0 = 1$$

   In other cases, when $T_1 \neq T_A$ or $T_2 \neq T_E$, the exponent becomes a negative number and thus the efficiency becomes less than 1.

   $$-a(T_1-50)^2 - b(T_2-80)^2 < 0, \quad e^{-a(T_1-50)^2-b(T_2-80)^2} < 1$$

(d) From $O(T) = e^{-a(T-50)^2 - b(T-80)^2}$,

$$\begin{aligned}
\frac{\mathrm{d}O}{\mathrm{d}T} &= e^{-a(T-50)^2 - b(T-80)^2} \cdot \big(-2a(T-50) - 2b(T-80) \big) \\
\operatorname{sign}\left(\frac{\mathrm{d}O}{\mathrm{d}T}\right) &= \operatorname{sign}\big(-2a(T-50) - 2b(T-80) \big) \\
&= \operatorname{sign}\big((-2a-2b)T + 100a + 160b \big) \\
\operatorname{sign}\left(\frac{\mathrm{d}O}{\mathrm{d}T}\right) &= 0 \quad \text{when } T = \frac{100a+160b}{2a+2b} = \frac{50a+80b}{a+b}
\end{aligned}$$

Thus, $T_O = \frac{50a+80b}{a+b}$. Note that T_O is a function of the parameters a and b.

(e) Plugging in $a = b$ gives:

$$T_O = \frac{50b+80b}{b+b} = \frac{130}{2} = 65.$$

(f) Plugging in $a = 4b$ gives:

$$T_O = \frac{50 \cdot 4b + 80b}{4b+b} = \frac{280}{5} = 56.$$

(g) I present two ways to show that $T_A < T_O < T_E$.

- **By replacing** $a = kb$

  Because both a and b are positive numbers, we can introduce a positive number k that satisfies $a = kb$. Then,

  $$T_O = \frac{50a+80b}{a+b} = \frac{50kb+80b}{kb+b} = \frac{50k+80}{k+1} = 50 + \frac{30}{k+1}.$$

  k is a positive number, thus

  $$k > 0, \quad 1+k > 1, \quad 0 < \frac{1}{1+k} < 1, \quad 0 < \frac{30}{1+k} < 30$$

  and thus $50 < T_O < 80$.

- **By geometric argument using weighted average and internal section point**

  For any given numbers A and B, the number that is at the midpoint of A and B is the average value:

  $$\mu = \frac{A+B}{2} = \frac{1 \cdot A + 1 \cdot B}{1+1}.$$

  The concept of "average" can be generalized to "weighted average". Here, "weight" means how important you consider each element. The general formula of weighted average of A and B with the weight m and n is given as:

  $$\omega = \frac{mA + nB}{m+n}$$

  Note that the average value μ coincides with the weighted average value ω if $m = n$; when the weights are given equally.

  For example, the weighted average of A and B with the weights 9 and 1 means that you consider 9 times more of A than B, and the weighted average will be 9 times closer to A than B (internal section point of A and B with the ratio 1 and 9). For instance, the weighted average of 100 and 200 with the weight 9 and 1 would be 110.

  Looking back to the quantity we are interested in,

  $$T_O = \frac{50a + 80b}{a+b}$$

  This is the weighted average of 50 and 80 with the weight a and b. Thus, given that a and b are positive, T_O is in between 50 and 80. In other words, T_O is the internal section point of 50 and 80 that divides them by the ratio b and a.

  It makes sense that T_O is in between T_A and T_E, because T_O should be the temperature that should let annealing and elongation work the best at the same time.

(h) Given that $a = b$, $T_O = \frac{50a+80b}{a+b} = 65$. Plugging it together with $a = b = \frac{1}{1000}$ to $O(T) = e^{-a(T-50)^2-b(T-80)^2}$ gives

$$O(65) = e^{-\frac{1}{1000}15^2-\frac{1}{1000}15^2} = e^{-0.45} \approx 0.63$$

The best overall emergency efficiency is lower than the best non-emergency efficiency $A(T_A) \cdot E(T_E) = 1$. The efficiency loss is $1 - e^{-0.45}$, which is approximately 0.37 or 37%