Contents

1 Bisection Method

1. Identify the interval $[a, b]$ that contains the root of the function $f(x)$. This means finding two points a and b such that $f(a)$ and $f(b)$ have opposite signs (i.e., $f(a) * f(b) < 0$). This interval can be obtained either graphically or algebraically.
2. Divide the interval $[a, b]$ into two equal sub-intervals by finding the midpoint

$$c = \frac{a+b}{2}$$

3. Evaluate the function $f(c)$ at the midpoint c. If $f(c) = 0$, then c is the root of the function and we are done.
4. If $f(c)$ has the same sign as $f(a)$, then the root must lie in the interval $[c, b]$. So, set $a = c$ and go to step 2.
5. If $f(c)$ has the same sign as $f(b)$, then the root must lie in the interval $[a, c]$. So, set $b = c$ and go to step 2.
6. Repeat steps $2-5$ until you obtain an interval $[a, b]$ that is small enough or until $f(c)$ is sufficiently close to zero.
7. The final value of c obtained is the approximate root of the function $f(x)$ within the interval $[a, b]$.

NOTE: The Bolzano method guarantees convergence to a root of the function as long as the function is continuous on the interval $[a, b]$. However, it does not guarantee uniqueness of the root, nor does it give an estimate of the error in the approximation.

2 Reguli Falsi Method

1. Identify the interval $[a, b]$ that contains the root of the function $f(x)$. This means finding two points a and b such that $f(a)$ and $f(b)$ have opposite signs (i.e., $f(a) * f(b) < 0$). This interval can be obtained either graphically or algebraically.
2. Evaluate the function $f(a)$ and $f(b)$ at the endpoints a and b.
3. Calculate the x-intercept of the straight line that connects the points $(a, f(a))$ and $(b, f(b))$. This x-intercept is given by the formula:

$$c = a - \frac{f(a)(b-a)}{f(b) - f(a)}$$

4. Evaluate the function $f(c)$ at the point c.
5. If $f(c) = 0$, then c is the root of the function and we are done.
6. If $f(c)$ has the same sign as $f(a)$, then the root must lie in the interval $[c, b]$. So, set $a = c$ and go to step 2.
7. If $f(c)$ has the same sign as $f(b)$, then the root must lie in the interval $[a, c]$. So, set $b = c$ and go to step 2.
8. Repeat steps $2-7$ until you obtain an interval $[a, b]$ that is small enough or until $f(c)$ is sufficiently close to zero.
9. The final value of c obtained is the approximate root of the function $f(x)$ within the interval $[a, b]$.

NOTE: The Regula Falsi method is a modified version of the Bolzano method that uses a linear approximation of the function to find the root. It also guarantees convergence to a root of the function as long as the function is continuous on the interval $[a, b]$. However, it may converge more slowly than the Bolzano method, especially for functions with steep slopes.

3 Newton-Raphson Method

1. Choose an initial guess x_0 for the root of the function $f(x)$.
2. Calculate the derivative $f'(x)$ of the function $f(x)$.
3. Evaluate the function $f(x)$ and its derivative $f'(x)$ at the initial guess x_0.
4. Calculate the next approximation x_1 of the root using the formula:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{1}$$

5. Evaluate the function $f(x)$ and its derivative $f'(x)$ at the new approximation x_1.
6. Calculate the next approximation x_2 of the root using the formula:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2}$$

7. Repeat steps $5 - 6$ until you obtain an approximation x_i that is sufficiently close to the root or until the maximum number of iterations is reached.
8. The final value of x_i obtained is the approximate root of the function $f(x)$.

NOTE: The Newton-Raphson method can converge faster than the Bolzano and Regula Falsi methods for functions with well-behaved derivatives. However, it requires an initial guess that is sufficiently close to the root and may fail to converge or converge to a different root if the function has multiple roots or if the derivative changes sign near the root.

4 Jacobi Iteration Method

1. This method is applicable to the system of equation in which leading diagonal elements of co-effcient matrix are dominant(large in magnitude) in their respective rows.

2. The system of equations is written in the form :

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

NOTE:

Diagonal dominance property must be satisfied :

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

3. Rewriting the equations for x, y, z respectively :

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z)$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z)$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y)$$

4. To solve the system of equations, we start with initial guesses for x, y, z.

$$x_0 = 0, y_0 = 0, z_0 = 0$$

5. Then we use the above equations to calculate the values of x, y, z.

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x_0 - a_{23}z_0)$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x_0 - a_{32}y_0)$$

6. These values are then used to calculate the new values of x, y, z.

7. This process is repeated until the values of x, y, z converge to the desired accuracy.

5 Gauss-Seidel Iteration Method

1. This method is applicable to the system of equation in which leading diagonal elements of co-effcient matrix are dominant(large in magnitude) in their respective rows.

2. The system of equations is written in the form :

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

NOTE: Diagonal dominance property must be satisfied :

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

3. Rewriting the equations for x, y, z respectively :

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z)$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z)$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y)$$

4. To solve the system of equations, we start with initial guesses for x, y, z.

$$x_0 = 0, y_0 = 0, z_0 = 0$$

5. Then we use the above equations to calculate the values of x, y, z.

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x_0 - a_{23}z_0)$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x_0 - a_{32}y_0)$$

NOTE: To calculate the values we use the updated values of x, y, z as soon as they are calculated.

6 Finite Differences

1. Finite difference is a method of approximating the derivative of a function at a point by using the function values at nearby points.
2. The finite difference method is used to solve ordinary differential equations that have conditions imposed on the boundary rather than at the initial point.
3. The finite difference method is used to solve partial differential equations.

NOTE:

(a) The finite difference method is used to solve ordinary differential equations that have conditions imposed on the boundary rather than at the initial point.

(b) The finite difference method is used to solve partial differential equations.

$$y = f(x)$$

Consider,

$$x : (a), (a+h), (a+2h), (a+3h), ...(a+nh)$$

$$y : y_0, y_1, y_2, y_3, ...y_n$$

$$y_1 = f(a+h)$$
$$y_2 = f(a+2h)$$
$$y_3 = f(a+3h)$$

$$\boxed{y_n = f(a+nh)}$$

Where,

$$x \rightarrow Arguments$$
$$y \rightarrow Entries$$
$$h \rightarrow Difference\ Interval$$

6.1 Forward Difference($\triangle$)

1. The forward difference is defined as the difference between the function values at two consecutive points.

2. The forward difference is denoted by Δ.

$$\Delta y = y_1 - y_0$$
$$\Delta y_0 = f(a+h) - f(a)$$
$$\Delta y_1 = f(a+2h) - f(a+h)$$
$$\Delta y_2 = f(a+3h) - f(a+2h)$$
$$\Delta y_3 = f(a+4h) - f(a+3h)$$
$$\Delta y_n = f(a+(n+1)h) - f(a+nh)$$

6.1.1 n^{th} Forward Difference(Δ^n)

$$\Delta^n(\Delta y_0) = \Delta^n(y_1 - y_0)$$

$\therefore$

$$\Delta^n y_0 = \Delta^n y_1 - \Delta^n y_0$$

Example :

$$\Delta(\Delta y_0) = \Delta(y_1 - y_0)$$
$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

6.1.2 Forward Difference Table

1. The forward difference table is a table that is used to calculate the forward difference of a function.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
a	y_0					
$a+h$	y_1	Δy_0				
$a+2h$	y_2	Δy_1	$\Delta^2 y_0$			
$a+3h$	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$		
$a+4h$	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	
$a+5h$	y_5	Δy_4	$\Delta^2 y_3$	$\Delta^3 y_2$	$\Delta^4 y_1$	$\Delta^5 y_0$

6.2 Backward Difference(∇)

1. The backward difference is defined as the difference between the function values at two consecutive points.

2. The backward difference is denoted by ∇.

$$\nabla y = y_0 - y_{-1}$$
$$\nabla y_0 = f(a-h) - f(a)$$
$$\nabla y_1 = f(a-2h) - f(a-h)$$
$$\nabla y_2 = f(a-3h) - f(a-2h)$$
$$\nabla y_3 = f(-4h) - f(a-3h)$$
$$\nabla y_n = f(a-(n-1)h) - f(a-nh)$$

6.2.1 n^{th} Backward Difference(∇^n)

$$\nabla^n(\nabla y_0) = \nabla^n(y_0 - y_{-1})$$

$\therefore$

$$\nabla^n y_0 = \nabla^n y_0 - \nabla^n y_{-1}$$

Example :

$$\nabla(\nabla y_0) = \nabla(y_0 - y_{-1})$$
$$\nabla^2 y_0 = \nabla y_0 - \nabla y_{-1}$$

6.2.2 Backward Difference Table

1. The backward difference table is a table that is used to calculate the backward difference of a function.

x	y	∇	∇^2	∇^3	∇^4	∇^5
$a-5h$	y_{-5}					
$a-4h$	y_{-4}	∇y_{-4}				
$a-3h$	y_{-3}	∇y_{-3}	$\nabla^2 y_{-3}$			
$a-2h$	y_{-2}	∇y_{-2}	$\nabla^2 y_{-2}$	$\nabla^3 y_{-2}$		
$a-1h$	y_{-1}	∇y_{-1}	$\nabla^2 y_{-1}$	$\nabla^3 y_{-1}$	$\nabla^4 y_{-1}$	
a	y_0	∇y_0	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$	$\nabla^5 y_0$

6.3 Shift Operator(E)

1. The shift operator is defined as the difference between the function values at two consecutive points.

2. The shift operator is denoted by E.

General Relation :

$$E^m y_n = y_{m+n}$$

$$Ef(a) = f(a+h)$$
$$Ef(a+h) = f(a+2h)$$
$$Ef(a+2h) = f(a+3h)$$
$$Ef(a+3h) = f(a+4h)$$
$$Ef(a+4h) = f(a+5h)$$

Example :

$$Esinx = sin(x+h)$$
$$Ee^{2x} = e^{2(x+h)}$$

6.3.1 n^{th} Shift Operator(E^n)

$$E^n f(a) = f(a+nh)$$

Example :

$$E^2 f(a) = f(a+2h)$$
$$E^3 f(a) = f(a+3h)$$
$$E^2 sinx = sin(x+2h)$$
$$E^3 e^{2x} = e^{2(x+3h)}$$

6.3.2 Some Examples of Shift Operator(E^1)

$$E^1 y_0 = y_1$$
$$E^1 y_1 = y_2$$
$$E^1 y_5 = y_6$$
$$E^2 y_0 = y_2$$
$$E^2 y_3 = y_5$$
$$E^3 y_6 = y_9$$
$$E^4 y_4 = y_8$$

6.4 Inverse Operator(E^{-1})

1. The inverse operator is defined as the difference between the function values at two consecutive points.

2. The inverse operator is denoted by E^{-1}.

General Relation :

$$E^{-m}y_n = y_{n-m}$$

$$E^{-1}f(a-h) = f(a)$$
$$E^{-1}f(a+2h) = f(a+h)$$
$$E^{-1}f(a+3h) = f(a+2h)$$
$$E^{-1}f(a+4h) = f(a+3h)$$
$$E^{-1}f(a+5h) = f(a+4h)$$

Example :

$$E^{-1}sin(x) = sin(x-h)$$
$$E^{-1}e^{2(x+h)} = e^{2(x-h)}$$

6.4.1 n^{th} Inverse Operator(E^{-n})

$$E^{-n}f(a+nh) = f(a)$$

Example :

$$E^{-2}f(a+h) = f(a-h)$$
$$E^{-3}f(a+3h) = f(a)$$
$$E^{-4}f(a) = f(a-4h)$$

6.4.2 Some Examples of Inverse Operator(E^{-1})

$$E^{-1}y_0 = y_{-1}$$
$$E^{-1}y_1 = y_0$$
$$E^{-1}y_5 = y_4$$
$$E^{-2}y_0 = y_{-2}$$

$$E^{-2}y_3 = y_1$$
$$E^{-3}y_6 = y_{-3}$$

7 Polynomial Interpolation

The technique or method of estimating unknown values from given set of observation is known as Polynomial Interpolation.

There are two types of Polynomial Interpolation:

1. Equal interval
2. Unequal interval

7.1 Equal Interval

Here we use the following formulae for estimating interpolation with equal interval :

1. **Newton's Forward Formula**
2. **Newton's Backward Formula**
3. **Gauss's Forward Formula**
4. **Gauss's Backward Formula**
5. **Stirling's Formula**

7.2 Unequal Interval

Here we use the following formulae for estimating interpolation with unequal interval :

1. **Lagrange's Formula**
2. **Newton's Divided Difference Formula**

7.3 Newton's Forward Formula

1. This formula is used for estimating the values from the top to bottom in a difference table.
2. As the top values must be near to the desired interval.

To solve and find the accurate value we must find out the $\Delta f(x)$ and $\Delta^2 f(x)$ and so on.
This is possible by creating a difference table.
Example:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
a	y_0					
$a+h$	y_1	Δy_0				
$a+2h$	y_2	Δy_1	$\Delta^2 y_0$			
$a+3h$	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$		
$a+4h$	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	
$a+5h$	y_5	Δy_4	$\Delta^2 y_3$	$\Delta^3 y_2$	$\Delta^4 y_1$	$\Delta^5 y_0$

Where,

$$x \to Arguments$$
$$y \to Entries$$
$$h \to Difference\ Interval$$
$$u \to Interpolation\ Difference$$

7.3.1 Formula

$$f(a+(u)h) = f(a)+\frac{u}{1!}(\Delta f(a))+\frac{u(u-1)}{2!}(\Delta^2 f(a))+\frac{u(u-1)(u-2)}{3!}(\Delta^3 f(a))+\frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 f(a))\cdots$$

7.4 Newton's Backward Formula

1. This formula is used for estimating the values from the bottom to top in a difference table.
2. As the bottom values must be near to the desired interval.

To solve and find the accurate value we must find out the $\Delta f(x)$ and $\Delta^2 f(x)$ and so on.
This is possible by creating a difference table.
Example:

x	y	∇	∇^2	∇^3	∇^4	∇^5
$a-5h$	y_{-5}					
$a-4h$	y_{-4}	∇y_{-4}				
$a-3h$	y_{-3}	∇y_{-3}	$\nabla^2 y_{-3}$			
$a-2h$	y_{-2}	∇y_{-2}	$\nabla^2 y_{-2}$	$\nabla^3 y_{-2}$		
$a-1h$	y_{-1}	∇y_{-1}	$\nabla^2 y_{-1}$	$\nabla^3 y_{-1}$	$\nabla^4 y_{-1}$	
a	y_0	∇y_0	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$	$\nabla^5 y_0$

Where,

$$x \to Arguments$$
$$y \to Entries$$
$$h \to Difference\ Interval$$
$$u \to Interpolation\ Difference$$

7.4.1 Formula

$$f(a+(u)h) = f(a)+\frac{u}{1!}(\nabla f(a))+\frac{u(u+1)}{2!}(\nabla^2 f(a))+\frac{u(u+1)(u+2)}{3!}(\nabla^3 f(a))+\frac{u(u+1)(u+2)(u+3)}{4!}(\nabla^4 f(a))\ldots$$

7.5 Gauss' Forward Formula

1. This formula is used for estimating the values from the top to bottom in a difference table.
2. As the top values must be near to the desired interval.
3. Used to centre interpolation difference.

To solve and find the accurate value we must find out the $\Delta f(x)$ and $\Delta^2 f(x)$ and so on.
This is possible by creating a difference table.
Example:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
a	y_0					
$a+h$	y_1	Δy_0				
$a+2h$	y_2	Δy_1	$\Delta^2 y_0$			
$a+3h$	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$		
$a+4h$	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	
$a+5h$	y_5	Δy_4	$\Delta^2 y_3$	$\Delta^3 y_2$	$\Delta^4 y_1$	$\Delta^5 y_0$

Where,

$$x \rightarrow Arguments$$

$$y \rightarrow Entries$$

$$h \rightarrow Difference\ Interval$$

$$u \rightarrow Interpolation\ Difference$$

7.5.1 Formula

$$f(a+uh) = y_0 + \frac{u}{1!}(\Delta y_0) + \frac{u(u-1)}{2!}(\Delta^2 y_{-1}) + \frac{u(u-1)(u+1)}{3!}(\Delta^3 y_{-1}) + \frac{u(u-1)(u+1)(u-2)}{4!}(\Delta^4 y_{-2}) \cdots$$

7.6 Guass' Backward Formula

1. This formula is used for estimating the values from the bottom to top in a difference table.
2. As the bottom values must be near to the desired interval.

To solve and find the accurate value we must find out the $\Delta f(x)$ and $\Delta^2 f(x)$ and so on.
This is possible by creating a difference table.
Example:

x	y	∇	∇^2	∇^3	∇^4	∇^5
$a-5h$	y_{-5}					
$a-4h$	y_{-4}	∇y_{-4}				
$a-3h$	y_{-3}	∇y_{-3}	$\nabla^2 y_{-3}$			
$a-2h$	y_{-2}	∇y_{-2}	$\nabla^2 y_{-2}$	$\nabla^3 y_{-2}$		
$a-1h$	y_{-1}	∇y_{-1}	$\nabla^2 y_{-1}$	$\nabla^3 y_{-1}$	$\nabla^4 y_{-1}$	
a	y_0	∇y_0	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$	$\nabla^5 y_0$

Where,

$$x \rightarrow Arguments$$
$$y \rightarrow Entries$$
$$h \rightarrow Difference\ Interval$$
$$u \rightarrow Interpolation\ Difference$$

7.6.1 Formula

$$f(a+uh) = y_0 + \frac{u}{1!}(\Delta y_0) + \frac{u(u-1)}{2!}(\Delta^2 y_{-1}) + \frac{u(u-1)(u+1)}{3!}(\Delta^3 y_{-1}) + \frac{u(u-1)(u+1)(u+2)}{4!}(\Delta^4 y_{-2}) \cdots$$

7.7 Lagrange's Interpolation

7.7.1 Formula

x	a_1	a_2	a_3	a_4
y	b_1	b_2	b_3	b_4

$$\begin{aligned} f(x) = &\frac{(x-a_2)(x-a_3)(x-a_4)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}(b_1) + \frac{(x-a_1)(x-a_3)(x-a_4)}{(a_2-a_1)(a_2-a_3)(a_2-a_4)}(b_2) \\ &+ \frac{(x-a_1)(x-a_2)(x-a_4)}{(a_3-a_1)(a_3-a_2)(a_3-a_4)}(b_3) + \frac{(x-a_1)(x-a_2)(x-a_3)}{(a_4-a_1)(a_4-a_2)(a_4-a_3)}(b_4) \end{aligned}$$

Example : Find the value of $f(x)$ at $x = 10$ from the following table using Lagrange's Interpolation Formula.

x	5	6	9	11
y	12	13	14	16

Sol. :

$$\begin{aligned} f(x) = &\frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(5-9)(6-11)}(13) \\ &+ \frac{(x-5)(x-6)(x-11)}{(9-6)(9-9)(9-11)}(14) + \frac{(x-5)(x-9)(x-11)}{(11-5)(11-6)(11-9)}(16) \end{aligned}$$

$$\begin{aligned} f(10) = &\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(5-9)(6-11)}(13) \\ &+ \frac{(10-5)(10-6)(10-11)}{(9-6)(9-9)(9-11)}(14) + \frac{(10-5)(10-9)(10-11)}{(11-5)(11-6)(11-9)}(16) \end{aligned}$$

$$\begin{aligned} f(10) = &\frac{(10-6)(10-9)(10-11)}{-24}(12) + \frac{(10-5)(10-9)(10-11)}{15}(13) \\ &+ \frac{(10-5)(10-6)(10-11)}{-24}(14) + \frac{(10-5)(10-9)(10-11)}{60}(16) \end{aligned}$$

$$f(x) = 4 + 12.666$$

$$\implies f(x) = 16.666$$

7.8 Newton's Divided Difference

7.8.1 Formula

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x_0) \cdots$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	a_0			
x_1	a_1	$\Delta f(x_1) = \frac{a_1 - a_0}{x_1 - x_0}$		
x_2	a_2	$\Delta f(x_2) = \frac{a_2 - a_1}{x_2 - x_1}$	$\Delta^2 f(x_1) = \frac{\Delta f(x_2) - \Delta f(x_1)}{x_2 - x_1}$	
x_3	a_3	$\Delta f(x_3) = \frac{a_3 - a_2}{x_3 - x_2}$	$\Delta^2 f(x_2) = \frac{\Delta f(x_3) - \Delta f(x_2)}{x_3 - x_2}$	$\Delta^3 f(x_1) = \frac{\Delta^2 f(x_2) - \Delta^2 f(x_1)}{x_3 - x_2}$

Example : Find the value of $f(x)$ at $x = 10$ from the following table using Newton's Divided Difference Formula.

x	5	6	9	11
y	12	13	14	16

Sol. :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
5	12			
6	13	$\Delta y = \frac{13-12}{6-5} = 1$		
9	14	$\Delta y = \frac{14-13}{9-6} = \frac{1}{3}$	$\Delta^2 y = \frac{\frac{1}{3}-1}{9-5} = -\frac{1}{6}$	
11	16	$\Delta y = \frac{16-14}{11-9} = 1$	$\Delta^2 y = \frac{1-\frac{1}{3}}{11-6} = \frac{1}{30}$	$\Delta^3 y = \frac{\frac{1}{30}-\left(-\frac{1}{6}\right)}{11-5} = \frac{1}{60}$

$$f(x) = 12+(x-5)(1)+(x-5)(x-6)(-0.222)+(x-5)(x-6)(x-9)(0.09258)$$

$$\implies f(x) = 16.666$$

8 Numerical Integration

- We first need to find h i.e. common difference between the intervals.
- To determine which rule is to be used, we need to find the number of intervals.
- If the numeber is divisible by 1 then it is eligible for Trapezoidal Rule.
- If the number is divisible by 2 then it is eligible for Simpson's $^1/_{3^{th}}$ Rule.
- If the number is divisible by 3 then it is eligible for Simpson's $^3/_{8^{th}}$ Rule.

$$h = \frac{(Upper\ Limit) - (Lower\ Limit)}{Total\ Number\ of\ Intervals}$$

Example : Find h for the following integral.

$$\int_1^2 \frac{1}{x}, dx$$

$$h = \frac{(Upper\ Limit) - (Lower\ Limit)}{Total\ Number\ of\ Intervals}$$

$$h = \frac{2-1}{6}$$

$$h = \frac{1}{6}$$

8.1 Trapezoidal Rule

8.1.1 Formula

- This formula is assuming that the function is divided into 6 intervals.

$$f(x) = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

8.2 Simpson's Rules

8.2.1 Simposn's $^1/_{3^{rd}}$ Rule

8.2.1.1 Formula

- This formula is assuming that the function is divided into 6 intervals.

$$f(x) = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

8.2.2 Simposn's $3/8^{th}$ Rule

8.2.2.1 Formula

- This formula is assuming that the function is divided into 6 intervals.

$$f(x) = \frac{3h}{8}[(y_0 + y_6) + 4(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

8.3 Differential Equations

8.3.1 Picard's Method

8.3.1.1 Formula

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n), dx$$

8.4 Euler's Rule/Runga-Kutta 1^{st} Order Method

8.4.1 Formula

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

8.5 Euler's Modified Rule/Runge-Kutta 2^{nd} Order Method

8.5.1 Formula

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

8.5.2 Euler's Modified Rule

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$$

8.6 Runge-Kutta 4^{th} Order Method

8.6.1 Formula

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$
$$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$
$$k_4 = hf(x_n + h, y_n + k_3)$$
$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

8.6.2 Runge-Kutta 4^{th} Order Method

$$
\begin{aligned}
y_{n+1} &= y_n + k \\
&= y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{aligned}
$$