Associativity Of Convolution

Proof: Let a, b, c functions which are measurable on the reals, & let us assume ~~$a*(b*h)$ exists~~, $a*(b*c)$ & $(a*b)*c$ exist.

To Show: $(a*b)*c = a*(b*c)$.

By the definition of convolution,

$$((a*b)*c)(u) = \int_R (a*b)(x)\, c(u-x)\, dx$$

$$= \int_R \left[\int_R a(y)\, b(x-y)\, dy\right] c(u-x)\, dx$$

$$= \int_R \int_R a(y)\, b(x-y)\, c(u-x)\, dy\, dx$$

By Fubini's Theorem we can switch the order of integration. Thus we get

$$((a*b)*c)(u) = \int_R \int_R a(y)\, b(x-y)\, c(u-x)\, dx\, dy.$$

$$= \int_R a(y) \left[\int_R b(x-y)\, c(u-$$

By translation invariance,

$$\int_R b(x-y)\, c(u-x)\, dx$$

$$= \int_R b((x+y)-y)\, c(u-(x+$$

$$= \int_R b(x)\, c((u-y)-x)\, dx$$

$$= (b*c)(u-y)$$

$$\Rightarrow \; = ((a*b)*c)(u) = \int_R f(y)\,(b*c)(u-y$$

Which by definition is $(a*(b*c))$

Hence Convolution is associative

Task 6

We know that convolving twice with 2 filters is convenient as convolving the two filters first to get a new filter then convolve signal with new filter. Hence we need to prove that convolution of gaussian kernel with another kernel with another gaussian kernel is also a gaussian kernel.

Suppose 2 1D gaussian kernels $g_1(x), g_2(x)$ are with standard deviation σ_1, σ_2 then their convolution is

$$\int_{-\infty}^{\infty} g_1(u) g_2(x-u)\,du$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(\frac{-u^2}{2\sigma^2}\right) \times \frac{1}{\sqrt{2\pi\sigma_2}} \exp\frac{-(x-u)^2}{2\sigma_2^2}\,du$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left\{-\,(\sigma_1^2+\sigma_2^2)u^2 + \sigma_1^2 x^2 - 2\sigma_1^2 x u\right\} du$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \left(\int_{-\infty}^{\infty} \exp\left\{-\frac{(\sigma_1^2+\sigma_2^2)u^2 - 2\sigma_1^2 x u}{2\sigma_1^2\sigma_2^2}\right\} du + \int_{-\infty}^{\infty} \exp\left\{\frac{-\sigma_1^2 x^2}{2\sigma_1\sigma_2}\right\} du \right)$$

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Let $A = (\sigma_1^2 + \sigma_2^2) / (\sigma_1^2 \sigma_2^2) > 0$ &

$Z = x / \sigma_2^2$

The $= \int_{-\infty}^{\infty} g_1(u) g_2(x-u) du.$

$$= \frac{1}{2\pi \sigma_1 \sigma_2} \exp\left\{ \frac{-x^2}{2\sigma^2} \cdot \right\} \sqrt{\frac{2\pi}{A}} \exp\left\{ \frac{z^2}{2A} \right\}$$

$$= \frac{1}{\sqrt{2\pi} \sqrt{\sigma_1^2 + \sigma_2^2}} \exp \frac{x^2}{2(\sigma_1^2 + \sigma_2^2)\sigma_2^2 / \sigma_1^2 \sigma_2^2} \frac{-x^2}{}$$

$$= \frac{1}{\sqrt{2\pi} \sqrt{\sigma_1^2 + \sigma_2^2}} \exp\left(\frac{-x^2}{2(\sigma_1^2 + \sigma_2^2)} \right)$$

So result is Gaussian function with Standard deviation $\sqrt{\sigma_1^2 + \sigma_2^2}$ which is larger than both of 2 deviations of 2 Gaussian filters.

Further g_1 & g_2 are same Gaussian kernels so we have

$$\int_{-\infty} g_1(u) g_2(x-u)\, du = \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} \exp\left\{ \frac{-x^2}{2(\sqrt{2}\sigma)^2} \right\}$$

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