Pontryagin Duality

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The work we are focusing on concerns the contributions of L.S. Pontriagyn and E.R. Van Kampen. In 1934, Pontriagyn announced and proved the Duality theorem for the case of compact abelian groups with a countable base. In the same year, J.W. Alexander dealt with the case of arbitrary discrete abelian groups. The following year, E.R. Van Kampen was able to complete the Proof of the Duality Theorem for all locally compact abelian groups. Roughly speaking, the duality theorem states that every **LCA** (locally compact abelian group) is the dual group of its dual group. From this, we deduce that every piece of information about a locally compact abelian group is stored as information about its dual. In the case of compact groups, this is particularly interesting as their dual is discrete, and vice versa. Thus, any compact Hausdorff abelian group can be completely described by the purely algebraic properties of its dual group. This scenario is the one interest us, namely, the duality between Abelian Group (discrete one) **Ab** and compact Hausdorff abelian group **KAb**. We will not discuss the general case of **LCA**, known as the Pontryagin-Van Kampen duality Theorem. Instead, we focus on the Duality of interest, which we refer to as the Pontryagin Duality. We divide our work into three main parts. The first one, called Categories and Functors, focuses on the category of compact groups and their principal characteristics, as well as defining the functor that induces the duality. The second section is more tedious as we attempt to demonstrate that the two natural transformations are indeed bijections in the most consistent way possible. In the last section, we enunciate the Duality theorem and provide two examples of how a purely algebraic notion determines some topological notion in the dual group, and vice versa.

Categories and Functors

To define our Duality we need two category and two functor, we obviously start from the category.

Definition 1. *Let G a group and a topological space, than G is called a topological group if the two operation* $\cdot : G \times G \longrightarrow G$*, mapping* (x,y) *to* xy *and* $(-)^{-1} : G \longrightarrow G$*, mapping* x *to* x^{-1}*, are continuos.*

Where G $\times$ G is equipped with the product topology

Let's expose two proposition to understand how strong the structure of group in a topological space is.

Proposition 1. *Let G be a topological group for each a* $\in$*G the left and right moltiplication are homeomorphism, the inversion is also a homeomorphism.*

Proposition 2. *Let G be a topological group and e the identity element. if U is a neighbourhood of e, then there exist V an open neighbourhood of e such that:*

1. V=V^{-1}
2. $V^2 \subseteq U$

(where $V^2 = \{ v_1 v_2 \mid v_1 \in V \ v_2 \in V \}$*)*

Definition 2. *A topological space is called a* T_1 *space if every point in the space is a closed set. A topological space is called a* T_2 *space, or Hausdorff, if for each pair of distinct points a and b, there exist two open sets* U_a *and* U_b *containing a and b respectively, such that* $U_a \cap U_b$ *is the empty set.*

For a topological group, the two conditions are equivalent. In fact:

Given two distinct points x and y of G, a topological group, then $xy^{-1} \neq e$, so $G \setminus xy^{-1}$ is an open set and a neighborhood of e. By Proposition 2, there exists an open set V such that $V^2 \subseteq G \setminus xy^{-1}$, thus $xy^{-1} \notin V^2$. Now, xV and yV are two open neighborhoods of x and y. Suppose $xV \cap yV \neq \emptyset$, then $xv = yv'$ for some v and v' in V, so $xy^{-1} = v^{-1}v' \in V^2$, which is a contradiction. Therefore, G is Hausdorff.

To check that a topological group is Hausdorff, it suffices to check that e is a closed set.

Every group can be seen as a topological group with the discrete topology

Definition 3. *Let G and H topological groups a map f from G to H is called continuous homomorphism if it is both a grups homomorphism and a continuous map. if f is also homeomorphism is called topological isomorphism*

We are interested in only a special kind of topological group:

Definition 4. *An abelian topological group whose topology is both compact and Hausdorff is called compact group*

The category of compact groups with their morphism is denoted from now on **KAb**.

Proposition 3. *If* (X,τ_1) *and* (Y,τ_2) *are compact Hausdorff spaces and* $f : (X,\tau_1) \longrightarrow (Y,\tau_2)$ *is a continuous mapping then f is a closed map.*

Thank to that proposition a continuous bijective homomorphism is always a topological isomorphism.

To complete our task we have to further inspect the scructure of the abelian compact groups.

Definition 5. *The connected component of a point x in X is the union of all connected subsets of X that contain x; it is the unique largest (with respect to $\subseteq$) connected subset of X that contains x. The maximal connected subsets (ordered by inclusion $\subseteq$) of a non-empty topological space are called the connected components of the space.*

The connected component of the identity in topological group i have some particular characteristics:

Proposition 4. *The identity component is a subgroup in any topological group.*

Proof. Let G_0 be this connected component. We need to show that this is a subgroup, we have a continuous map $H \times H \to G$ given by $(x,y) \longmapsto xy^{-1}$, and we want to show that the image is contained in H, but: A product of two connected spaces is connected, so $H \times H$ is connected. The image of a connected space under a continuous map is connected. A point is in the connected component of e if and only if there exists a connected subset of G containing that point and e. so the image of the continuous map is G_0. So G_0 is subgroup. $\square$

In our case is always a normal subgroup. Let G be a topological group and H a normal subgroup of G. Consider the quotient G/H with the quotient topology, namely the finest topology on G/H that makes the canonical projection $q : G \to G/H$ continuous(that is, U is open in G/H if and only if $p^{-1}(U)$ is open in G). Since we have a group topology on G, the quotient topology consists of all sets q(U), where U runs over the family of all open sets of G (as $q^{-1}(q(U))$ is open in G in such a case). In particular, we can prove the following important properties of the quotient topology.

Proposition 5. *Let G be a topological group, let H be a normal subgroup of G and let G/H be equipped with the quotient topology. Then the canonical projection $q : G \longrightarrow G/H$ is open.*

Proof. Let $U \neq \emptyset$ be an open set in G. Then $q^{-1}(q(U)) = HU = \bigcup_{h \in H} hU$ is open, since each hU is open. Therefore, q(U) is open in G/H. $\square$

Remembering that a connected component of a space is necessarily a closed subset of that space. So the connected component of the identity is closed normal subgroup. The connected component of an element $x \in G$ is simply the coset $xG_0 = G_0x$. Now we are interested in studying the quotient group G/G_e, the first interesting propriety is that, If G is a topological group, then the group G/G_0 is totally disconnected.

Let G be a compact group, H a normal subgroup of G, and G/H equipped with the quotient topology, then G/H is again a compact group, simply noting that the canonical quotient map is an open map. Noting that G_0 is a compact subset in any compact group G, so we have that the canonical quotient map is a closed map.In fact, let $C \subseteq G$ a closed set such that $p^{-1}pC = G_0C$, which is a closed set, since p is an open map, we have that pC is a closed set in the quotient space. We lastly observe that the quotient G/G_e is an Hausdorff space noting that G_0 is closed set and $p(G_0) = 1_{G/G_0}$ is a closed set too, being p closed map. characteristic of the topological group, G/G_0 is a Compact group.

We remeber a topological fact that is used in the last chapter.

Proposition 6. *A continuous surjective map between two compact Hausdorff space is a quotient map.*

We can now start to talk about the functor. Let first consider the circle group $\mathbb{T}$, The circle group is defined as the group, under multiplication, of complex numbers of modulus one. In other words, it

is the group of complex numbers on the unit circle, under multiplication. An equivalent description of the circle group is as the quotient group $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ is the additive group of real numbers and $\mathbb{Z}$ is the subgroup of $\mathbb{R}$ defined as the additive group of integers. Note that the isomorphism between the group $\mathbb{R}/\mathbb{Z}$ and the complex numbers of modulus one is given by: $\theta \longmapsto e^{2\pi i\theta}$.
This set is clearly an abelian group and is a compact group with the subspace topology inherited from $\mathbb{R}$.

This object is going to have a particular role in the Pontriagyn Duality.
From now on let A an abelian group, the set of the group homomorphisms between A and T are known as the characters of the group A. Characters of Abelian groups are the key ingredients of Pontryagin Duality.
So a character of an Abelian group A is a homomorphism from A to $\mathbb{T}$. Any Abelian group A has a trivial character $0_{\mathrm{Hom}_{Ab}(A,\mathbb{T})}$: A $\longrightarrow \mathbb{T}$ that sends each element to the neutral element $1_{\mathbb{T}} \in \mathbb{T}$.

Let **Ab** the category of abelian group and their morphism. In first place the functor Hom_{Ab}(-,$\mathbb{T}$), is a functor that takes values in $\mathbf{Set}^{op}$. But we can lift this functor to $\mathbf{Ab}^{op}$. In fact, for each $\nu,\xi \in$ Hom_{Ab}(A,$\mathbb{T}$), we can define:

$$\begin{aligned} \chi+\xi \ :A &\longrightarrow \mathbb{T} \\ g &\longmapsto \chi g \cdot \xi g \end{aligned}$$

and clearly this function is again a character. Given ξ a character of A we can define

$$\begin{aligned} -\xi \ :A &\longrightarrow \mathbb{T} \\ g &\longmapsto (\xi g)^{-1} \end{aligned}$$

and clearly the trivial character work as identity. so Hom_{Ab}(A,$\mathbb{T}$) is an abelian group.

Our goal is to topologize this abelian group, and to achieve that, we need to introduce the notion of compact-open topology.
Let X,Y be topological spaces, and consider the set Hom_{Top}(X,Y) of continuous functions from X to Y. For a compact subset K $\subseteq$ X and an open subset U $\subseteq$ Y , we can consider the set:

$$\text{V (K,U)} := \left\{ f \in Hom_{Top}(X,Y) \mid f[K] \subseteq U \right\}$$

The collection of all sets of the form V (K, U) can be taken as a subbasis for a topology on Hom_{Top}(X,Y).

In our case, with A abelian group and considering A as a discrete space, and T as a topological space, we can equip Hom_{Ab}(A,$\mathbb{T}$) with the compact-open topology. Since the compact subsets of A are exactly the finite ones, this means that we take as a subbasis for the topology the sets of characters

$$\text{V } (\{a_1,\ldots,a_n\}, \text{U}) := \text{f} \in \mathrm{Hom}_{Ab}(\text{A},\mathbb{T}) \mid \text{ f}(a_i) \in \text{U}.$$

When the family of the compact subset of X are exactaly the finite ones, this topology is called the topology of pointwise convergence. Hom_{Ab}(A,$\mathbb{T}$) is a subgroup of $\mathbb{T}^A$ that is a topological group. the subspace topology induced by the product topology is exactly the pointwise convergence topology. Therefore, Hom_{Ab}(A,$\mathbb{T}$) is a topological group by the following proposition:

Proposition 7. *Let G a topological group, H a subgroup of G, with is relative subspace topology as a subset of G, then H is a topological group.*

Proof. The mapping (x,y) $\mapsto$ xy^{-1} from H $\times$ H to H is continuous since is the restiction of the corresponding maps of G. $\square$

So Hom$_{Ab}$(A,$\mathbb{T}$) $\subseteq$ $\mathbb{T}^A$ has the topology inherited from the product topology of $\mathbb{T}^A$ which is compact by the Tychonoff's Theorem, since $\mathbb{T}$ compact.
Since A is discrete, the group of all algebric homomorphism from A to $\mathbb{T}$ is a closed subset of $\mathbb{T}^A$: For any pair (a,b) $\in$ A $\times$ A, the set M(a,b) = $\left\{ \xi \in \mathbb{T}^A \mid \xi(\mathrm{a}+\mathrm{b}) = \xi(\mathrm{a}) + \xi(\mathrm{b}) \right\}$ is closed since $\xi \mapsto \xi(c) : \mathbb{T}^A \longrightarrow \mathbb{T}$ is continuos by definition of product topology. But then Hom$_{Ab}$(A,$\mathbb{T}$) = $\bigcap_{(a,b)\in A\times A} M(a,b)$ is closed and therefore a compact space.
Thanks to the folliwing proposition we reach our goal.

Proposition 8. *Let A an abelian group, Hom$_{Ab}$(A,$\mathbb{T}$) equipped with the compact-open topology is an Hausdorff space*

Proof. let $f,g \in$ Hom$_{Ab}$(A,$\mathbb{T}$), $f \neq g$ so exist a such that $fa \neq ga$, since $\mathbb{T}$ is an Hausdorff space, exist two open set U_f and U_g such that $fa \in U_f$, $ga \in U_g$ and $U_f \cap U_g = \emptyset$. So $f \in V(\{a\}, U_f)$ and $g \in V(\{a\}, U_g)$ which are open set in the compact-open topology. Also, $V(\{a\}, U_f) \cap V(\{a\}, U_g) = \emptyset$, because if not, there would exist $h \in V(\{a\}, U_f) \cap V(\{a\}, U_g)$ that send a in $V(\{a\}, U_g)$ and $V(\{a\}, U_f)$ which is impossibile. So Hom$_{Ab}$(A,$\mathbb{T}$) is a Hausdorff space. $\square$

Remark that by a compact group we shall mean, in these notes, an Abelian topological group whose topology is both compact and Hausdorff. Thanks to the previous dissertion we can lift our functor to **KAb**^{op}

$$
\begin{array}{rcl}
\mathrm{Hom}_{Ab}(-,\mathbb{T}) : \mathbf{Ab} & \longrightarrow & \mathbf{KAb}^{op} \\
A & \longmapsto & \mathrm{Hom}_{Ab}(A,\mathbb{T}) \\
f\big\downarrow & & \big\downarrow \mathrm{Hom}_{Ab}(f,\mathbb{T}) \\
B & \longmapsto & \mathrm{Hom}_{Ab}(B,\mathbb{T})
\end{array}
$$

where $\mathrm{Hom}_{Ab}(f,\mathbb{T}) : \mathrm{Hom}_{Ab}(B,\mathbb{T}) \longrightarrow \mathrm{Hom}_{Ab}(A,\mathbb{T})$ is defined by mapping a group homomorphism γ from B to $\mathbb{T}$ to the group homomorphism γf from A to $\mathbb{T}$, this map is a well defined continuous homorphism. In fact:

To see that $\mathrm{Hom}_{Ab}(f,\mathbb{T})$ is continuous, let V(K,U) be a subbasis open set of $\mathrm{Hom}_{Ab}(A,\mathbb{T})$, the continuity follow from the fact $\mathrm{Hom}_{Ab}(f,\mathbb{T})^{-1}$ (V(K,U)) = V(f(K),U) is an open set of $\mathrm{Hom}_{Ab}(B,\mathbb{T})$. And $\mathrm{Hom}_{Ab}(f,\mathbb{T})(\xi - \nu)$(b) = $(\xi - \nu)(f\mathrm{b})$ = ξf(b) - νf(b) = $\mathrm{Hom}_{Ab}(f,\mathbb{T})(\xi)$(b) - $\mathrm{Hom}_{Ab}(f,\mathbb{T})(\nu)$(b) so is a continuous homomorphism.

So $\mathrm{Hom}_{Ab}(-,\mathbb{T})$ is a well defined functor.

To define the other functor, we need to consider the circle group $\mathbb{T}$ as a compact group. indeed, if G is a compact group, a morphism from G to T in **KAb** is called a continuous character. This is a group homomorphism that is also continuous with respect to the given topologies. Using the circle group $\mathbb{T}$ as a dualising object, we then look at the represented functor:

$$\mathrm{Hom}_{KAb}(-,\mathbb{T})\text{: } \mathbf{KAb} \longrightarrow \mathbf{Set}^{op}$$

It is easy to lift the functor above to takes values in the opposite of Ab; indeed, this is done exactly as in the case of the represented functor $\mathrm{Hom}_{Ab}(-,\mathbb{T})$ that we started with. The sum of two continuous characters out of a compact group is again a continuous character of G and the pointwise opposite of a continuous character is also a continuous character of G. The trivial character $0_{\mathrm{Hom}_{KAb}(G,\mathbb{T})}$ is always continuous.

We have for H,G compact group and $f \in \mathrm{Hom}_{KAb}(G,H)$:

$$\begin{aligned}\mathrm{Hom}_{KAb}(-,\mathbb{T}) : \mathbf{KAb} &\longrightarrow \mathbf{Ab}^{op}\\ G &\longmapsto \mathrm{Hom}_{KAb}(G,\mathbb{T})\\ f\Big\downarrow \quad & \qquad \Big\downarrow \mathrm{Hom}_{KAb}(f,\mathbb{T})\\ H &\longmapsto \mathrm{Hom}_{KAb}(H,\mathbb{T})\end{aligned}$$

And like the previous argoument on $\mathrm{Hom}_{Ab}(f,\mathbb{T})$, we have that $\mathrm{Hom}_{KAb}(-,\mathbb{T})$ is a well defined functor.

Units

We can start to talk about the Pontriagyn Duality, let's first give the definition of equivalence of category:

Definition 6. *An equivalence of categories C and D is a pair of functors $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ and G: $\boldsymbol{D} \longrightarrow \boldsymbol{C}$ together with natural isomorphisms $\eta : 1_C \Longrightarrow G \circ F$ and $\zeta : 1_D \Longrightarrow F \circ G$, called the units of the equivalence.*

In our case we already define the two category and the two functor, let's talk about the two units. Let A be an abelian group. We wish to define a group morphism η_A : A $\longrightarrow \mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})$. For an element a $\in$ A, let's consider the map:

$$\begin{aligned}ev_a :\ \mathrm{Hom}_{Ab}(A,\mathbb{T}) &\longrightarrow \mathbb{T}\\ \chi &\longmapsto \chi a\end{aligned}$$

Proposition 9. *let $A \in \boldsymbol{Ab}$, $a \in A$, the map ev_a just define is a continuous homorphism.*

Proof. $ev_a(\chi - \xi) = (\chi - \xi)(a) = \chi a(\xi a)^{-1} = ev_a(\chi)(ev_a(\xi))^{-1}$ so it is a group homomorphism. For the continuity, let $U \subseteq \mathbb{T}$ an open set, by definition $V(a,U)$ is a subbasis element so is an open set. We have that $ev_a(V(a,U)) \subseteq U$ because $\forall f \in V(a,U)$, $ev_a(f) = fa \in U$, so ev_a is a continuous map i.e. $ev_a \in \mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})$. □

For all $A \in \mathbf{Ab}$ we define:

$$\begin{aligned}\eta_A : A &\longrightarrow \mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})\\ a &\longmapsto ev_a\end{aligned}$$

It is easy to check that η_A is group homomorphism. In fact, for all a,b $\in$ A:
$(\eta_A(a+b))f = ev_{a+b}(f) = f(a+b) = f(a)f(b) = ev_a(f)ev_b(f) = \eta_A(a)f\eta_A(b)f = (\eta_A(a)+\eta_A(b))f$

where $f \in \mathrm{Hom}_{Ab}(A,\mathbb{T})$ and $\eta_A(0_A)f = ev_{0_A}f = f0_A = 1_{\mathbb{T}}$, so is a group homomorphism between A and $\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})$.

We are ready to define the first natural trasformation. The homomorphisms η_A assemble into a transformation

$$\eta : 1_{Ab} \Longrightarrow \mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(-,\mathbb{T}),\mathbb{T})$$

Proposition 10. *η is natural trasformation, i.e. for all A,B $\in$ **Ab** and for all $f \in Hom_{Ab}(A,B)$, defining $C_f := Hom_{KAb}(Hom_{Ab}(f,\mathbb{T}),\mathbb{T})$, the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & Hom_{KAb}(Hom_{Ab}(A,\mathbb{T}),\mathbb{T}) \arrow[d, "C_f"] \\ B \arrow[r, "\eta_B"] & Hom_{KAb}(Hom_{Ab}(B,\mathbb{T}),\mathbb{T}) \end{tikzcd}$$

Proof. Remark that $\mathrm{Hom}_{Ab}(f,\mathbb{T}) : \mathrm{Hom}_{Ab}(B,\mathbb{T}) \longrightarrow \mathrm{Hom}_{Ab}(A,\mathbb{T})$ send a group morphism $g \longmapsto gf$, so it is the precomposition with f, which we can denote as $-\circ f$. For $a \in A$ the composition of the bottom and left row yields to $\eta_B(fa) = ev_{fa}$ and the top row sends a to the continuous homomorphism ev_a. $\mathrm{Hom}_{KAb}(-,\mathbb{T})$ is controvariant functor so $(\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(f,\mathbb{T}),\mathbb{T}))(ev_a) = (ev_a)\circ \mathrm{Hom}_{Ab}(f,\mathbb{T}) = ev_a(-\circ f)$. For the commutativity, we have to prove this equality $ev_{fa} = ev_a(-\circ f)$. So $\forall g \in \mathrm{Hom}_{Ab}(B,\mathbb{T})$ we have: $ev_{fa}(g) = g(fa) = ev_a(gf)$. The diagramm commute, so η is a natural trasformation. □

It's left to prove that for all $A \in$ **Ab,** η_A is an isomorfhism i.e. is a group isomorphism, because **Ab** is a category of algebras a morphism that is bijective on underlying sets is always an isomorphism, so we need to show that η_A is a bijective morphism.

The injectivity of η_A hinges on the assertion that any Abelian group possesses enough characters to differentiate its elements. This means that for any Abelian group A and a nonzero element a $\in$A, there exists a character $\chi \in \mathrm{Hom}_{Ab}(A,\mathbb{T})$ such that $\chi a \neq 1_{\mathbb{T}}$. This statement can be established using non-constructive methods like Zorn's Lemma or through arguments involving Baer's characterization of injective objects in **Ab**.

Given that the prove of the injectivity is easy. In fact, given a,b $\in$ A such that a $\neq$ b we have that ab$^{-1} \neq 0_A$ so exist a character $\chi \in \mathrm{Hom}_{Ab}(A,\mathbb{T})$ such that $\chi ab^{-1} \neq 1_{\mathbb{T}}$ so $\chi a \neq \chi b$. considering η_A we have that $\eta_A(a) \neq \eta_A(b)$ infact $ev_a(\chi) \neq ev_b(\chi)$. Therefore, η_A is indeed injective.

To further solidify the proof, we need to introduce some definitions and a theorem.

Definition 7. *An abelian group E is injective if it satisfies the following condition:*
If A and B are abelian group, $f : A \longrightarrow B$ is an injective group homomorphism and $g : A \longrightarrow E$ is an arbitrary group homomorphism, then there exists a group homomorphism $h : B \longrightarrow E$ such that hf = g, i.e. such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[dl, "h"] \\ & E & \end{tikzcd}$$

*where 0 is zero object of **Ab***

Indeed, to demonstrate that $\mathbb{T}$ is an injective object in the category of abelian groups, we can consider the category of abelian groups (**Ab**) as equivalent to the category of $\mathbb{Z}$-modules (**ZMod**), where $\mathbb{Z}$ denotes the ring of integers.

Theorem 1 (Baer's criterion)**.** *An object $E \in$ **ZMod** is injective precisely if for $n\mathbb{Z}$ any $\mathbb{Z}$-ideal regarded as an $\mathbb{Z}$-module, any morphism $f : n\mathbb{Z} \longrightarrow E$ in **ZMod** can be extended to $\mathbb{Z}$ along the inclusion $n\mathbb{Z} \hookrightarrow \mathbb{Z}$.*

Lemma 11. *$\mathbb{T}$ is an injective abelian group.*

Proof. Using the Baer's criterion, we need to show that for n$\mathbb{Z}$ any $\mathbb{Z}$-ideal ,regarded as an $\mathbb{Z}$-module, and any morphism f : n$\mathbb{Z} \longrightarrow \mathbb{T}$,there exist $\bar{f} : \mathbb{Z} \longrightarrow \mathbb{T}$ that make the following diagramm commute

$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r] \arrow[d, "f"] & \mathbb{Z} \arrow[dl, "\bar{f}"] \\ & \mathbb{T} & \end{tikzcd}$$

n$\mathbb{Z}$ is an addictive subgroup of $\mathbb{Z}$ generate by n, the morphism f is uniquelly determinate by $f(n) \in \mathbb{T}$, so we can extend f to $\bar{f}$ by defining $\bar{f}(1) = \sqrt[n]{f(n)}$ and $\bar{f}(0) = 1_{\mathbb{T}}$. This $\bar{f}$ is indeed a well-defined group homomorphism and extend f, satisfying the requirements of Baer's Criterion. $\square$

Lemma 12. *Any Abelian group has enough characters to separate its elements. That is, for any Abelian group A and a non-zero element $a \in A$, there exist a character $\chi \in Hom_{Ab}(A, \mathbb{T})$ such that $\chi a \neq 1_{\mathbb{T}}$.*

Proof. We have to distinguish two case: the first one when $ord(a)$ = n, so we can define $f : < a > \longrightarrow \mathbb{T}$ that send a to $\sqrt[n]{1} \neq 1_{\mathbb{T}}$. By the fact that $\mathbb{T}$ is injective we have a character $\chi = \bar{f}$ such that $\chi a \neq 1_{\mathbb{T}}$. The other case is when $ord(a) = \infty$ in the same way as before we can send a to any number of $\mathbb{T}$ that have infinite order, as before we find a character $\chi = \bar{f}$ such that $\chi a \neq 1_{\mathbb{T}}$. $\square$

So we have the following theorem:

Theorem 2. *η_A for all $A \in$ **Ab** is an injective group homomorphism.*

Before discussing the surjectivity of this map, we shift our focus to the other side of the duality. Let G a compact group. We aim to define a continuous homomorphism ζ_G : A $\longrightarrow$ Hom$_{Ab}$(Hom$_{KAb}(G, \mathbb{T}), \mathbb{T})$. For an element g $\in$ G, let's consider the map:

$$ev_g : \text{Hom}_{KAb}(G, \mathbb{T}) \longrightarrow \mathbb{T}$$
$$\chi \longmapsto \chi g$$

As before, we need to verify some proprties

Proposition 13. *let $G \in$ **KAb**, $g \in G$, the map ev_g just define is a homomorphism.*

Proof. $ev_g(\chi - \xi) = (\chi - \xi)(g) = \chi g(\xi g)^{-1} = ev_g(\chi)(ev_g(\xi))^{-1}$ so it is a group homorphism. Thus $ev_g \in \text{Hom}_{Ab}(\text{Hom}_{KAb}(G, \mathbb{T}), \mathbb{T})$. $\square$

For all $G \in$ **KAb** we define:

$$\zeta_G : G \longrightarrow \text{Hom}_{Ab}(\text{Hom}_{KAb}(G, \mathbb{T}), \mathbb{T})$$
$$g \longmapsto ev_g$$

It easy to verify that ζ_G is group homorphism. For all g,h $\in$ G:
$(\zeta_G(g+h))f = ev_{g+h}(f) = f(g+h) = f(g)f(h) = ev_g(f)ev_h(f) = \zeta_G(g)f\zeta_G(h)f = (\zeta_G(g)+\zeta_G(h))f$
where $f \in \mathrm{Hom}_{KAb}(G,\mathbb{T})$ and $\zeta_G(1_G)f = ev_{1_G}f = f1_G = 1_{\mathbb{T}}$, so is a group homorphism between G and $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$.
We now need to show that ζ_G is a continuous map.
Lets's consider a subbasis open set of $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$, without losing of generality we can consider

$$V(\{f\},U) = \{h \in \mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T}) \mid h(f) \in U\ \}$$

where $f : G \longrightarrow \mathbb{T}$ is a continuous map. Let $V = f^{-1}(U)$ which is an open set of G, we claim that $\zeta_G(V) \subseteq V(\{f\},U)$, implying that ζ_G is a continuous map. Indeeed, for $g \in V$, $\zeta_G(g) = ev_g$ and $ev_g \in V(\{f\},U)$ because $ev_g(f) = fg$, which is in U by definition of V. Thus, ζ_G is a contiuous homorphism.

The contiuous homomorphisms ζ_A assemble into a transformation

$$\zeta : 1_{KAb} \Longrightarrow \mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(-,\mathbb{T}),\mathbb{T})$$

Proposition 14. *ζ is natural trasformation i.e. for all G,H $\in$ **KAb** and for all $f \in Hom_{KAb}(G,H)$, defining $C_f := Hom_{Ab}(Hom_{KAb}(f,\mathbb{T}),\mathbb{T})$*

$$\begin{array}{ccc} G & \xrightarrow{\zeta_G} & Hom_{Ab}(Hom_{KAb}(G,\mathbb{T}),\mathbb{T}) \\ \big\downarrow f & & \big\downarrow C_f \\ H & \xrightarrow{\zeta_H} & Hom_{Ab}(Hom_{KAb}(H,\mathbb{T}),\mathbb{T}) \end{array}$$

this diagramm commute.

Proof. The Proof is the literally the same as in proposition 6. □

To demostrate that ζ_G for all $G \in$ **KAb** is a topological isomorphism, thank to the previous obsevation, we only need to show that ζ_G is a contiuous bijective homorphism. Therefore,it remanins to prove that ζ_G is injective and surjective map.

As previously discussed, the injectivity of ζ_G amounts to the statement that any compact group has enough continuous characters to separate its elements, i.e. for any $G \in$ **KAb** and any $0 \neq$ g $\in G$, there exist a continuous character χ : G $\longrightarrow \mathbb{T}$ such that χg $\neq 1$. The traditional proof of this statement relies on the Peter-Weyl Theorem concerning the representations of a compact (not necessarily Abelian) topological group. The result on the existence of enough continuous characters of a compact (Abelian) group needed for Pontryagin Duality is a specific consequence of the Peter-Weyl Theorem. However, the proof of the Peter-Weyl Theorem is quite deep and goes beyond the scope of this discussion, so we will omit it here.

Theorem 3 (Peter-Weyl)**.** *Let G be a compact Hausdorff group. Then G has sufficiently many irreducile contiuous representations by unitary matrices. In other words, for each g $\in$ G, $g \neq 1_G$, there is a continuous homomorphism χ of G into the unitary group U(n), for some n, such that $\chi(g) \neq 1_G$.*
If G is abelian then, without loss of generality, it can be assumed that n=1. As U(1) = $\mathbb{T}$ we obtain the desired propriety.

Corollary 15. *Every compact Hausdorff abelian topological group has enough contiuous character to separate points.*

So ζ_G is an injective map.
To continue our exploration of duality, we first need to establish that η_A is an isomorphism for a specific class of Abelian groups: the finitely generated ones.

Theorem 4. *Every finitely generated abelian group A is isomorphic to a direct sum of primary cyclic groups and infinite cyclic groups. A primary cyclic group is one whose order is a power of a prime. In other words, every finitely generated Abelian group is isomorphic to a group of the form:*

$$\mathbb{Z}^n \oplus \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_t\mathbb{Z}$$

where $n \geq 0$ is the rank, and the numbers q_1, ..., q_t are powers of (not necessarily distinct) prime numbers. In particular, G is finite if and only if n = 0. The values of n, q_1, ..., q_t are uniquely determined by G, that is, there is one and only one way to represent G as such a decomposition.

Let's first considered A a primary cyclic abelian group. Thus, A is generated by some element a of order p, where p is a prime. Then the character in $\mathrm{Hom}_{Ab}(A,\mathbb{T})$ are unique determinate by the imagine of a, which needs to be an element of order p, $\sqrt[p]{1}$. So $\mathrm{Hom}_{Ab}(A,\mathbb{T})$ is a cyclic group genetare by the unique morphism that send a to one of the primitive p-rooth. Clearly, A and $\mathrm{Hom}_{Ab}(A,\mathbb{T})$ are isomorphic.

$$\mathbb{Z}_p \cong \mathrm{Hom}_{Ab}(\mathbb{Z}_p,\mathbb{T})$$

For a finite group we obviously have:

$$\mathbb{Z}_p \cong \mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(\mathbb{Z}_p,\mathbb{T}),\mathbb{T})$$

Now let's consider $\mathbb{Z}$. Clearly we have $\mathrm{Hom}_{Ab}(\mathbb{Z},\mathbb{T}) \cong \mathbb{Z}$, beacuse every morpfism is determinate by the image of 1, and there is one for every element of $\mathbb{T}$.
To obtain the desired isomorphism $\mathrm{Hom}_{KAb}(\mathbb{T},\mathbb{T}) \cong \mathbb{Z}$, we have to go in detail of the topological group $\mathbb{T}$ as a quotient of the topological group $(\mathbb{R},+)$. We need to study the proper closed subgroup of $\mathbb{T}$, starting with those of $\mathbb{R}$.

Proposition 16. *Every non-discrete subgroup G of $\mathbb{R}$ is dense.*

We need to show that for every $x \in \mathbb{R}$, and for every ε>0, we can find g $\in G \cap [x-\varepsilon, x+\varepsilon]$. As G is not discrete, 0 is not an isolated point. Therefore, there exist a point $x_\varepsilon \in (G \setminus \{0\}) \cap [0,\varepsilon]$, then the intervals $[nx_\varepsilon,(n+1)x_\varepsilon]$ for n=0,$\pm$1,$\pm$2,... cover $\mathbb{R}$ with lenghts less than or equal to ε. For some n, we have $nx_\varepsilon \in [x-\varepsilon, x+\varepsilon]$ and of course $nx_\varepsilon \in G$.

So we have a strong conseguence:

Proposition 17. *let G a closed subgroup of $\mathbb{R}$, then G or is trivial or is a discrete subgroup of the form $\alpha\mathbb{Z}$ for some $\alpha > 0$.*

Proof. Assume $G \neq \mathbb{R}, 0$, so it's not dense, meaning G needs to be discrete because is closed. Then G contains some positive real number b, hence $G \cap [0,b]$ is a non-empty subset of the compact set $[0,b]$. Thus $G \cap [0,b]$ is a compact and discrete, hence finite, and so there exist a least element $\alpha >$ 0 in G. For each $x \in G$, let's denote $[x/\alpha]$ the interger part of x/α. So we have that $x - [x/\alpha]\alpha \in G$ and we have that $x - [x/\alpha]\alpha \in [0,\alpha]$ so $x - [x/\alpha]\alpha = 0$ so we have $x = [x/\alpha]\alpha$, as desired. $\square$

Noting that the map x $\mapsto 1/\alpha$x is topological group isomorphism of $\mathbb{R}$ onto itself, such that maps $\mathbb{R}/\alpha\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$, so every proper Hausdroff quotient of $\mathbb{R}$ with a closed subgroup G is topological isomorphic to $\mathbb{T}$.

Corollary 18. *Every proper closed subgroup of* $\mathbb{T}$ *is finite*

Proof. Identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$ and let $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ be the quotient canonical map. If G is proper closed subgroup of $\mathbb{T}$ so $\pi(G)^{-1}$ is a proper closed subgroup of $\mathbb{R}$. So $\pi(G)^{-1}$ is discrete. By a result of general topology, the restriction of the quotient map to the $\pi(G)^{-1}$ is an open map. Than G is discrete, and being a closed subgroup in a compact is finite. $\square$

We are now ready to demonstrate the desired isomorphism $\mathrm{Hom}_{KAb}(\mathbb{T},\mathbb{T}) \cong \mathbb{Z}$.

Let $\chi \in \mathrm{Hom}_{KAb}(\mathbb{T},\mathbb{T})$. Let denote K the kernel of this continuous character, K = $\chi(1)^{-1}$ so K is a closed subgroup of $\mathbb{T}$, it follows K = $\mathbb{T}$ or is a finite cyclic group. If K = $\mathbb{T}$, χ is the trivial character. if K is a finite cyclic group of order r than by the previous observation, we have that $\mathbb{T}/K$ is topological isomorphic to $\mathbb{T}$. Let π the canonical quotient map, and let θ the topolical isomorphism from $\mathbb{T}/K$ to $\mathbb{T}$. Consider the following diagramm:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\chi} & \mathbb{T} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle id_T/-id_T} \\ \mathbb{T}/K & \xrightarrow{\theta} & \mathbb{T} \end{array}$$

where id_T is the identity map. We must have that $\theta\pi(x) = rx$ or x^r in the moltiplicative notation, as θ is a topological isomorphism. So we have that χ is in the form or $\chi(x) = rx$ or $\chi(x) = -rx$. This yields the following isomorphism:

$$\mathrm{Hom}_{KAb}(\mathbb{T},\mathbb{T}) \cong \mathbb{Z}$$

and consequently,

$$\mathbb{Z} \cong \mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(\mathbb{Z},\mathbb{T}),\mathbb{T})$$

as desired.

So we have the following important lemma:

Lemma 19. *For an abelian finitely generate group A, the unit map*
$\eta_A : A \longrightarrow Hom_{KAb}(Hom_{Ab}(A,\mathbb{T}),\mathbb{T})$ *is an isomorphism.*

Proof. The structure theorem for finitely abelian group tell us that A is isomorphic to a direct sum of primary cyclic groups and infinite cyclic groups. Clearly both the functor $\mathrm{Hom}_{Ab}(-,\mathbb{T})$ and $\mathrm{Hom}_{KAb}(-,\mathbb{T})$ preserve finite direct sum and by the previous proposition and observation we have A *cong* $\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})$. The unit map η_A sends 1_A to the trivial continuous character of $\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})$. In fact $\eta_A(1_A) = ev_{1_A}$ that send every character to $1_{\mathbb{T}}$ so is the trivial one, as we want. $\square$

To proceed, we need to delve a bit into the structure of topological subgroups in a topological group.

Proposition 20. *The closure of a topologica subgroup H in G is again a subgroup*

Proof. Let x,y $\in \bar{H}$ Suppose xy $\notin \bar{H}$. By openness of $\bar{H}^C$, there is a neighbourhood of U of (x,y) $\in$ G$\times$G such that uv $\notin \bar{H}$ for all (u,v) $\in$ U. U contains some V$_x$ $\times$ V$_y$ where V$_x$,V$_y$ are neighbourhoods of x and y, respectively. Pick u $\in V_x \cap H$, v $\in V_y \cap H$ and we have a contraddiction. □

Lemma 21. *Let A an abelian group, let $B \subseteq Hom_{Ab}(A,\mathbb{T})$ be a subgroup that separate the point i.e., $\forall$ $a \in A \setminus 0$ exist a character f $\in$ B such that fa $\neq 0$. Than B is dense in $Hom_{Ab}(A,\mathbb{T})$.*

Proof. Let's first suppose that A is finitely generated. without loss of generaly, thanks to the previuos proposition, we can suppose that B is a closed subgroup. Suppose B $\neq \mathrm{Hom}_{Ab}(A,\mathbb{T})$ (which is equivalent to say that B is not dense). So, the quotient group $\mathrm{Hom}_{Ab}(A,\mathbb{T})/$B is not trivial. Thanks to the Peter-Weyl we know that have a non trivial continuous character, i.e. $\mathrm{Hom}_{Ab}(A,\mathbb{T})$ have a continuous character $\bar{\chi}$ whose restriction to B is zero. Thus the restriction map $(-)_{|B}$: $\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T}) \longrightarrow \mathrm{Hom}_{KAb}(B,\mathbb{T})$ that sends $\chi \longmapsto \chi_{|B}$ is not injective. Precomposing with η_A that is an isomphorphism we have that $(-)_{|B} \circ \eta_A$: A $\longrightarrow \mathrm{Hom}_{KAb}(B,\mathbb{T})$ is not injective. There exist an a $\in$ A such that $(-)_{|B} \circ \eta_A(a) = 1_{\mathrm{Hom}_{KAb}(B,\mathbb{T})}$, where $1_{\mathrm{Hom}_{KAb}(B,\mathbb{T})}$ is the continuous character that send all the elements of B in $1_{\mathbb{T}}$, this means that for all $f \in$ B character of A fa = $1_{\mathbb{T}}$ i.e. B fails to separate the points. for the general case, let A be an abelian group. Consider f a character of A and consider an open basic neighbourhood of f,N = V(a_i,U) for some a_i in A with i=1,2,...,n and U is an open neighbourhood of $f(a_i)$ for all i. We must show that B $\cap$ N $\neq \emptyset$. Let A' be the subgroup of A generated by $\{a_1,\ldots,a_n\}$ and define:

$$\text{N'} = \{g \in \mathrm{Hom}_{Ab}(A',\mathbb{T}) \mid ga_i \in U \; for \; i = 1,\ldots,n\ \}$$
$$\text{B'} = \left\{g_{|A'} \mid g \in B\right\}$$

Then N' is a non-empty open set in $\mathrm{Hom}_{Ab}(A',\mathbb{T})$ since it contains the restiction of f to A', and B' is a subgroup of $\mathrm{Hom}_{Ab}(A',\mathbb{T})$ that separates the points of A'. Since A' is finitely generated, by the first part, we can find an element in B' $\cap$ N', but by defition, any such element must be the restiction to A' of an element of B $\cap$ N. So B is dense in $\mathrm{Hom}_{Ab}(A,\mathbb{T})$. □

Finally, we are ready to prove one of the isomorphism.

Theorem 5. *For any $G \in$ **KAb**, the unit map $\zeta_G : G \longrightarrow Hom_{Ab}(Hom_{KAb}(G,\mathbb{T}),\mathbb{T})$ is an isomorphism.*

Proof. Thanks to the Peter-Weyl Theorem, we already proven that ζ_G is injective. $\zeta_G(G)$ is a subgroup of $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$ that separates the points of $\mathrm{Hom}_{KAb}(G,\mathbb{T})$, which is an abelian group. In fact $\forall$ $f \in \mathrm{Hom}_{KAb}(G,\mathbb{T})$, with $f \neq 1_{\mathrm{Hom}_{KAb}(G,\mathbb{T})}$, there exist $h \in \zeta_G(G)$ such that $h(f)$ $\neq 1_{\mathbb{T}}$. This because if $f \neq 1_{\mathrm{Hom}_{KAb}(G,\mathbb{T})}$, there exist $\bar{g} \in$ G such that $f(\bar{g}) \neq 1_{\mathbb{T}}$. So it suffices to take $h = ev_{\bar{g}}$, which is in $\zeta_G(G)$, and we have $ev_{\bar{g}}(f) \neq 1_{\mathbb{T}}$, as desired. We obtain that $\zeta_G(G)$ is a dense subgroup of $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$, but G is compact so its image is closed because ζ_G is a continuous map. Thus $\zeta_G(G) = \mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$ so the unit map is surjective, thus a bijection, and hence a isomorphism in **KAb**. □

From this follows pretty easily the second isomorphism:

Theorem 6. *For any $A \in$ **Ab**, the unit map $\eta_A : A \longrightarrow Hom_{KAb}(Hom_{Ab}(A,\mathbb{T}),\mathbb{T})$ is an isomorphism.*

Proof. Thanks to the theorem 2 we know that η_A is injective. If the image is not the whole $\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})$ than the quotient group $\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})/A$ would have a non zero character. That is $\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T})$ have a charater that restiction to A is zero. However, this would be an element of $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T}),\mathbb{T})$ that is not an element of the image of the unit map $\zeta_{\mathrm{Hom}_{Ab}(A,\mathbb{T})} : \mathrm{Hom}_{Ab}(A,\mathbb{T}) \longrightarrow \mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(\mathrm{Hom}_{Ab}(A,\mathbb{T}),\mathbb{T}),\mathbb{T})$, because such a character doesn't exist in $\mathrm{Hom}_{Ab}(A,\mathbb{T})$. This is impossible because it contradicts the previous theorem. So the unit map is surjective, thus a bijection, and hence a isomorphism in **Ab**. □

Duality

Finally we reach our main goal.

Pontryagin Duality. *The two contravariant functors*

$$Hom_{Ab}(-,\mathbb{T}) : \boldsymbol{Ab} \longrightarrow \boldsymbol{KAb}^{op}$$
$$Hom_{KAb}(-,\mathbb{T}) : \boldsymbol{KAb} \longrightarrow \boldsymbol{Ab}^{op}$$

together with the natural isomorphism

$$\eta : 1_{Ab} \Longrightarrow Hom_{KAb}(Hom_{Ab}(-,\mathbb{T}),\mathbb{T})$$
$$\zeta : 1_{KAb} \Longrightarrow Hom_{Ab}(Hom_{KAb}(-,\mathbb{T}),\mathbb{T})$$

are a dual equivalence.

Thus we established the Duality $\mathbf{Ab} \cong \mathbf{Kab}^{op}$. This allows us to trasfer theorem and construction between these categories. Let's explore some examples:

Property I. *Let $G \in$ **KAb***
G is connected $\Longrightarrow Hom_{KAb}(G,\mathbb{T})$ is torsion-free
$Hom_{KAb}(G,\mathbb{T})$ is torsion-free $\Longrightarrow$ G is connected

For the first one, we need to demostarte two things: if G is connected, then every contiuous character χ is surjective, and if every character is surjective, then $\mathrm{Hom}_{KAb}(G,\mathbb{T})$ is torsion free.
Let χ a continuous character. If is not surjective, then $\chi(G)$ is a proper (beacuse is not surjective) closed subgroup of $\mathrm{Hom}_{KAb}(G,\mathbb{T})$, because G is compact and $\mathbb{T}$ is Hausdorff. By Corollary 15, we know that every proper closed subgroup of $\mathbb{T}$ is finite so $\chi(G) = \{\chi(g_1),\chi(g_2),\dots,\chi(g_n)\}$ for some $g_1,\dots,g_n \in$G. Thus, we have that $\chi^{-1}(g_1),\chi^{-1}(g_2),\dots,\chi^{-1}(g_n)$ are disjoint closed in G, beacuse χ is continuous and $\mathbb{T}$ is Hausdorff.The union of this disjoint closed subset is G, so we have that $(\bigcup_{i=1}^{n-1}\chi^{-1}(g_i))^C$ and $\chi^{-1}(g_n)$ are disjoint open whose union is G, implying that G is not connected.
If $\mathrm{Hom}_{KAb}(G,\mathbb{T})$ is not torsion-free, there exist χ such that $n\chi$ is equal to the zero of $\mathrm{Hom}_{KAb}(G,\mathbb{T})$, so for all $g \in$ G we have $n\chi(g) = 0_{\mathrm{Hom}_{KAb}(G,\mathbb{T})}(g) = 1_{\mathbb{T}}$. Consequently, $\forall g \in$G, $Ord(g)|\, n$. However, $\mathbb{T}$ have almost an element with infinite period, so χ cannot be surjective.
This establishes the first sentence of the property I.

If G is not connected, then G$_e$, the connected component of the identity, is not equal to G. G$_0$ is a closed normal subgroup of G. By previuos proposition we have that G/G$_0$ is a totally disconnected compact group. Let's study its character. Let χ : G/$G_0 \longrightarrow \mathbb{T}$ a continuos character. Any continuos characters cannot be surjective: assume that $\chi(G/G_0) = \mathbb{T}$. Then χ : G/$G_0 \longrightarrow \mathbb{T}$ would be a quotient map by the proposition 6, so $\mathbb{T}$ would be a quotient of G. Since total disconnectedness is inherited by quotiens of compact groups, we conclude that $\mathbb{T}$ must be totally disconnected, which is a contradiction. Therefore, no character can be surjective. Hence, $\chi(G/G_0)$ is a closed subgroup of $\mathbb{T}$, so is finite by corollary 15. Let m = $|\chi(G/G_0)|$, so χ is a torsion element of $\mathrm{Hom}_{KAb}(G/G_0,\mathbb{T})$. In fact for every element g $\in$ G/G_0 we have m$\chi(g)$ = $1_{\mathbb{T}}$, and so mχ = $1_{\mathrm{Hom}_{KAb}(G/G_0,\mathbb{T})}$. By composing with canonical quotient map p : G $\longrightarrow$ G/G_0 we obtain χp : G $\longrightarrow$ $\mathbb{T}$ that is continuous character of G. we obtain χp $\in$ $\mathrm{Hom}_{KAb}(G,\mathbb{T})$ and is it a continuous torsion character, because for every g $\in$ G, m$\chi(p(g)$ = $1_{\mathrm{Hom}_{KAb}(G/G_0,\mathbb{T})}(p(g))$ = $1_{\mathbb{T}}$. We have tha mχp = $1_{\mathrm{Hom}_{KAb}(G,\mathbb{T})}$ so χp is a torsion element, and $\mathrm{Hom}_{KAb}(G,\mathbb{T})$ is not torsion free. This establishes the second sentence of the property I.

For the second property, we will assume a standard fact about topological metrizable groups:

Theorem 7 (Birkhoff-Kakutani)**.** *A topological group is metrizable if and only if it has a countable base of neighborhoods of* 0_G

Additionally, we will assume a fundamental characteristic for **LCA**-group in the Pontriagyn-Van Kampen Duality

Proposition 22. *Let G be an* ***LCA****-group* $Hom_{LCA}(G,\mathbb{T})$ *its dual group and K any compact neighbourhood of 0 in G . If U is a "small" neighbourhood of 0 in T (more precisely if* $U \subseteq \{exp(2\pi ix) : -1/4 < x < 1/4\}$*), then* $\overline{V(K,U)}$ *the closure of the set* $V(K,U)$*, is a compact neighbourhood of 0 in* $Hom_{LCA}(G,\mathbb{T})$*.*

This proposition bring withs it a corollary that justifies our Duality:

Corollary 23. *If G is an abelian discrete topological group, then* $Hom_{LCA}(G,\mathbb{T})$ *is a compact group. If G is a compact Hausdorff abelian topological group, then* $Hom_{LCA}(G,\mathbb{T})$ *is discrete.*

Property II. *Let* $G \in$ ***KAb***
G is metrizable $\Longrightarrow$ $Hom_{KAb}(G,\mathbb{T})$ *is countable*
$Hom_{KAb}(G,\mathbb{T})$ *is countable* $\Longrightarrow$ *G is metrizable*

Firstly, assume that G is metrizable, Then G has a countable $\{U_n\}$ base of open neighbourhoods of 0. Than $\{\overline{U_n}\}$ forms a base of compact neighbouhoods of 0. If U is a "small" neighbourhood of 0 in T, so V($\overline{U_n}$,U) are open sets in $\mathrm{Hom}_{KAb}G,\mathbb{T})$. Thanks to the proposition, we have that $\overline{V(\overline{U_n},U)}$ are compact set of 0 in $\mathrm{Hom}_{KAb}(G,\mathbb{T})$. As each $\chi \in \mathrm{Hom}_{KAb}(G,\mathbb{T})$ is continuous, there exists j such that $\chi(\overline{U_j}) \subseteq$ U. So $\chi \in$ V($\overline{U_j}$,U), Thus we have that $\mathrm{Hom}_{KAb}(G,\mathbb{T}) = \cup_{i=1}^{\infty}$V($\overline{U_i}$,U), but we have $\mathrm{Hom}_{KAb}(G,\mathbb{T}) = \cup_{i=1}^{\infty}\overline{V(\overline{U_i},U)}$. Thanks to the corollary we have that is a discrete space, so the compact subset are exactally the finite ones, so we can write $\mathrm{Hom}_{KAb}G,\mathbb{T})$ as a conutable union of finite set, thus is a countable set too.

Conversely, assume that $\mathrm{Hom}_{KAb}(G,\mathbb{T})$ is countable. For each positive integer n , let U_n = $\{exp(2\pi ix) \in T : -1/n < x < 1/n\}$. Then each U_n is an open neighbourhood of 0 in T , and so the sets V(K,U_n) are open in $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$, for any finite subset K of $\mathrm{Hom}_{KAb}(G,\mathbb{T})$. Indeed, it is easily verified that if we allow K to range over all finite subsets of $\mathrm{Hom}_{KAb}(G,\mathbb{T})$ and n to range over all natural numbers then the sets V(K,U_n) form a subbase of neighbourhoods of 0 in $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$. Since $\mathrm{Hom}_{KAb}(G,\mathbb{T})$ is countable, $\mathrm{Hom}_{KAb}(G,\mathbb{T})$ has only a countable number of finite subsets. Thus there are only a countable number of sets V(K,U_n) in the subbase of neighbourhoods of 0 in $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$, Hence $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$ is metrizable. The duality theorem tells us that G is topologically isomorphic to $\mathrm{Hom}_{Ab}(\mathrm{Hom}_{KAb}(G,\mathbb{T}),\mathbb{T})$, and so G too is metrizable.

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