EXPLORING FRACTAL DIMENSION
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Maxwell Seay and Aaron Huntley
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[Man82] [Fal85]

2. Notion of Length, Area, Volume . . .

3. Notion of Dimension

I would like to know how to determine the dimension of arbitrary objects. First I will look at simple examples to find out more about a notion of dimension.

Example 3.1 (Line segment)**.** I know what a line segment is. I also know every line segment is 1 dimensional.

I take an approximation. I will cover a line segment with boxes. Each box has diameter (or radius) r. (To me it doesn't matter much since the difference between radius and diameter is always just a factor of 2.)

Take the line segment to be the interval $[0, 1]$. If $r = 1$, then I need only 1 box to cover the line segment.

If $r = 1/2$, I will need at least 2 boxes.

If $r = 1/3$, I will need at least 3 boxes.

If $r = 1/n$, I will need at least n boxes to cover the line segment.

I will define a function that takes in a radius, r, and outputs the least number of boxes of radius r needed to cover the line segment. From the above examples it can be seen that this function is:

$$\mathcal{N}(r) = \frac{1}{r}$$

Another idea to think on. How to double the *length* of the line segment? To double the length of the line segment, copy and paste it once, and append the copy onto the end of the original.

So in other words, to double the length of the line segment, add 1, (or $2^1 - 1$ you could say,) extra copy.

Example 3.2 (Square)**.** I know what a square is. I also know every square is 2 dimensional.

I take an approximation. I will cover a square with boxes. Each box has diameter (or radius) r.

Take the square to be the unit square, $[0, 1] \times [0, 1]$. If $r = 1$, then I need only 1 box to cover the square. (The box and square are the same.)

If $r = 1/2$, I will need at least 4 boxes.

If $r = 1/3$, I will need at least 9 boxes.

If $r = 1/n$, I will need at least n^2 boxes to cover the unit square.

I will define a function that takes in a radius, r, and outputs the least number of boxes of radius r needed to cover the square. From the above examples it can be seen that this function is:

$$\mathcal{N}(r) = \frac{1}{r^2}$$

Now how to double the *area* of the square? To double the area of the square, copy and paste it three times, and append the copies to three sides of the original.

So in other words, to double the area of the square, add 3, (or $2^2 - 1$ you could say,) extra copies.

So for the line segment we have that the number of squares of radius r needed to cover it as

$$\mathcal{N}(r) = \frac{1}{r}$$

And for the square we have that the number of squares of radius r needed to cover it as

$$\mathcal{N}(r) = \frac{1}{r^2}$$

From these couple examples it seems that, in general

$$\mathcal{N}(r) = c\left(\frac{1}{r}\right)^d$$

It seems that it would be appropriate to call the d in this expression the dimension. This function $\mathcal{N}(r)$ is called a power law since $1/r$ is always raised to some power. [Fal13]

4. Cantor Set

Let $\mathcal{C}$ be the Cantor Set.

Definition 4.1 (Cantor Set)**.** $\mathcal{C}$ can be described as an intersection of an infinite sequence of sets. The sequence of sets, $\{\mathcal{C}_i\}_{i=0}^{\infty}$, is defined recursively. For example, the first three are:

$$C_0 = [0,1]$$

$$C_1 = [0, 1/3] \cup [2/3, 1]$$

$$C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$$

$$C_3 = [0, 1/27] \cup [2/27, 3/27] \cup [6/27, 7/27] \cup [8/27, 9/27] \cup [18/27, 19/27] \cup [20/27, 21/27] \cup [24/27, 25/27] \cup [26/27, 1]$$

And the recursive definition for a set $\mathcal{C}_i$ is:

$$\mathcal{C}_i = \frac{1}{3}\mathcal{C}_{i-1} \cup \frac{2}{3} + \frac{1}{3}\mathcal{C}_{i-1}$$

Or an alternate (non-recursive) definition for a set $\mathcal{C}_i$ is:

$$\mathcal{C}_i = \bigcup_{j=0}^{i-1} \frac{2j}{3^{i-1}} + \left[j\frac{1}{3^i}, (j+1)\frac{1}{3^i}\right] \cup \left[(j+2)\frac{1}{3^i}, (j+3)\frac{1}{3^i}\right]$$

(idk I just came up with this. I think it works.)

Corollary 4.2. *$\mathcal{C}_i \neq \mathcal{C}$ for all $i \in \mathbb{N}$. No set in the sequence $\{\mathcal{C}_i\}$ is the Cantor Set, however the further you go (the larger the i), the "closer" $\mathcal{C}_i$ becomes $\mathcal{C}$.*

Corollary 4.3. *If $x \notin [0,1]$, then $x \notin \mathcal{C}$.*

Corollary 4.4. *The Cantor Set is non-empty.*

Proof. Consider the point 0.

$0 \in [0,1]$ so $0 \in \mathcal{C}_0$.

Now make the assumption that $0 \in \mathcal{C}_k$.

And since $\frac{1}{3} \cdot 0 = 0$ it must be true that

$$0 \in \frac{1}{3} \cdot \mathcal{C}_k$$

Note that by definition we have that

$$\mathcal{C}_{k+1} = \frac{1}{3}\mathcal{C}_k \cup \frac{2}{3} + \frac{1}{3}\mathcal{C}_k$$

Well since $0 \in \frac{1}{3} \cdot \mathcal{C}_k$ it must be true that

$$0 \in \frac{1}{3}\mathcal{C}_k \cup \frac{2}{3} + \frac{1}{3}\mathcal{C}_k$$

or

$$0 \in \mathcal{C}_{k+1}$$

So by induction we have that

$$0 \in \mathcal{C}$$

Thus $\mathcal{C}$ is non-empty. $\square$

Corollary 4.5. *The Cantor Set is a proper subset of* $[0, 1]$.

Corollary 4.6. $\mathcal{C}_i$ *is always a covering of* $\mathcal{C}_{i+1}$ *and also always a covering of* $\mathcal{C}$.

Theorem 4.7 (Theorem Example)**.**

References

[Fal13] Kenneth Falconer, *Fractals. A very short introduction*, Very Short Introd., vol. 367, Oxford: Oxford University Press, 2013 (English).

[Fal85] K. J. Falconer, *The geometry of fractal sets*, Camb. Tracts Math., vol. 85, Cambridge University Press, Cambridge, 1985 (English).

[Man82] Benoit B. Mandelbrot, *The fractal geometry of nature. Rev. ed. of "Fractals", 1977*, 1982 (English).