Theorem:

$\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^-} f(x) = L$ and $\lim\limits_{x \to a^+} f(x) = L$.

Limit Laws

Let f and g be real-valued functions and let $c \in \mathbb{R}$ be a constant. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.
2. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$.
3. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$.
4. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$.
5. $\lim_{x \to a} c = c$.
6. $\lim_{x \to a} x = a$.

Definition of continuity

Let f be a real-valued function.

The function f is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Continuity Theorem 1:

The following function types are continuous at every point in their domains:

Polynomial, Trig+h, exp, logs, roots, mag

Let f and g be real-valued functions and $c \in \mathbb{R}$ be a constant.

Continuity Theorem 2:

If the functions f and g are continuous at $x = a$, then the following functions are continuous at $x = a$:

1. $f + g$,
2. cf,
3. fg,
4. $\dfrac{f}{g}$ if $g(a) \neq 0$.

Let f and g be real-valued functions and $c \in \mathbb{R}$ be a constant.

Continuity Theorem 2:

If the functions f and g are continuous at $x = a$, then the following functions are continuous at $x = a$:

1. $f + g$,
2. cf,
3. fg,
4. $\dfrac{f}{g}$ if $g(a) \neq 0$.

Recall that $(g \circ f)(x) = g(f(x))$.

Continuity Theorem 3:

If f is continuous at $x = a$ and g is continuous at $x = f(a)$, then $g \circ f$ is continuous at $x = a$.

Continuity rule for limits

Theorem:

Let f and g be real-valued functions and $b \in \mathbb{R}$.

If $\lim_{x \to a} g(x) = b$ and f is continuous at b then

$$\lim_{x \to a} f\big(g(x)\big) = f\left(\lim_{x \to a} g(x)\right) = f(b).$$

Note:

This theorem also holds for limits as $x \to \infty$.

L'Hôpital's Rule

L'Hôpital's rule is a technique for evaluating limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ when f and g are differentiable.

Theorem:

Let f, g be real-valued functions. If

- $\lim_{x \to a} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and
- f and g are differentiable near $x = a$, and
- $g'(x) \neq 0$ at all points x near a with $x \neq a$, and
- $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists,

then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.}$$

L'Hôpital's Rule

Note:

- L'Hôpital's Rule can only be used to show that a limit exists. It cannot be used to show that a limit does not exist.
- Remember to check that the limit is of the form 0/0 or ∞/∞ before using L'Hôpital's Rule.
- L'Hôpital's Rule also holds for limits as $x \to \infty$, and for one-sided limits $x \to a^+$ and $x \to a^-$ for $a \in \mathbb{R}$.

Theorem (Limit Laws):

Let (a_n) and (b_n) be sequences of real numbers and $c \in \mathbb{R}$ a constant.

If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist, then

1. $\lim\limits_{n\to\infty} [a_n + b_n] = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n.$
2. $\lim\limits_{n\to\infty} [ca_n] = c \lim\limits_{n\to\infty} a_n.$
3. $\lim\limits_{n\to\infty} [a_n b_n] = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n.$
4. $\lim\limits_{n\to\infty} \left[\frac{a_n}{b_n}\right] = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ provided $\lim\limits_{n\to\infty} b_n \neq 0.$
5. $\lim\limits_{n\to\infty} c = c.$

Standard Limits

(1) $\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0)$

(2) $\lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$

(3) $\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0)$

(4) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

(5) $\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a \in \mathbb{R})$

(6) $\lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$

(7) $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \mathrm{e}^a \quad (a \in \mathbb{R})$

(8) $\lim_{n\to\infty} \frac{n^p}{a^n} = 0 \quad (p \in \mathbb{R}, a > 1)$

(9) $\lim_{n\to\infty} \arctan(cn) = \frac{\pi}{2} \quad (c > 0)$

Note:

Standard limits (1), (3), (4), (6), (7), (8), (9) also hold for limits of real-valued functions as $x \to \infty$ by replacing n with x.
Standard limit (2) also holds for $x \to \infty$ when $0 \le r < 1$.

Note:

The order hierarchy can be used to help identify the fastest growing term in an expression as $n \to \infty$:

$$\log n \ll n^p \ll a^n \ll n!$$

where $p > 0$ and $a > 1$.

Sandwich Theorem for sequences

Let (a_n), (b_n) and (c_n) be sequences of real numbers.

If $a_n \leq c_n \leq b_n$ for all $n > N$ for some N, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L.$$

Continuity theorem for sequences

Let $f(x)$ be a real-valued function, (a_n) a sequence of real numbers and $b \in \mathbb{R}$.

Theorem:

If $\lim_{n\to\infty} a_n = b$ and f is continuous at $x = b$ then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(b).$$

The only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is a natural number whereas x is a real number.

Theorem:

Let $f(x)$ be a real-valued function and (a_n) be a sequence of real numbers such that $a_n = f(n)$.

$$\text{If} \quad \lim_{x\to\infty} f(x) = L \quad \text{then} \quad \lim_{n\to\infty} a_n = L.$$

This means that we can use the techniques for evaluating limits of functions to evaluate limits of sequences.

Note:

$$\lim_{n\to\infty} a_n = L \quad \not\Rightarrow \quad \lim_{x\to\infty} f(x) = L.$$

eg. $a_n = \sin(2\pi n)$, $f(x) = \sin(2\pi x)$.

Geometric Series

A geometric series is a series of the form

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$$

where $a \in \mathbb{R}\backslash\{0\}$ and $r \in \mathbb{R}$.

The series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

If $|r| < 1$, we have

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Note:

This follows from the fact that $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$ for $r \neq 1$.

Harmonic p Series

A harmonic p series is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

The series converges if $p > 1$ and diverges if $p \leq 1$.

Example

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges BUT $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Properties of Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series, and $c \in \mathbb{R}\backslash\{0\}$ a constant.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then

1. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ converges.
2. $\sum_{n=1}^{\infty}(ca_n) = c \sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty}(ca_n)$ diverges.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Note:

$\lim_{n\to\infty} a_n \neq 0$ includes the case that the limit $\lim_{n\to\infty} a_n$ does not exist.

If $\lim_{n\to\infty} a_n = 0$ then

1. $\sum_{n=1}^{\infty} a_n$ may converge or diverge.
2. The Divergence Test is not applicable, so we need to use another test to determine if $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with non-negative terms (i.e. $a_n \geq 0$).

1. If $\sum_{n=1}^{\infty} b_n$ is another series such that $\sum_{n=1}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ also diverges.
2. If $\sum_{n=1}^{\infty} c_n$ is another series such that $\sum_{n=1}^{\infty} c_n$ converges and $c_n \geq a_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ also converges.

To apply the comparison test we compare a given series to a harmonic p series or geometric series.

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms (i.e. $a_n > 0$ for all n).
Let

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

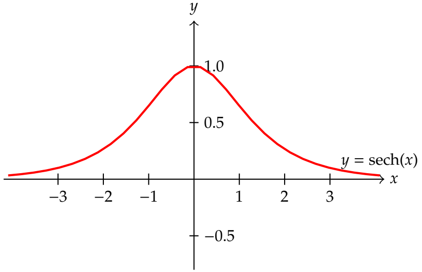
1. If $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the ratio test is inconclusive.

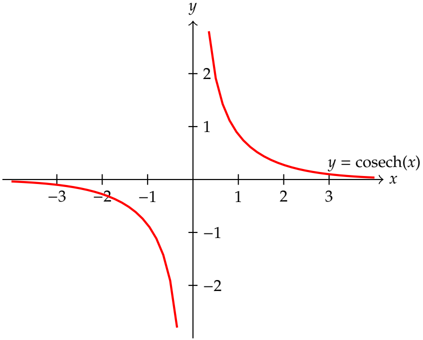
The ratio test is useful if a_n contains an exponential or factorial function of n.

Reciprocal Hyperbolic Functions

We define the three reciprocal hyperbolic functions:

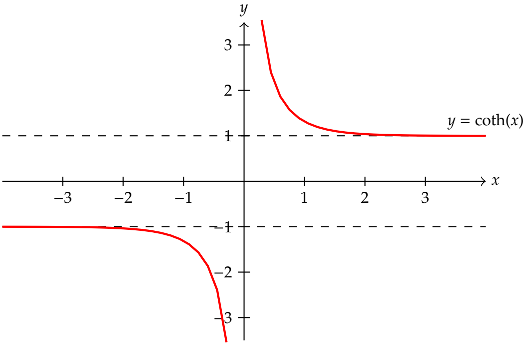
$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad x \in \mathbb{R} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}, x \in \mathbb{R} \setminus \{0\}$$





Reciprocal Hyperbolic Functions

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} \; ,$$
$$x \in \mathbb{R} \setminus \{0\}$$



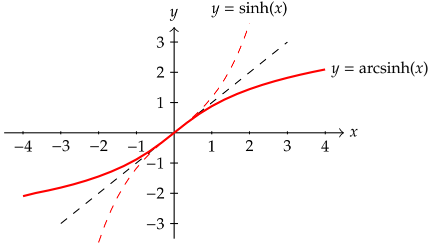
Inverses of Hyperbolic Functions

We define three inverse hyperbolic functions.

1. Inverse hyperbolic sine function: $\operatorname{arcsinh} x$

Since $\sinh x$ is a 1-1 function

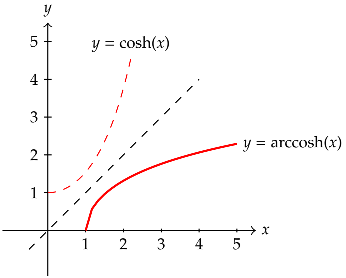
$$\begin{aligned}
\text{domain } \operatorname{arcsinh} x &= \text{range } \sinh x = \mathbb{R}. \\
\text{range } \operatorname{arcsinh} x &= \text{domain } \sinh x = \mathbb{R}. \\
\operatorname{arcsinh}(\sinh x) &= x, \quad x \in \mathbb{R}. \\
\sinh(\operatorname{arcsinh} x) &= x, \quad x \in \mathbb{R}.
\end{aligned}$$



2. Inverse hyperbolic cosine function: $\operatorname{arccosh} x$

Restrict domain of $\cosh x$ to be $[0, \infty)$ to give a 1-1 function. Then

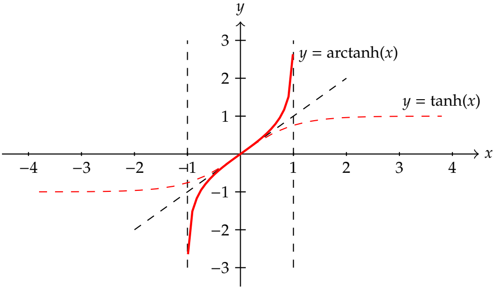
$$\begin{aligned} \text{domain } \operatorname{arccosh} x &= \text{range } \cosh x = [1, \infty). \\ \text{range } \operatorname{arccosh} x &= \text{restricted domain } \cosh x = [0, \infty). \\ \cosh(\operatorname{arccosh} x) &= x, \quad x \geq 1. \\ \operatorname{arccosh}(\cosh x) &= x, \quad x \geq 0. \end{aligned}$$



3. Inverse hyperbolic tangent function: $\operatorname{arctanh} x$

Since $\tanh x$ is a 1-1 function

$$\begin{aligned}
\text{domain } \operatorname{arctanh} x &= \text{range } \tanh x = (-1,1). \\
\text{range } \operatorname{arctanh} x &= \text{domain } \tanh x = \mathbb{R}. \\
\tanh(\operatorname{arctanh} x) &= x, \quad -1 < x < 1. \\
\operatorname{arctanh}(\tanh x) &= x, \quad x \in \mathbb{R}.
\end{aligned}$$



The inverse hyperbolic functions can be expressed in terms of natural logarithms.

$$\boxed{\begin{aligned}
\operatorname{arcsinh} x &= \log\left(x + \sqrt{x^2+1}\right), && x \in \mathbb{R} \\
\operatorname{arccosh} x &= \log\left(x + \sqrt{x^2-1}\right), && x \geq 1 \\
\operatorname{arctanh} x &= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), && -1 < x < 1
\end{aligned}}$$

We can also define inverse reciprocal hyperbolic functions:

- $\operatorname{arcsech} x \qquad (0 < x \leq 1)$
- $\operatorname{arccosech} x \qquad (x \neq 0)$
- $\operatorname{arccoth} x \qquad (x < -1$ or $x > 1)$

Differentiation via the Complex Exponential

If $z = x + yi$ where $x, y \in \mathbb{R}$ then we define

$$e^z = e^{x+iy} = e^x \, e^{iy} = e^x(\cos y + i \sin y).$$

Derivatives of functions from $\mathbb{R}$ to $\mathbb{C}$ are defined similarly as those from $\mathbb{R}$ to $\mathbb{R}$.

Differentiation to functions from $\mathbb{R}$ to $\mathbb{C}$ is also linear and follows the product law.

Show that $\dfrac{d}{dt}\left(e^{kt}\right) = ke^{kt}$ when $k = a + bi \in \mathbb{C}$.

$$\frac{d}{dt}\left[e^{(a+bi)t}\right] = \frac{d}{dt}\left[e^{at}e^{ibt}\right]$$

$$
\begin{aligned}
&= \frac{d}{dt}\left[e^{at}\left(\cos(bt) + i\sin(bt)\right)\right] \\
&= ae^{at}\left[\cos(bt) + i\sin(bt)\right] + e^{at}\left[-b\sin(bt) + bi\cos(bt)\right] \\
&= ae^{at}\left[\cos(bt) + i\sin(bt)\right] + e^{at}\left[bi^2\sin(bt) + bi\cos(bt)\right] \\
&= ae^{at}\left[\cos(bt) + i\sin(bt)\right] + bie^{at}\left[\cos(bt) + i\sin(bt)\right] \\
&= (a + bi)e^{at}\left[\cos(bt) + i\sin(bt)\right] \\
&= (a + bi)e^{at}e^{ibt} \\
&= (a + bi)e^{(a+ib)t}.
\end{aligned}
$$

Section 4: Integral Calculus

Review of integration

Integration by Parts

The product rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Integrate

$$\int \frac{d}{dx}(uv)\,dx = \int \left(\frac{du}{dx}v + u\frac{dv}{dx}\right) dx$$

$$\Rightarrow uv = \int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx$$

$$\Rightarrow \quad \boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

Trigonometric and Hyperbolic Substitutions

We can use trigonometric and hyperbolic substitutions to integrate expressions containing

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2},$$

where a is a positive real number.

Method:

Put $x = g(\theta)$. Then

$$\int f(x)\,dx = \int f[g(\theta)]g'(\theta)\,d\theta$$

Integrand	Substitution
$\sqrt{a^2 - x^2}, \quad \dfrac{1}{\sqrt{a^2 - x^2}}, \quad \left(a^2 - x^2\right)^{\frac{3}{2}}$ etc.	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}, \quad \dfrac{1}{\sqrt{a^2 + x^2}}, \quad \left(a^2 + x^2\right)^{-\frac{3}{2}}$ etc.	$x = a\sinh\theta$
$\sqrt{x^2 - a^2}, \quad \dfrac{1}{\sqrt{x^2 - a^2}}, \quad \left(x^2 - a^2\right)^{\frac{5}{2}}$ etc.	$x = a\cosh\theta$
$\dfrac{1}{a^2 + x^2}, \quad \dfrac{1}{(a^2 + x^2)^2}$ etc.	$x = a\tan\theta$

Denominator Factor	Partial Fraction Expansion
$(x-a)$	$\dfrac{A}{x-a}$
$(x-a)^r$	$\dfrac{A_1}{x-a}+\dfrac{A_2}{(x-a)^2}+\cdots+\dfrac{A_r}{(x-a)^r}$
(x^2+bx+c)	$\dfrac{Ax+B}{x^2+bx+c}$
$(x^2+bx+c)^r$	$\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{(x^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(x^2+bx+c)^r}$

A linear first order ODE has the form:

$$\frac{dy}{dx} + \mathcal{P}(x)y = \mathcal{Q}(x)$$

To solve:

Multiply ODE by $\mathcal{I}(x)$

$$\mathcal{I}(x)\frac{dy}{dx} + \mathcal{P}(x)\mathcal{I}(x)y = \mathcal{Q}(x)\mathcal{I}(x)$$

If the left side can be written as the derivative of $y(x)\mathcal{I}(x)$, then

$$\frac{d}{dx}\left[y(x)\mathcal{I}(x)\right] = \mathcal{Q}(x)\mathcal{I}(x)$$

which can be solved by integrating with respect to x.

So one integrating factor is

$$\boxed{\mathcal{I}(x) = e^{\int \mathcal{P} dx}}$$

Note:

Since we only need one integrating factor $\mathcal{I}$, we can neglect the '$+c$' and modulus signs when calculating $\mathcal{I}$.

Solving ODEs by substitution

Sometimes it is possible to make a substitution to reduce a general first order ODE to a separable or linear ODE.

- A homogeneous type ODE has the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Substituting $u = \dfrac{y}{x}$ reduces the ODE to a separable ODE.

- Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Substituting $u = y^{1-n}$ reduces the ODE to a linear ODE.

Equilibrium Solutions

Definition

An equilibrium solution is a constant solution of an ODE.

Note:

For the ODE $\dfrac{dx}{dt} = f(x,t)$, this means

- $x(t) = C$ where C is a constant
- $\dfrac{dx}{dt} = 0$

Terminology

We often simply say equilibrium instead of *equilibrium solution*.
The plural form of equilibrium is equilibria.

Phase plots

A phase plot is a plot of $\frac{dx}{dt}$ as a function of x.

A phase plot will give

- the equilibria
- the behaviour of solutions close to the equilibria

Note:

Phase plots are only useful for ODEs of the form

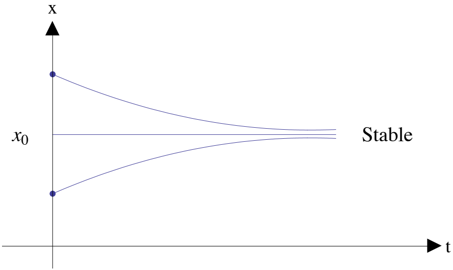
$$\frac{dx}{dt} = f(x)$$

i.e., when the right-hand side has no explicit dependence on t.

ODEs of this form are called autonomous.

Stability of equilibria

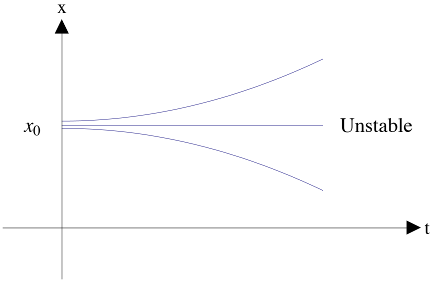
An equilibrium is stable if solutions that start nearby move closer to the equilibrium as t increases.



On a phase plot:

Stability of equilibria

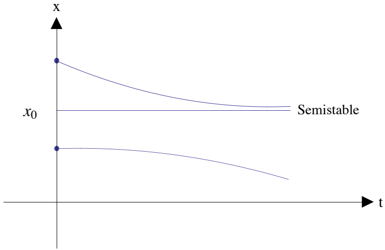
An equilibrium is unstable if solutions that start nearby move further away as t increases.

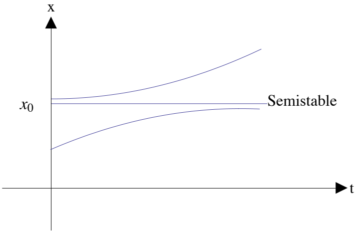


On a phase plot:

Stability of equilibria

An equilibrium is semistable if on one side of the equilibrium, solutions that start nearby move closer as t increases, whereas on the other side, solutions move further away as t increases.





On a phase plot:

Population Models

Malthus (Doomsday) model

Rate of growth is proportional to the population p at time t.

$$\begin{aligned} \frac{dp}{dt} &\propto p \\ \Rightarrow \frac{dp}{dt} &= kp \qquad \text{(separable/linear)} \end{aligned}$$

where k is a constant of proportionality representing net births per unit population per unit time.

If the initial population is $p(0) = p_0$, then the solution is

$$p(t) = p_0 e^{kt}$$

You should check that you can derive this on your own!

Note:

The Doomsday model predicts:

- $k > 0$: unbounded exponential growth
- $k < 0$: population dies out
- $k = 0$: population stays constant

Unbounded exponential growth is unrealistic in the long term.

Doomsday model with harvesting.

Remove some of the population at a constant rate.

$$\frac{dp}{dt} = kp - h, \quad h > 0.$$

Logistic model.

Include “competition” term in Malthus’ model since overcrowding, disease, lack of food and natural resources will cause more deaths.

$$\frac{dp}{dt} = \underbrace{kp}_{\text{net birth rate}} - \underbrace{\frac{k}{a}p^2}_{\text{competition term}} = kp\left(1 - \frac{p}{a}\right)$$

where $a > 0$ is the carrying capacity.

Logistic model with harvesting.

Remove some of the population at constant rate:

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{a}\right) - h, \quad h > 0, \; a > 0$$

Definitions

1. Transient terms: terms decaying to 0 as $t \to \infty$.
2. Steady state terms: terms NOT decaying to 0 as $t \to \infty$.

The solution for the concentration can be classified as follows.

Section 6: Second Order Differential Equations

A second order ODE has the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

The general form of a linear second order ODE is

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

- If $\mathcal{R}(x) = 0$, the ODE is homogeneous (H).
- If $\mathcal{R}(x) \neq 0$, the ODE is inhomogeneous (IH).

Note:

A homogeneous linear ODE is different to a homogeneous type first order ODE.

The general solution of a second order ODE typically has two arbitrary constants.

Initial value problem for a second order ODE

Solve

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

subject to the conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$.

Boundary value problem for a second order ODE

Solve

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

subject to the conditions $y(a) = y_0$ and $y(b) = y_1$.

Definition:

Two functions y_1 and y_2 are linearly independent if

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \Rightarrow \quad c_1 = c_2 = 0$$

or equivalently, if neither function is a non-zero constant multiple of the other function.

Example 6.1:

(a) Are $y_1(x) = x^2,\ y_2(x) = 2x^2$ linearly independent?

(b) Are $y_1(x) = e^{2x},\ y_2(x) = xe^{2x}$ linearly independent?

Case 1: $b^2 - 4ac > 0$

- 2 distinct real values λ_1, λ_2
- 2 linearly independent solutions

$$e^{\lambda_1 x}, \quad e^{\lambda_2 x}$$

- General Solution:

$$\boxed{y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}}$$

Case 2: $b^2 - 4ac = 0$

- 1 real value $\lambda = \dfrac{-b}{2a}$

- 1 solution is $e^{\lambda x}$

- 2^{nd} linearly independent solution is $xe^{\lambda x}$ (found using variation of parameters — not in syllabus).

- General Solution:

$$\boxed{y(x) = Ae^{\lambda x} + Bxe^{\lambda x}}$$

Case 3: $b^2 - 4ac < 0$

- 2 complex conjugate values

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$

- 2 linearly independent complex solutions

$$e^{(\alpha+i\beta)x}, \quad e^{(\alpha-i\beta)x}$$

- General solution over the complex numbers:

$$y(x) = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} \qquad \text{where } C_1, C_2 \in \mathbb{C}$$

To find the general solution over the real numbers, consider

$$y_c = e^{(\alpha+i\beta)x} = e^{\alpha x}\Big(\cos(\beta x) + i\sin(\beta x)\Big).$$

Because y_c is a solution of $ay'' + by' + cy = 0$ over the complex numbers, we have

$$ay_c'' + by_c' + cy_c = 0 + 0i$$

Take the real part of this equation:

$$\mathrm{Re}\,(ay_c'' + by_c' + cy_c) = 0$$

$$a\mathrm{Re}\Big(y_c''\Big) + b\mathrm{Re}\Big(y_c'\Big) + c\mathrm{Re}\Big(y_c\Big) = 0$$

$$a\Big(\mathrm{Re}(y_c)\Big)'' + b\Big(\mathrm{Re}(y_c)\Big)' + c\Big(\mathrm{Re}(y_c)\Big) = 0$$

So $Re(y_c) = e^{\alpha x}\cos(\beta x)$ is a real solution of the ODE.

Similarly, $\mathrm{Im}(y_c) = e^{\alpha x}\sin(\beta x)$ is a real solution of the ODE.

$y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$ are two linearly independent real solutions of the ODE.

Therefore the general solution over the real numbers is

$$y = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x) \qquad \text{where } A, B \in \mathbb{R}.$$

Inhomogeneous 2^{nd} Order Linear ODEs

Theorem:

The general solution of

$$y'' + \mathcal{P}(x)y' + \mathcal{Q}(x)y = \mathcal{R}(x)$$

is the function y given by

$$y(x) = y_{\mathcal{H}}(x) + y_{\mathcal{P}}(x)$$

where

• $y_{\mathcal{H}}(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution of the homogeneous ODE (called the homogeneous solution, GS(H)),

• $y_{\mathcal{P}}(x)$ is a solution of the inhomogeneous ODE (called a particular solution, PS(IH)),

Superposition of Particular Solutions

Theorem:

A particular solution of

$$ay'' + by' + cy = \alpha\,\mathcal{R}_1(x) + \beta\,\mathcal{R}_2(x)$$

is

$$y_{\mathcal{P}}(x) = \alpha y_1(x) + \beta y_2(x)$$

where

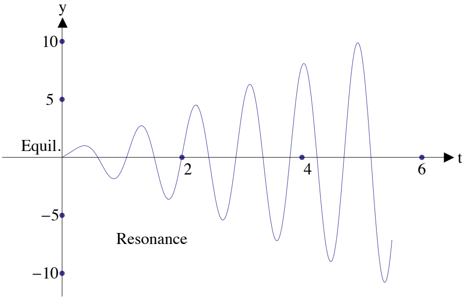
- $y_1(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_1(x)$,
- $y_2(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_2(x)$,
- a, b, c, α, β are constants.

To solve, try $y(t) = e^{\lambda t}$

$$\Rightarrow \quad m\lambda^2 + \beta\lambda + k = 0$$

$$\Rightarrow \quad \lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

- If $\beta = 0$: $\lambda = \pm ib$ simple harmonic motion
- If $0 < \beta < 2\sqrt{mk}$: $\lambda = a \pm ib$ underdamped, weak damping
- If $\beta = 2\sqrt{mk}$: $\lambda = a, a$ critical damping
- If $\beta > 2\sqrt{mk}$: $\lambda = a, b$ overdamped, strong damping



Definition

Resonance: Resonance occurs when the external force f has the same form as one of the terms in the $GS(H)$.

If $\beta = 0$, then the $PS(IH)$ will grow without bound as $t \to \infty$.

Limits

Let $f(x, y)$ be a function of two variables.

The limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) is L, written

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if $f(x, y)$ gets arbitrarily close to L whenever (x, y) is close enough to (x_0, y_0) but $(x, y) \neq (x_0, y_0)$.

Note:

1. If it exists, L must be a unique finite real number.
2. The limit can exist even if f is undefined at (x_0, y_0).
3. The usual limit laws apply.

Continuity

Let $f(x, y)$ be a function of two variables.

f is continuous at $(x, y) = (x_0, y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

Note:

The continuity theorems for functions of one variable can be generalised to functions of two variables.

First Order Partial Derivatives

Let $f(x,y)$ be a function of two variables. The first order partial derivatives of f with respect to the variables x and y are defined by the limits:

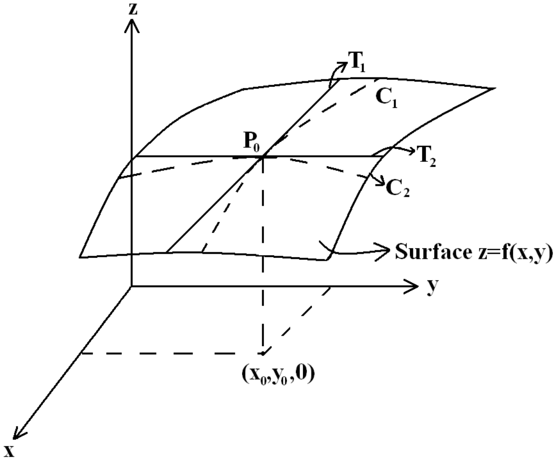
$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Note:

- $\dfrac{\partial f}{\partial x}$ measures the rate of change of f with respect to x when y is held constant.
- $\dfrac{\partial f}{\partial y}$ measures the rate of change of f with respect to y when x is held constant.

Geometric Interpretation of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$



The tangent line T_1 has equation ($y = y_0$ fixed):

$$z - z_0 = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0)$$

The tangent line T_2 has equation ($x = x_0$ fixed):

$$z - z_0 = \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0)$$

Since a plane passing through (x_0, y_0, z_0) has the form

$$z - z_0 = \alpha(x - x_0) + \beta(y - y_0)$$

the tangent plane has equation

$$z - z_0 = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0)$$

or equivalently,

$$\boxed{z = z_0 + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0).}$$

Linear Approximations

If f is differentiable at (x_0, y_0), we can approximate $z = f(x, y)$ by its tangent plane at (x_0, y_0, z_0), when (x, y) is close to (x_0, y_0).

That is:

$$\boxed{f(x,y) \approx \underbrace{z_0 + \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)}(y - y_0)}_{\text{equation of tangent plane}}}$$

when (x, y) is close to (x_0, y_0).

This is called the linear approximation to f near (x_0, y_0).

Approximate Change

Rearranging the linear approximation equation, we get

$$f(x,y) - f(x_0,y_0) \approx \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0).$$

Let $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta f = z - z_0 = f(x,y) - f(x_0,y_0)$.

The approximate change in f near (x_0,y_0), for small changes Δx and Δy in x and y, is:

$$\boxed{\Delta f \approx \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\Delta y}$$

Second Order Partial Derivatives

Let $f(,y)$ be a function of two variables. The second order partial derivatives of f with respect to x and y are defined by:

- $f_{xx} = \left(f_x\right)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$
- $f_{yy} = \left(f_y\right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$
- $f_{xy} = \left(f_x\right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$
- $f_{yx} = \left(f_y\right)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$

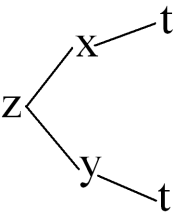
Theorem:

If the second order partial derivatives of f exist and are continuous then $f_{xy} = f_{yx}$.

Chain Rule

1. If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$ are differentiable functions, then $z = f(g(t), h(t))$ is a function of t, and

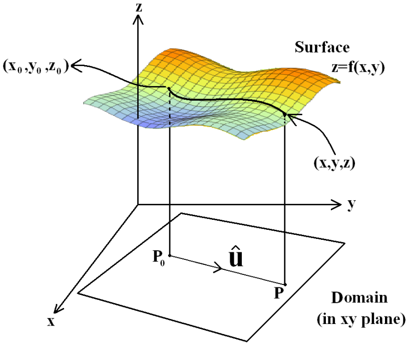
$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$



Directional Derivatives

Let $\hat{\mathbf{u}} = (u_1, u_2)$ be a unit vector in the xy-plane (so $u_1^2 + u_2^2 = 1$). The rate of change of f at $P_0 = (x_0, y_0)$ in the direction $\hat{\mathbf{u}}$ is the directional derivative $D_{\hat{\mathbf{u}}} f \Big|_{P_0}$.

Geometrically this represents the slope of the surface $z = f(x, y)$ above the point P_0 in the direction $\hat{\mathbf{u}}$.



The straight line starting at $P_0 = (x_0, y_0)$ with velocity $\hat{\mathbf{u}} = (u_1, u_2)$ has parametric equations:

$$x = x_0 + tu_1, \quad y = y_0 + tu_2.$$

Hence,

$$\begin{aligned} D_{\hat{\mathbf{u}}} f \Big|_{P_0} &= \text{rate of change of } f \text{ along the straight line at } t = 0 \\ &= \text{value of } \frac{d}{dt} f(x_0 + tu_1, y_0 + tu_2) \text{ at } t = 0 \\ &= f_x(x_0, y_0) x'(0) + f_y(x_0, y_0) y'(0) \qquad \text{by the chain rule} \\ &= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2. \end{aligned}$$

We can also write this as a dot product

$$D_{\hat{\mathbf{u}}} f \Big|_{P_0} = \left(\frac{\partial f}{\partial x}\bigg|_{P_0}, \frac{\partial f}{\partial y}\bigg|_{P_0} \right) \cdot (u_1, u_2).$$

Gradient Vectors

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, we can define the gradient of f to be the vector

$$\boxed{\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

Then the directional derivative of f at the point P_0 in the direction $\hat{\mathbf{u}}$ is the dot product

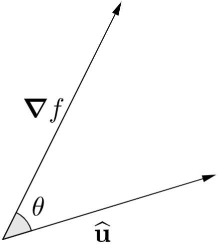
$$\boxed{D_{\hat{\mathbf{u}}} f\,\Big|_{P_0} = \nabla f\Big|_{P_0} \cdot \hat{\mathbf{u}}}$$

Properties of ∇f

The directional derivative of f is

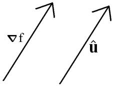
$$\begin{aligned} D_{\hat{\mathbf{u}}} f &= \nabla f \cdot \hat{\mathbf{u}} \\ &= \|\nabla f\| \, \|\hat{\mathbf{u}}\| \cos\theta \\ &= \|\nabla f\| \cos\theta \end{aligned}$$

where θ is the angle between ∇f and $\hat{\mathbf{u}}$.



So for fixed ∇f:

- $D_{\hat{\mathbf{u}}} f$ is maximum when $\cos\theta = 1$ so $\theta = 0$.

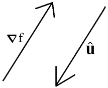


$\Rightarrow \nabla f$ points in the direction in which f increases the fastest

In this direction, $D_{\hat{\mathbf{u}}} f = \|\nabla f\|$

$\Rightarrow \|\nabla f\|$ is the fastest rate of increase of f.

- $D_{\hat{\mathbf{u}}}f$ is minimum when $\cos\theta = -1$ so $\theta = \pi$



$\Rightarrow -\nabla f$ points in the direction in which f decreases the fastest

In this direction, $D_{\hat{\mathbf{u}}}f = -||\nabla f||$

- $D_{\hat{\mathbf{u}}}f = 0$ when $\cos\theta = 0$ so $\theta = \dfrac{\pi}{2}$ and $\nabla f \perp \hat{\mathbf{u}}$.

But $D_{\hat{\mathbf{u}}}f = 0$, whenever $\hat{\mathbf{u}}$ is tangent to a level curve of f (where f = constant).

$$\Rightarrow \nabla f \perp \text{ level curves of } f$$

This gives a geometrical interpretation of ∇f:

• the *direction* of ∇f is the direction of steepest ascent of f.

• the *length* of ∇f, $\|\nabla f\|$, is the slope of the surface in the direction of steepest ascent.

• the direction of $-\nabla f$ is the direction of steepest descent of f.

• ∇f is perpendicular to the level curves of f.

Note:

The direction of steepest ascent is sometimes also called:

- the direction of fastest increase
- the direction of steepest increase

and similarly for the direction of steepest descent.

Stationary Points

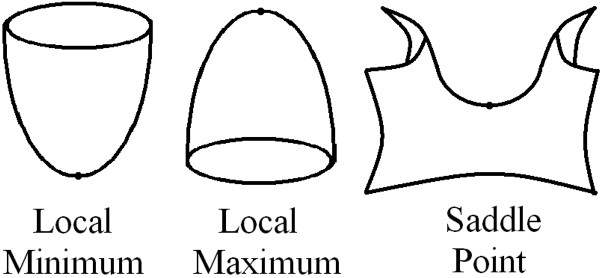
A stationary point of f is a point (x_0, y_0) at which

$$\nabla f = \mathbf{0}$$

So $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously at (x_0, y_0).

Geometrically, this means that the tangent plane to the graph $z = f(x, y)$ at (x_0, y_0) is horizontal, i.e. parallel to the xy-plane.

Three important types of stationary points are



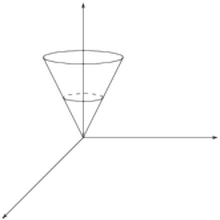
A function f has a

1. local maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in some disk centred at (x_0, y_0),
2. local minimum at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in some disk centred at (x_0, y_0),
3. saddle point at (x_0, y_0) if (x_0, y_0) is a stationary point, and there are points near (x_0, y_0) with $f(x, y) > f(x_0, y_0)$ and other points near (x_0, y_0) with $f(x, y) < f(x_0, y_0)$.

Any local maximum or minimum of f will occur at a critical point (x_0, y_0) such that

1. $\nabla f(x_0, y_0) = \mathbf{0}$ <u>or</u>

2. $\dfrac{\partial f}{\partial x}$ and/or $\dfrac{\partial f}{\partial y}$ do not exist at (x_0, y_0).



$z = \sqrt{x^2 + y^2}$. Minimum at $(0, 0)$ BUT ∇f does not exist at $(0, 0)$.

Second Derivative Test

If $\nabla f(x_0, y_0) = \mathbf{0}$ and the second partial derivatives of f are continuous on an open disk centred at (x_0, y_0), consider the Hessian function

$$H(x, y) = f_{xx}f_{yy} - (f_{xy})^2$$

evaluated at (x_0, y_0).

Then (x_0, y_0) is a

1. local minimum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$.
2. local maximum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$.
3. saddle point if $H(x_0, y_0) < 0$.

Note: Test is inconclusive if $H(x_0, y_0) = 0$.

Partial Integration

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

The partial indefinite integrals of f with respect to the first and second variables (say x and y) are denoted by:

$\int f(x,y)\,dx$ and $\int f(x,y)\,dy$.

- $\int f(x,y)\,dx$ is evaluated by holding y fixed and integrating with respect to x.

- $\int f(x,y)\,dy$ is evaluated by holding x fixed and integrating with respect to y.

Example 7.19: Evaluate $\int_0^1 (3x^2y + 12y^2x^3)\, dy$.

Solution:

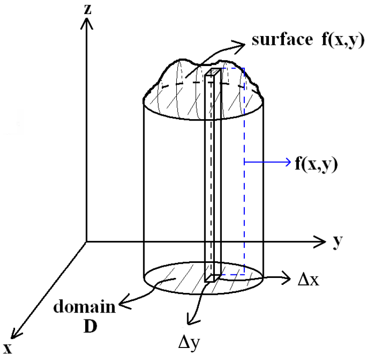
Double Integrals

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

We can evaluate the double integral:

$$\boxed{\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy}$$

$\iint_D f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$ that lies above the domain D in the xy plane, if $f(x,y) \geq 0$ in D.



$$\text{Volume of thin rod} \quad = \quad \underbrace{\text{(Area base)}}_{\substack{\parallel \\ \Delta x \Delta y}} \cdot \underbrace{\text{(height)}}_{\substack{\parallel \\ f(x,y)}}$$

The double integral is defined as the limit of sums of the volumes of the rods:

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy$$

$$= \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{i=1}^{n} [f(x,y)\Delta x \Delta y]_i$$

Note:

If $f(x,y) = 1$ then

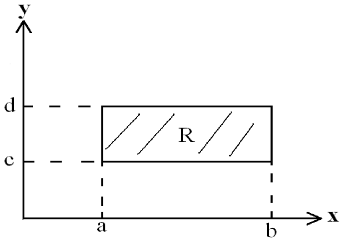
$$\iint_D dA = \iint_D dx\,dy$$

gives the area of the domain D.

Double Integrals Over Rectangular Domains

Definitions

1. $R = [a,b] \times [c,d]$ is a rectangular domain defined by
   $a \le x \le b, \quad c \le y \le d$.



2. $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$ means integrate with respect to x first and then integrate with respect to y.

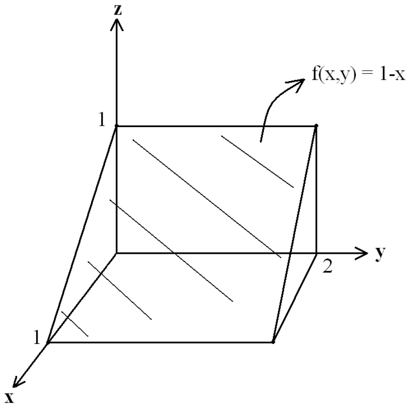
Fubini's Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over the domain $R = [a,b] \times [c,d]$. Then

$$\begin{aligned} \iint_R f(x,y)dA &= \int_c^d \int_a^b f(x,y)\,dx\,dy \\ &= \int_a^b \int_c^d f(x,y)\,dy\,dx \end{aligned}$$

So order of integration is not important.

Example 7.20: Using double integrals, find the volume of the wedge shown below.



Solution:

This can also be calculated as

$$\int_0^1 \int_0^2 \left(1 - x\right) dy\, dx$$

This gives the same answer, as expected by Fubini's Theorem. (Working omitted.)