1 General problem

We assume that the mean horizontal velocity components $\bar{u}$ and $\bar{v}$ and buoyancy $b = g\overline{\delta\rho}/\rho$ are independent of the horizontal position, as well as the $\overline{p}$. By the geostrophic balance, the latter condition implies that the mean horizontal velocity vanishes in the lower part of the stratified region. We use the Boussinesq approximation, so that $\nabla.\overline{\mathbf{u}} = 0$. Since $\bar{u}$ and $\bar{v}$ do not depend on x, y, this implies $\partial\overline{w}/\partial z = 0$ so that $\overline{w} = 0$ everywhere since it is equal to zero at the free surface at $z = 0$. With these assumptions, the advective terms desappear, so the dynamics is governed by the turbulent vertical transport of momentum and buoyancy.

$$\begin{array}{c} \frac{\partial u}{\partial t} + fv = \frac{\partial}{\partial z}(\nu_t \frac{\partial u}{\partial z}) \\ \frac{\partial v}{\partial t} - fu = \frac{\partial}{\partial z}(\nu_t \frac{\partial v}{\partial z}) \end{array} \tag{1}$$

By vertical integration of (7) (8), we get equations for the integrals $U \equiv \int_{-H}^{0} udz$ and $V = \int_{-H}^{0} vdz$.

$$\begin{array}{c} \frac{dU}{dt} + fV = u_*^2 \\ \frac{dV}{dt} - fU = 0 \end{array} \tag{2}$$

Considering that the vertical flux vanish at large depth $z = -H$. The solutions describes inertial oscillations starting from $U = V = 0$ at $t = 0$,

$$\begin{array}{c} U = \frac{u_*^2}{f}\sin(ft) \\ V = \frac{u_*^2}{f}(1 - \cos(ft)) \end{array} \tag{3}$$

We shall assume that the fluid is well mixed in a layer of depth h(t). Assuming this layer evolves slowly, the flow inside this layer depends only on the parameters u_*, f, h. The flow structure should depend on the Rossby number $Ro = u_*/(fh)$, so the velocity should take the non-dimensonal form

$$\begin{array}{c} u/u_* = F_{(Ro,Z)} \\ v/u_* = G_{(Ro,Z)} \end{array} \tag{4}$$

where $Z = -z/h$.

2 Near surface behaviour

Eq. 1 is linear in u and v for a given profile of ν_t. We can tansform it into an equation for the complex variable $w = u + iv$,

$$\frac{\partial w}{\partial t} - ifw = \frac{\partial}{\partial z}(\nu_t \frac{\partial w}{\partial z}) \tag{5}$$

Near the surface $z = 0$, we have $\nu_t = -\kappa u_* z(1 + Az)$, going to next order in z , and $\partial w/\partial z$ diverges, such that $\nu_t \partial w/\partial z = u_*^2$ which integrates as

$$w = -\frac{u_*}{\kappa}\ln(-z) + cte \tag{6}$$

To avoid such singularity, it is useful to consider the new variable $\psi(t,z)$ such that $w = -\frac{u_*}{\kappa}\psi(t,z)\ln(-z)$. Then $\partial w/\partial z = -\frac{u_*}{\kappa}[(\partial\psi/\partial z)\ln(-z) + \psi/z]$ and 5 becomes

$$[\frac{\partial\psi}{\partial t} - if\psi]\ln(-z) = -\kappa u_*[(\frac{\partial}{\partial z}(z\frac{\partial\psi}{\partial z}))\ln(-z) + 2\frac{\partial\psi}{\partial z}] \quad (7)$$

For $z \to 0$, the terms in $ln(-z)$ dominate, ans by equating its coefficients we get

$$\frac{\partial\psi}{\partial t} - if\psi = -\kappa u_*\frac{\partial}{\partial z}(z\frac{\partial\psi}{\partial z} + Az^2\frac{\partial\psi}{\partial z}) \quad (8)$$

with the boundary condition $\psi(t,z) = 1$ at $z = 0$. In the limit $z \to 0$,we can solve this as a Taylor expansion in z, in the forme $\psi(z) = 1 + az + bz^2$. This yields at the first order in z,

$$-if + (da/dt - ifa)z + (db/dt - ifb)z^2 = -\kappa u_*(a + 4bz + 2Aaz) \quad (9)$$

this yields $a = \frac{if}{\kappa u_*}$ and $b = -(1/4)\frac{f^2}{(\kappa u_*)^2} - Aa/2$. By taking the real and imaginary parts, we get the behaviour near the surface, taking the limit $z \to 0$.

$$\begin{array}{l} u = -\frac{u_*}{\kappa}\ln(-z) + C(t) \\ v = -\frac{fu_*}{\kappa^2}z\ln(-z) + B(t) \end{array} \quad (10)$$

3 Extrapolation to the whole layer

We can attempt to extrapolate it as an approximation in the whole layer. Here z_0 is a roughness height characterising the surface roughness which we shall assume constant, while $C_{(t)}$ corresponds to the possibility of free oscillations. The choice of z_0 influences the velocity very close to the surface, but we expect that the behaviour of the mixed layer depends only on the imposed shear stress. Assuming that the velocity follows this law 10 until $z = -h$,and drops to 0 at this point, we can express (considering the reduced variable $Z = -z/h$ and taking the primitive$\int lnZdZ = Z(lnZ - 1)$)

$$U = \int_{-h}^{0} \overline{u}dz = h\ [C_{(t)} - \frac{u_*}{\kappa}(\ln h - 1)]$$

Combining with 3, we get the expression of $C_{(t)} = \frac{u_*^2}{hf}\sin(ft) + \frac{u_*}{\kappa}(\ln h - 1)$, so that 10 yields

$$u_{(z,t)}/u_* = F_{(Z,t)} = Ro\sin(ft)\ -\ \frac{1}{\kappa}(\ln Z + 1) \quad (11)$$

For v, we can write the condition 10 in the form $zd(v/z)/dz = -fu_*/\kappa^2$ so the non-dimensional form 4 must satisfy $Zd(G/Z)/dZ = -(h/u_*)zd(v/z)/dz = hf/(u_*\kappa^2)$. By integration this yields

$$G_{(Z,t)} = \frac{1}{Ro\kappa^2}Z\ln(Z) + B(t)Z \quad (12)$$

Applying the integral condition, we get

$$V = \int_{-H}^{0} \overline{v} dz = u_* h \int_{0}^{1} G dZ = \frac{u_* h}{2} \left(B_{(t)} - \frac{1}{2Ro\kappa^2}\right) \tag{13}$$

Combining with 3, we get the expression of $B_{(t)} = 2Ro(1 - \cos(ft)) + (2Ro\kappa^2)^{-1}$, so that 12 yields

$$\frac{v_{(z,t)}}{u_*} = G_{(Z,t)} = [2Ro\,(1 - \cos(ft)) + \frac{1}{2Ro\kappa^2}]\, Z + \frac{1}{Ro\kappa^2} Z \ln Z \tag{14}$$

4 Lower boundary layer

At the contact with the thermocline, we have entrainment of the lower fluid, corresponding to a downward momentum flux equal to $(dh/dt)\mathbf{u}$ where $\mathbf{u}$ is given by the previous expressions of F and G at $Z = 1$. Introducing a friction velocity $u_{*e}^2 = (dh/dt)|\mathbf{u}|$, we get a log profile with prefactor $u_{*e}/\kappa = \kappa^{-1}(dh/dt)^{1/2}|\mathbf{u}|^{1/2}$. Since it is directed along the unit vector $\mathbf{u}/|\mathbf{u}|$, we can write the shear rate in terms of $F(1,t)$ and $G(1,t)$ as

$$\begin{aligned} \frac{\partial u}{\partial z} &= \kappa^{-1}(dh/dt)^{1/2} u_*^{1/2} ({F_{(1,t)}}^2 + {G_{(1,t)}}^2)^{-1/4} \tfrac{1}{z+h}\, F_{(1,t)} \\ \frac{\partial v}{\partial z} &= \kappa^{-1}(dh/dt)^{1/2} u_*^{1/2} ({F_{(1,t)}}^2 + {G_{(1,t)}}^2)^{-1/4} \tfrac{1}{z+h}\, G_{(1,t)} \end{aligned} \tag{15}$$

so that

$$|\partial \mathbf{u}/\partial z|^2 = \kappa^{-2}(dh/dt) u_* ({F_{(1,t)}}^2 + {G_{(1,t)}}^2)^{1/2} \frac{1}{(z+h)^2} \tag{16}$$

Using the results 11 and 14 at $Z = 1$,we have

$$\begin{aligned} F_{(1,t)} &= Ro \sin(ft) \; - \; \tfrac{1}{\kappa} \\ G_{(1,t)} &= 2Ro\,(1 - \cos(ft)) + \tfrac{1}{2Ro\kappa^2} \end{aligned} \tag{17}$$

The rate of energy dissipation associated with 15 is equal to $\epsilon_e = \kappa u_{*e}(z + h)\,|\partial \mathbf{u}/\partial z|^2 = u_{*e}^3/(\kappa(z+h))$. This yields

$$\epsilon_e = (dh/dt)^{3/2} ({F_{(1,t)}}^2 + {G_{(1,t)}}^2)^{3/2} \frac{u_*^{3/2}}{\kappa(z+h)} \tag{18}$$

If we just add the turbulent dissipation from each of the boundary layers, we get in total

$$\epsilon = \frac{u_*^3}{-\kappa z} + (dh/dt)^{3/2} ({F_{(1,t)}}^2 + {G_{(1,t)}}^2)^{3/2} \frac{u_*^{3/2}}{\kappa(z+h)} \tag{19}$$

The dissipation rate is partly converted to buoyancy flux, with a proportion Γ equal to the mixing efficiency. This efficiency cannot exceed a value of the order of $\Gamma_0 \simeq 0.25$. At the interface the buoyancy flux must evacuate a buoyancy amount $N^2 h dh/2$ in a time dt, so that the buoyancy flux at the interface is

$B_e = (1/2)N^2 hdh/dt$. Note that the buoyancy in the mixed layer is equal to $N^2h/2$, and its rate of change is equal to the divergence $\partial B/\partial z = (1/2)N^2 dh/dt$. At the interface $z+h = h_e$, we therefore have the relation $(1/2)N^2 hdh/dt = \Gamma_0\epsilon$. This gives dh/dt provided we know the typical width h_e. This can be estimated by the condition of marginal stability

$$Ri = \frac{g(\partial\rho/\partial z)}{|\partial\mathbf{u}/\partial z|^2} \simeq \frac{N^2 h}{2h_e\, |\partial\mathbf{u}/\partial z|^2} = Ri_c \tag{20}$$

From 16 , this yields

$$h_e = \frac{2Ri_c(dh/dt)u_*}{\kappa^2 N^2 h}({F_{(1,t)}}^2 + {G_{(1,t)}}^2)^{1/2} \tag{21}$$

Introducing $z = -h + h_e \simeq -h$ in 19, the condition on mixing efficiency thus yields

$$(1/2)\Gamma_0^{-1} N^2 hdh/dt = \frac{u_*^3}{\kappa h} + h(dh/dt)^{1/2}\frac{\kappa N^2}{2Ri_c}u_*^{1/2}({F_{(1,t)}}^2 + {G_{(1,t)}}^2) \tag{22}$$

or

$$(\frac{dh}{u_* dt})^{1/2}[\Gamma_0^{-1}(\frac{dh}{u_* dt})^{1/2} - \frac{\kappa}{Ri_c}({F_{(1,t)}}^2 + {G_{(1,t)}}^2)] = \frac{2u_*^2}{\kappa N^2 h^2} \tag{23}$$

Numerically, taking $\Gamma_0 = 0.25$ and $Ri_c = 0.25$, $u_* = 10^{-2}$m/s, $N = 10^{-2}s^{-2}$, h=25 m, this gives

$$(\frac{dh}{u_* dt})^{1/2}[4(\frac{dh}{u_* dt})^{1/2} - 0.16({F_{(1,t)}}^2 + {G_{(1,t)}}^2)] = 0.008 \tag{24}$$

$Ro = u_*/(fh) = 4$, so

$$\begin{aligned} F_{(1,t)} &= 4\sin(ft)\ -\ 2.5 \\ G_{(1,t)} &= 8\,(1-\cos(ft)) + 0.78 \end{aligned} \tag{25}$$

$< F^2 >= 14.25, < G^2 >= 109$, $T_f dh/dt/L_{P73} \simeq 0.25$ so that $dh/dt = 2.5f/(4\pi) = 0.210^{-4}$ m/s, so that $(dh/u_* dt)$ =0.002. The result would be good with only the first term, but the negative term is far too high.

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