EXPLORING FRACTAL DIMENSION
DRP SPRING 2025

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1. Preface

This was written by Max Seay of CWRU with direction from PhD student Aaron Huntley, as part of the Directed Reading Program in Spring 2025. As of writing, it has not been proofread. Date of latest update on cover page.

2. Introduction

The following notes describe concepts of measurement, dimensionality and fractal objects living within metric spaces. The goal is to uderstand how to compute the *dimensionality* of arbitrarily defined objects. It's already assumed that common objects attain dimensionality of whole numbers. For example: the point is of dimension 0, the line is of dimension 1, the filled square is of dimension 2, the filled cube is of dimension 3, etc. However these notes also describe how dimensionality can be computed for other, more strangely behaved objects. Objects like *fractals*, which can attain more unexpected measurements and dimensionality. For example, the Cantor Set of the real number line $\mathbb{R}$, which takes on non-integer dimensionality of $\frac{\log 2}{\log 3}$, while also having a length of 0. Or the Dragon Curves, a family of space-filling fractal curves derived from the repition of bending of a line towards infinity, which acheive positive area and have the ability to cover and tile the plane of real numbers, $\mathbb{R}^2$.

Definition 2.1 (Object)**.** A finite or infinite *compact* set S living within some metric space.

Below are listed further examples of objects, defined from sets, and their corresponding dimensionality:

- $\{0\} \in \mathbb{R}$

  dimensionality: 0
- $\{1\} \in \mathbb{R}$

  dimensionality: 0
- $\{2, 4, 6, 8\} \in \mathbb{R}$

  dimensionality: 0
- $[0, 1] \in \mathbb{R}$

  dimensionality: 1
- $(0, 1) \in \mathbb{R}$

  dimensionality: 1
- $(0, 1] \in \mathbb{R}$

  dimensionality: 1
- $[0, 1] \times [0, 1] \in \mathbb{R}^2$

  dimensionality: 2

3. How to measure the real numbers
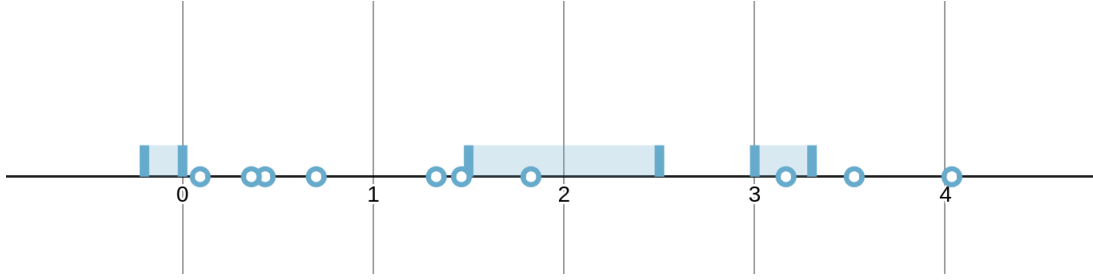
FIGURE 1. Points and line segments living within the real number line $\mathbb{R}$.

On the real number line $\mathbb{R}$, common objects can be defined. For example:

Example 3.1 (Point). A point on the real number line is a single real number. Examples include: $1, 2, 3, 17, \sqrt{2}, \pi$.

The dimensionality of a point is commonly known to be 0. A point can't have length.

Example 3.2 (Line segment). A line segment on the real number line is the defined to be the set of all real numbers that lie between two points. A line segment that starts at 0 and reaches 1 would be defined as the set

$$\{x : 0 \leq x \leq 1\}$$

equivalently expressed in *interval form*:

$$[0, 1]$$

In other words, a set containing every real number between and including 0 and 1.

The dimensionality of a line is commonly known to be 1.

Definition 3.3 (Interval length). Given an interval I of the form $[a, b]$, (a, b), $(a, b]$, or $[a, b)$ where $a \leq b$, the length of I, notated as $|I|$, is defined to be

$$|I| := b - a$$

This is intuitive and useful. But how can this be justified with some definition of measure? And how can this idea of length be extended to a more general way of measurement?

3.1. **Coverings.** Most definitions of measure and dimension involve the idea of a *covering*.

A *covering* is some collection of sets that cover some other set. If the covering is made from open sets, it's called an open covering.

Example 3.4 (Open covering for an interval). Let $I = [1, 3]$. An example of an open cover for I is

$$C = \{(-50, 0), (-20, 70), (50, 100)\}$$

since
$$I \subseteq \bigcup C_i$$
and every set C_i in C is open.

You could say that C is "overkill" as an open cover for I since it spans a much longer length than I. In this case, the length of the open cover $\bigcup C_i$ is
$$100 - (-50) = 150$$
and the length of I is
$$3 - 1 = 2$$

3.2. Infinite sequences of sets.

Example 3.5 (Stick cutting). Let there be some stick s to measure. Say the stick begins at a length of 1. Now do the following:

Cut the stick to $\frac{1}{2}$ its original length. Then take the stick and cut it to $\frac{1}{3}$ its original length. Then cut the stick to $\frac{1}{4}$ its original length. Continue this forever.

At the end of the cutting process, how long is the stick?

Unfortunately this question doesn't make sense. There is no "end" to the cutting process since it goes on forever. First the object should be defined in a valid way. Say s_0 is the stick before cutting, s_1 is the stick at the first cutting step etc.

A valid definition of the object to be measured is:
$$\bigcap_{i=0}^{\infty} s_i$$
In other words, the intersection of every stick. The object to be measured is the set of points that are inside each and every stick s_i. Now there is no longer any ambiguity.

Another object which represents this length value is the limit
$$\lim_{i \to \infty} |s_i|$$
where $s_i = [0, \frac{1}{i+1}]$ represents the stick after step i of the cutting process.

In other words, the length of each stick in the sequence as the sequence tends towards infinity.

On one hand, the stick is cut forever, so it must be reduced to nothing by the end. On the other hand, the amount that gets cut off decreases each time and so at some point there may be no more cut off at all.
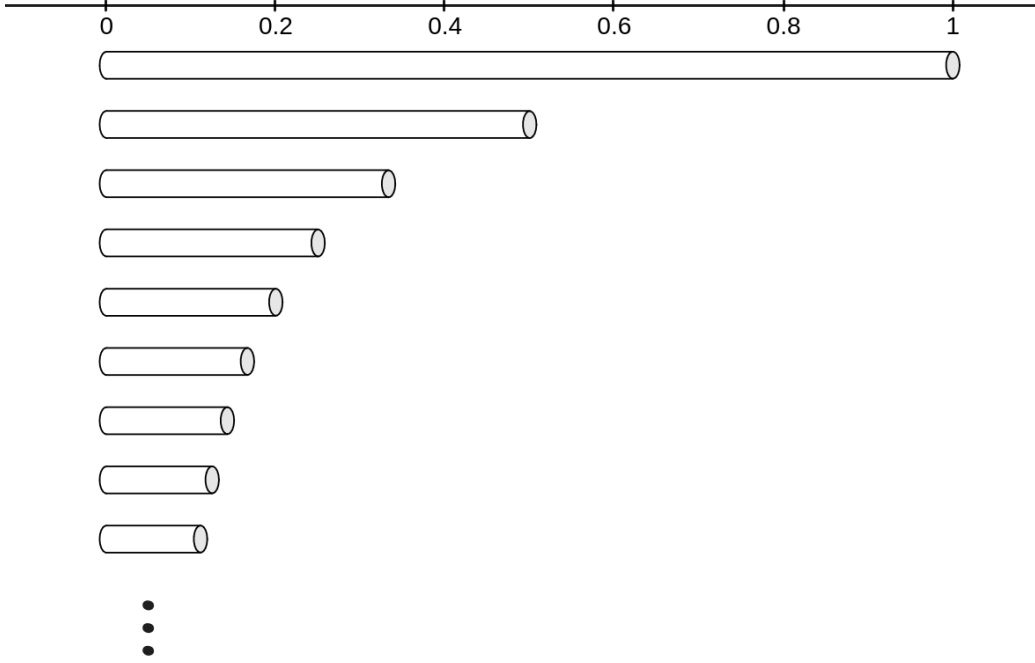
Diagram of the first 9 steps of stick cutting.

Corollary 3.6. *For each stick s_i in the sequence of sticks $\{s_i\}_{i=1}^{\infty}$, $s_i \subset s_{i-1}$. Each stick s_i is covered by stick s_{i-1}.*

Corollary 3.7.
$$\lim_{i \to \infty} |s_i| = 0$$
where $s_i = [0, \frac{1}{i+1}]$ represents the stick after step i of the cutting process.

Proof. Rewrite the limit with the definition of the length of s_i:
$$\lim_{i \to \infty} |s_i| = \lim_{i \to \infty} \left(\frac{1}{i+1} - 0 \right)$$
So we have
$$\lim_{i \to \infty} |s_i| = \lim_{i \to \infty} \left(\frac{1}{i+1} - 0 \right) = \lim_{i \to \infty} \left(\frac{1}{i+1} \right)$$
This is a common limit known to be
$$\lim_{i \to \infty} \left(\frac{1}{i+1} \right) = 0$$
Thus we have that
$$\lim_{i \to \infty} |s_i| = 0$$
as required. □

Corollary 3.8. *The length of $\cap_{i=1}^{\infty} s_i$ is 0.*

To prove this, I will show that given any real number $\varepsilon > 0$, the length of the stick is less than ε. Thus the length must be 0.

Proof. Let the stick be an interval (line segment). Let s_0 be the initial stick; a closed interval $[0, 1]$.

Step 1 is to cut the interval in half. So the interval becomes $\left[0, \frac{1}{2}\right]$.

Step 2 is to cut the interval into a third. So the interval becomes $\left[0, \frac{1}{3}\right]$.

Step 3 is to cut the interval into a fourth. So the interval becomes $\left[0, \frac{1}{4}\right]$.

Step k is to cut the interval $\left[0, \frac{1}{k}\right]$ into the interval $\left[0, \frac{1}{k+1}\right]$.

After every step of the cutting process, the stick is cut; the length of the interval is decreased. This can be seen by comparing the lengths of the stick before and after some cutting step k:

$$\left|0 - \frac{1}{k+1}\right| \leq \left|0 - \frac{1}{k}\right|$$

Let $\varepsilon \in \mathbb{R}^+$ be some arbitrary positive real number.

I will show that there always exists a cutting step k' such that after this step, the length of the stick is less than the arbitrarily small ε.

Choose the k' step to be a whole number such that

$$k' > \frac{1}{\varepsilon}$$

(By the archimedian principle this is always possible.)

I know that by definition of the cutting process, for this step k', the stick is cut to a length l that is

$$l \leq \frac{1}{k'}$$

And note that by the definition of k' we have that

$$l \leq \frac{1}{k'} < \frac{1}{1/\varepsilon} = \varepsilon$$

Or simply

$$l < \epsilon$$

Since the length of the stick l at some step k' is always less than ε, and ε can be made arbitrarily small, it must be true that
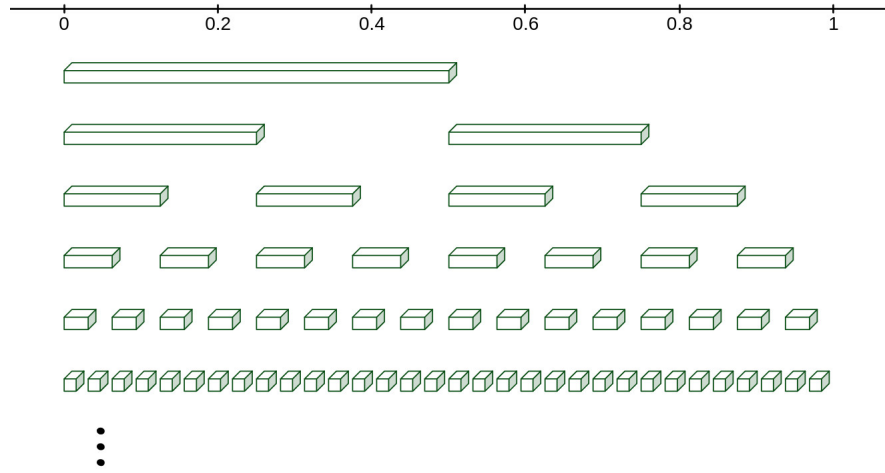
$$l = 0$$

As required. $\qquad\square$

FIGURE 2. The infinite construction of an object living in the real number line $\mathbb{R}$. On each step, each interval is cut and half and the right half is moved towards the right.

A common way of measuring an object is with the construction a covering. When the cover is wrapped around the object, the measure of the cover should be similar to that of the object itself.

For example, to measure the surface area of a sphere, a cover could take on the form of a 2-dimensional blanket or sheet, which when tighly wrapped around the sphere covers the entire object. The trivial measurement of the area of the sheet can then be compared to that of the sphere. The idea of covers are often involved with the many varied definitions of measure and dimension.

3.3. **Lebesgue measure.** Lebesgue measure on the real number line is a method of measuring objects within $\mathbb{R}$. The object is nearly wrapped within a collection of intervals, aimed to cover the entire object. As the cover tightens, shrinks or expands, the sum of measurements of the collection approaches that of the object itself.

Outer Lebesgue measure is to begin with an outside cover that initially overestimates the length of the object, then shrinking it, and approaching the desired measurement.[Edg08]

Inner Lebesgue measure is to begin with an inside cover that underestimates the length of the object, then expanding it, and approaching the desired measurement.[Edg08]

Definition 3.9 (Lebesgue measure)**.**

4. Notion of dimension: Minkowski–Bouligand or box-counting dimension

I would like to know how to estimate the dimension of arbitrary objects on the plane $\mathbb{R}^2$. First I will look at simple examples to find out more about the notion of dimension in 2D space.

Example 4.1 (Line segment)**.** I know what a line segment is. I also know every line segment is 1 dimensional.

I take an approximation. I will cover a line segment with boxes. Each box has diameter (or radius) r. (To me it doesn't matter much since the difference between radius and diameter is always just a factor of 2.)

Take the line segment to be the interval $[0, 1]$. If $r = 1$, then I need only 1 box to cover the line segment.

If $r = 1/2$, I will need at least 2 boxes.

If $r = 1/3$, I will need at least 3 boxes.

If $r = 1/n$, I will need at least n boxes to cover the line segment.

I will define a function that takes in a radius, r, and outputs the least number of boxes of radius r needed to cover the line segment. From the above examples it can be seen that this function is:
$$\mathcal{N}(r) = \frac{1}{r}$$
Another idea to think on. How to double the *length* of the line segment? To double the length of the line segment, copy and paste it once, and append the copy onto the end of the original.

So in other words, to double the length of the line segment, add 1, (or $2^1 - 1$ you could say,) extra copy.

Example 4.2 (Square)**.** I know what a square is. I also know every square is 2 dimensional.

I take an approximation. I will cover a square with boxes. Each box has diameter (or radius) r.

Take the square to be the unit square, $[0, 1] \times [0, 1]$. If $r = 1$, then I need only 1 box to cover the square. (The box and square are the same.)

If $r = 1/2$, I will need at least 4 boxes.

If $r = 1/3$, I will need at least 9 boxes.

If $r = 1/n$, I will need at least n^2 boxes to cover the unit square.

I will define a function that takes in a radius, r, and outputs the least number of boxes of radius r needed to cover the square. From the above examples it can be seen that this function is:
$$\mathcal{N}(r) = \frac{1}{r^2}$$
Now how to double the *area* of the square? To double the area of the square, copy and paste it three times, and append the copies to three sides of the original.

So in other words, to double the area of the square, add 3, (or $2^2 - 1$ you could say,) extra copies.

So for the line segment we have that the number of squares of radius r needed to cover it as
$$\mathcal{N}(r) = \frac{1}{r}$$
And for the square we have that the number of squares of radius r needed to cover it as
$$\mathcal{N}(r) = \frac{1}{r^2}$$
From these couple examples it seems that, in general
$$\mathcal{N}(r) = c \left(\frac{1}{r}\right)^d$$
Where c is a constant. It seems that it would be appropriate to call the d in this expression the dimension. This function $\mathcal{N}(r)$ is called a power law since $1/r$ is always raised to some power. [Fal13]

5. Notion of dimension: Hausdorff measure and dimension

Definition 5.1 (Diameter of a set)**.** Let S be a non-empty set living in $\mathbb{R}^n$. The diameter of S is defined to be
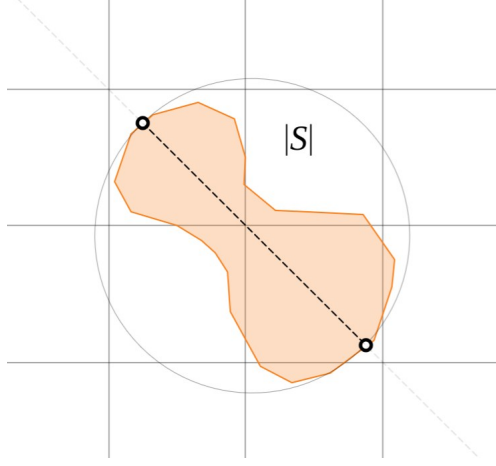
$$|S| = \sup\{|x - y| : x, y \in S\}$$

FIGURE 3. Diameter of a set S living in $\mathbb{R}^2$.

Definition 5.2 (δ-cover)**.** Let S be a set living in $\mathbb{R}^n$. A set of sets $\{C_i\}$, (where each C_i is itself a set) is said to be a δ-*cover* of S if

$$S \subseteq \bigcup C_i$$

and $|C_i| \leq \delta$ for every C_i.[Fal85]

Definition 5.3 (Hausdorff measure)**.** Let S be a set living in $\mathbb{R}^n$. First define the function

$$\mathcal{H}^d_\delta(S) = \inf \sum_{i=1}^{\infty} |C_i|^d$$

where the infimum is taken over each and every possible δ-cover $\{C_i\}$ of the set S.[Fal85]

In other words, this function picks the "tightest" δ-cover of S and sums the diameter of everything in the covering. The "tightest" δ-covering is the δ-covering with the smallest sum.

Define the *Hausdorff measure* of the set S to be the limit:

$$\lim_{\delta \to 0} \mathcal{H}^d_\delta(S)$$

And denote this as simply

$$\mathcal{H}^d(S)$$
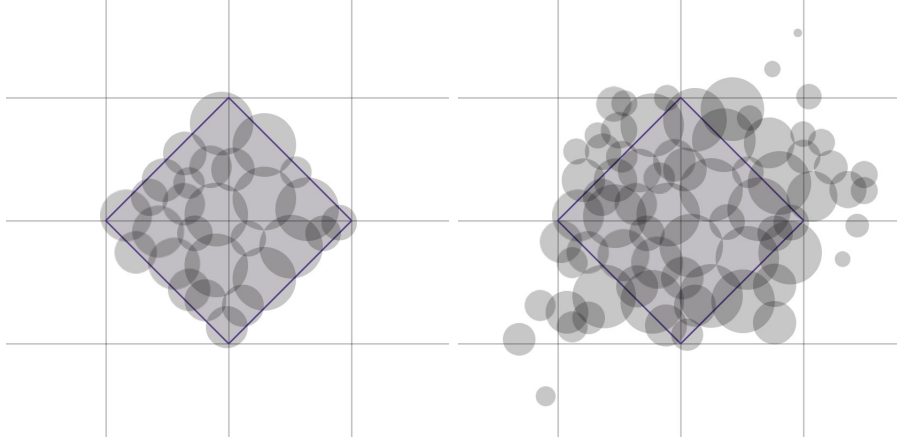
Let's measure some objects using Hausdorff measure.

FIGURE 4. Two valid δ-covers of an object living in $\mathbb{R}^2$. A relatively tight δ-cover (left) compared to a much less tight δ-cover (right). The sum of the diameters of the tight δ-cover (left) is less than the sum of the diameters of the much less tight δ-cover (right).

Example 5.4 (Line segment). Let S be a line segment living in $\mathbb{R}^2$. Say that

$$S = [0, 1]$$

The length of S is 1. Though now I would like to compute the measure of this set S using Hausdorff measure. Recall the function

$$\mathcal{H}_\delta^d(S) = \inf \sum_{i=1}^{\infty} |C_i|^d$$

where the infimum is taken over every possible δ-cover $\{C_i\}$ of S.

Ok for now let

$$\delta = {}^1/_{10}$$
$$d = 1$$

With these values we now have the function

$$\mathcal{H}_{{}^1/_{10}}^1(S) = \inf \sum_{i=1}^{\infty} |C_i|$$

where the infimum is taken over every possible δ-cover $\{C_i\}$ of S where $\delta = {}^1/_{10}$.

In other words, the infimum is now taken over every possible δ-cover $\{C_i\}$ of S where

$$|C_i| \leq {}^1/_{10}$$

for all C_i.

For simplicity choose a cover like that of the figure (top image.) That is, arrange the cover such that there are 10 balls each with size $^1/_{10}$ covering the line segment side-by-side.
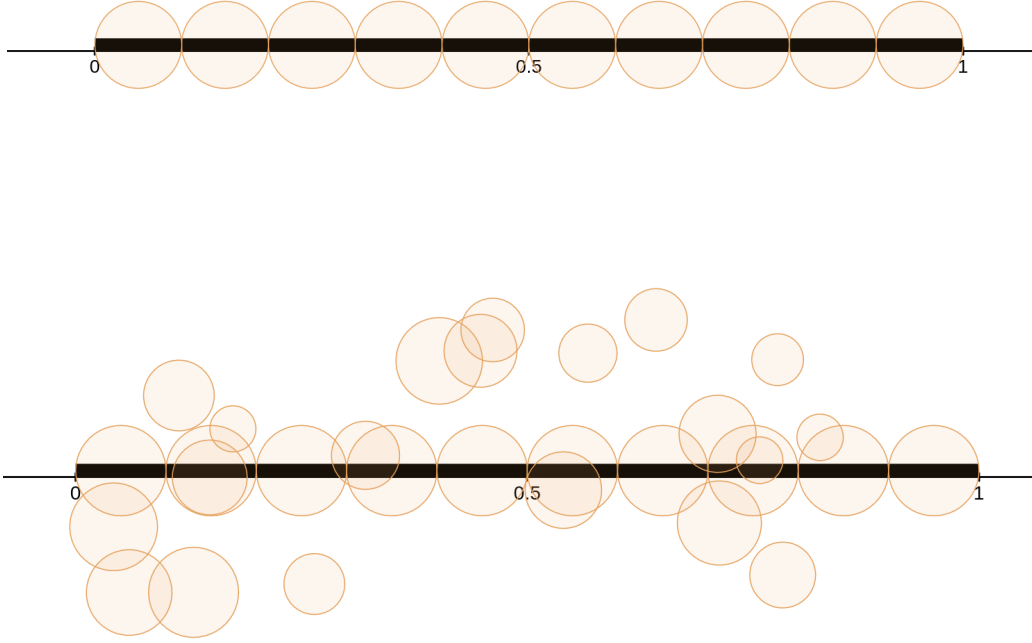
For each ball in the cover

$$|C_i| = {}^1/_{10}$$

FIGURE 5. Two ways to cover the line segment [0,1] with a δ-cover where $\delta = 1/10$. Each orange circle represents an element $C_i \in \{C_i\}$ of the δ-covering. Notice $[0,1] \subseteq \bigcup\{C_i\}$ and $|C_i| \leq 1/10$ for all C_i (as required of a δ-cover.) Let the top image show the cover $\{C_i\}$ and the bottom image show the cover $\{D_i\}$. It can be seen that

$$\sum |C_i| < \sum |D_i|$$

(Roughly speaking because there are less balls in the top cover $\{C_i\}$ than the bottom cover $\{D_i\}$.)

So we have

$$\mathcal{H}^1_{1/10}(S) = \sum_{i=1}^{10} |C_i| = \sum_{i=1}^{10} 1/10 = \left(\frac{1}{10} + \frac{1}{10} + \dots \right) = 1$$

So this "approximatation" of Hausdorff measure, where $\delta = 1/10$, agrees with intuitition.

Ok now try $\delta = 1/20$ and consider the same configuration of balls for the δ-cover of S. That is, arrange 20 balls side-by-side all of diameter $1/20$. Now we have that

$$\mathcal{H}^1_{1/20}(S) = \sum_{i=1}^{20} |C_i| = \sum_{i=1}^{20} 1/20 = \left(\frac{1}{20} + \frac{1}{20} + \dots \right) = 1$$

So this "approximatation" of Hausdorff measure, where δ is even smaller, still agrees with intuitition. Now using the same kind of covers as before, send δ to 0. To approach 0, it

suffices to let

$$\delta = \frac{1}{N}$$

where $N \in \mathbb{N}$ and $\delta \to 0$ means $N \to \infty$.

So now observe the limit

$$\lim_{\delta \to 0} \mathcal{H}^1_\delta(S) = \lim_{\delta \to 0} \sum_{i=1}^{1/\delta} |C_i| = \lim_{\delta \to 0} \sum_{i=1}^{1/\delta} \delta$$

Now replace δ with its definition from above

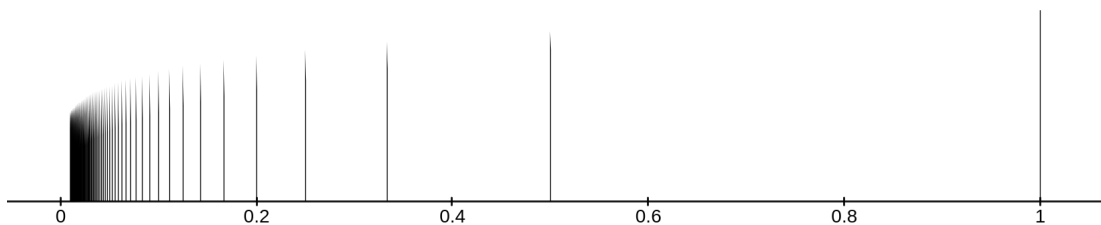$$\lim_{\delta \to 0} \mathcal{H}^1_\delta(S) = \lim_{N \to \infty} \sum_{i=1}^{N} \frac{1}{N}$$

I know for any and all $N \in \mathbb{N}$ that

$$\sum_{i=1}^{N} \frac{1}{N} = 1$$

Thus we have that the Hausdorff measure of the set S where $S = [0, 1]$ is

$$\mathcal{H}^1(S) = \lim_{\delta \to 0} \mathcal{H}^1_\delta(S) = \lim_{N \to \infty} \sum_{i=1}^{N} \frac{1}{N} = 1$$
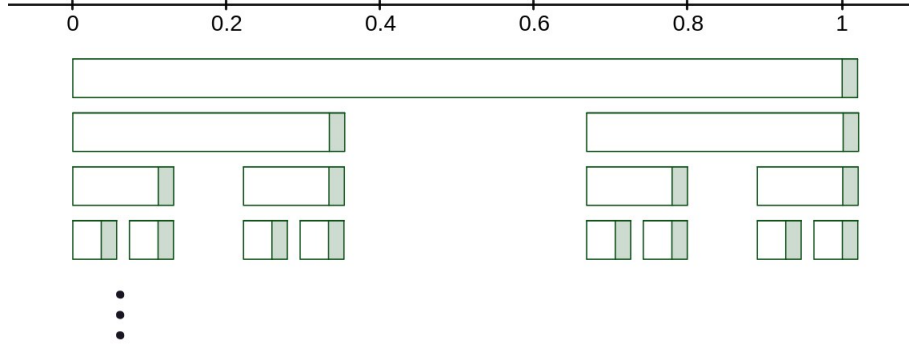
6. Cantor Set

FIGURE 6. The infinite construction of the Cantor Set in $\mathbb{R}$. On each step, the middle third of each interval is deleted.

Let $\mathcal{C}$ be the Cantor Set.

Definition 6.1 (Cantor Set). $\mathcal{C}$ can be described as an intersection of an infinite sequence of sets. The sequence of sets, $\{\mathcal{C}_i\}_{i=0}^{\infty}$, is defined recursively. For example, the first three are:

$$C_0 = [0, 1]$$
$$C_1 = [0, 1/3] \cup [2/3, 1]$$
$$C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$$
$$C_3 = [0, 1/27] \cup [2/27, 3/27] \cup [6/27, 7/27] \cup [8/27, 9/27] \cup [18/27, 19/27] \cup [20/27, 21/27] \cup [24/27, 25/27] \cup [26/27, 1]$$

And the recursive definition for a set $\mathcal{C}_i$ is:

$$\mathcal{C}_i = \frac{1}{3}\mathcal{C}_{i-1} \cup \frac{2}{3} + \frac{1}{3}\mathcal{C}_{i-1}$$

Corollary 6.2. $\mathcal{C}_i \neq \mathcal{C}$ for all $i \in \mathbb{N}$. No set in the sequence $\{\mathcal{C}_i\}$ is the Cantor Set, however the further you go (the larger the i), the "closer" $\mathcal{C}_i$ becomes $\mathcal{C}$.

Corollary 6.3. The Cantor Set is non-empty.

Proof. Consider the point 0.

$0 \in [0, 1]$ so $0 \in \mathcal{C}_0$.

Now make the assumption that $0 \in \mathcal{C}_k$.

And since $\frac{1}{3} \cdot 0 = 0$ it must be true that

$$0 \in \frac{1}{3} \cdot \mathcal{C}_k$$

Note that by definition we have that

$$\mathcal{C}_{k+1} = \frac{1}{3}\mathcal{C}_k \cup \frac{2}{3} + \frac{1}{3}\mathcal{C}_k$$

Well since $0 \in \frac{1}{3} \cdot \mathcal{C}_k$ it must be true that

$$0 \in \frac{1}{3}\mathcal{C}_k \cup \frac{2}{3} + \frac{1}{3}\mathcal{C}_k$$

or

$$0 \in \mathcal{C}_{k+1}$$

So by induction we have that

$$0 \in \mathcal{C}$$

Thus $\mathcal{C}$ is non-empty. $\qquad\square$

Corollary 6.4. $\mathcal{C}_i$ *is always a covering of* $\mathcal{C}_{i+1}$ *and also always a covering of* $\mathcal{C}$.

Corollary 6.5. *There Cantor Set has an infinite number of elements.*

Proof. To show the Cantor Set has an infinite number of elements, it suffices to define a injective map between all the natural numbers $\mathbb{N}$ and some (or all) of the elements in the Cantor Set $\mathcal{C}$. $\qquad\square$
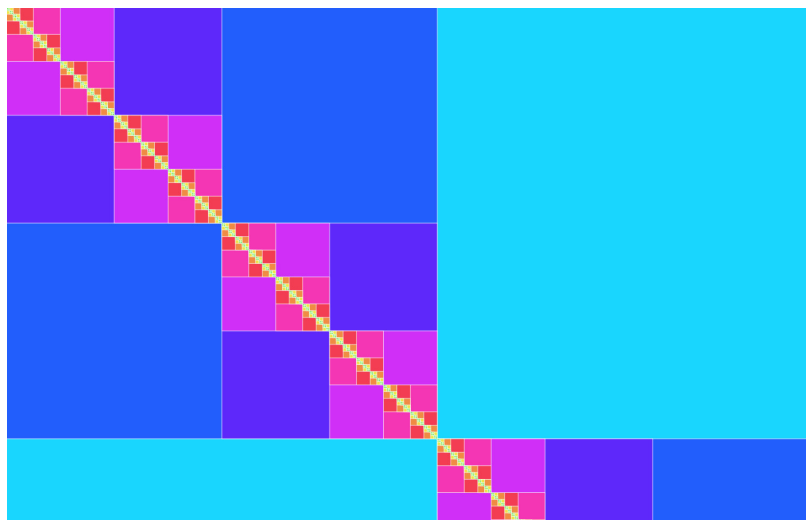
FIGURE 7. A set visualized in 2D with 10 iterations constructed the same way as the Cantor Set. However the $\frac{1}{3}$ in the Cantor Set definition is replaced with $\frac{1}{2}$ instead.
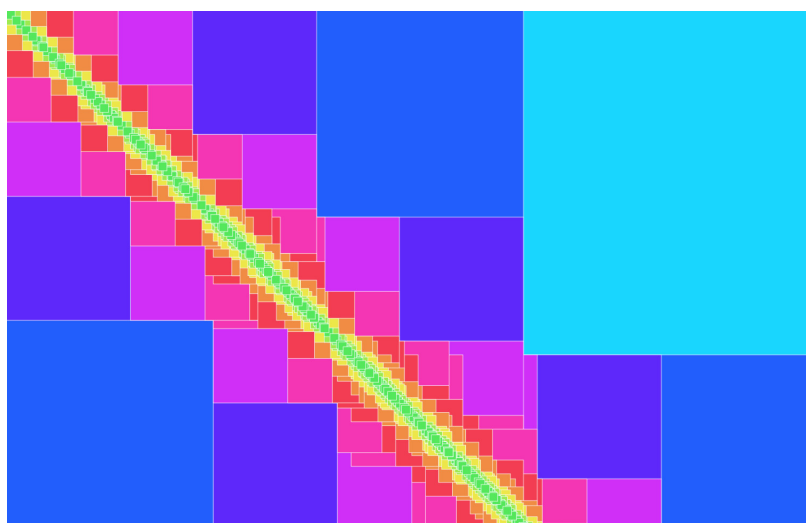

FIGURE 8. A set visualized in 2D with 10 iterations constructed the same way as the Cantor Set. However the $\frac{1}{3}$ in the Cantor Set definition is replaced with $\frac{3}{4}$ instead.
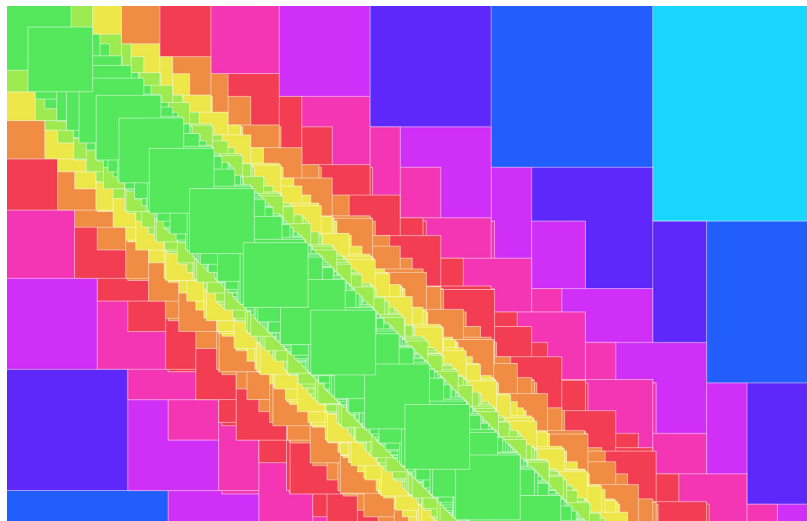
FIGURE 9. A set visualized in 2D with 10 iterations constructed the same way as the Cantor Set. However the $\frac{1}{3}$ in the Cantor Set definition is replaced with $\frac{4}{5}$ instead.

7. DRAGON

References

[Edg08] Gerald Edgar, *Measure, topology, and fractal geometry*, 2nd ed., Undergraduate Texts Math., New York, NY: Springer, 2008 (English).

[Fal13] Kenneth Falconer, *Fractals. A very short introduction*, Very Short Introd., vol. 367, Oxford: Oxford University Press, 2013 (English).

[Fal85] K. J. Falconer, *The geometry of fractal sets*, Camb. Tracts Math., vol. 85, Cambridge University Press, Cambridge, 1985 (English).