Test of Hypothesis (2)

❖ Test of Significance

Test of significance is a statistical procedure to arrive at a conclusion or decision on the basis of samples and to test whether the formulated hypothesis can be accepted or rejected in probability sense. The aim of test of significance is to reject the null hypothesis.

❖ Steps in Solving Testing of Hypothesis Problem:

The major steps involved in the solution of a "testing of hypothesis" problem may be outlined as follows:

- Explicit knowledge of the nature of the population distribution and the parameter(s) of interest, i.e., the parameter(s) about which the hypothesis are set up.
- Setting up of the null hypothesis H_0 and the alternative hypothesis H_1 in terms of the range of the parameter values each one embodies.
- Choose the appropriate level of significance (α) depending on the reliability of the estimates and permissible risk.
- Choose the suitable test statistic and compute the test statistic under the null hypothesis.
- We compare the computed value of test statistic with the significant value (tabulated value) at the given level of significance α.
    - If the computed value of test statistic (in modulus value) is greater than the significant value (tabulated value) then we reject the null hypothesis at level of significance α and we say that it is significant.
    - If the computed value of test statistic is less than the tabulated value then we accept the null hypothesis at level of significance α and we say that it is not significant.

❖ The p-Value Approach to Hypothesis Testing:

The $p-value$ is the probability of obtaining a test statistic equal to or more extreme than the result obtained from the sample data, given that the null hypothesis H_0 is really true.

The $p-value$ is often referred to as the observed level of significance, which is the smallest level at which H_0 can be rejected for a given set of data. The decision rule for rejecting H_0 in the $p-value$ approach is:

- If the $p-value$ greater than or equal to α, the null hypothesis is not rejected.
- If the $p-value$ is smaller than α, the null hypothesis is rejected.

P-Value Approach

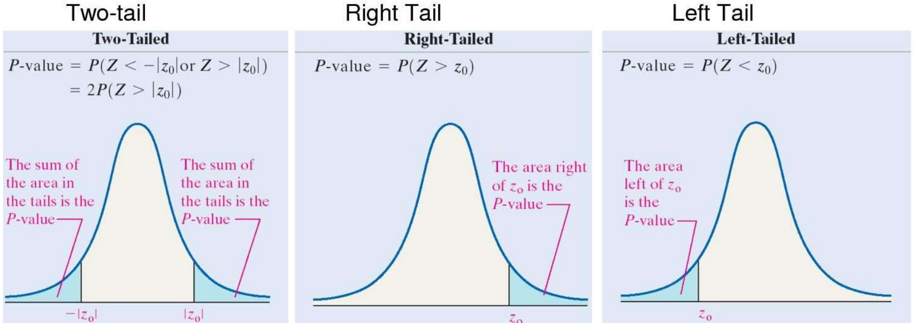
Assume that the null hypothesis is true.

The P-Value is the probability of observing a sample mean that is as or more extreme than the observed.

How to compute the P-Value for each type of test:

Step 1: Compute the test statistic $z_0 = \dfrac{\overline{x} - \mu_0}{\sigma / \sqrt{n}}$

Two-tail Right Tail Left Tail



❖ **Steps in Determining the** $p-value$

- State the null hypothesis H_0
- State the alternative hypothesis H_1.
- Choose the level of significance α.
- Choose the sample size n.
- Determine the appropriate statistical technique and corresponding test statistic to use.
- Collect the data and compute the sample value of the appropriate test statistic.
- Calculate the $p-value$ based on the test statistic. This involves
  - → Sketching the distribution under the null hypothesis H_0.
  - → Placing the test statistic on the horizontal axis.
  - → Shading in the appropriate area under the curve, on the basis of the alternative hypothesis H_1.
- Compare the $p-value$ to α.

- Make the statistical decision. If the $p-value$ is greater than or equal to α, the null hypothesis is not rejected. If the $p-value$ is smaller than α, the null hypothesis is rejected.
- Express the statistical decision in terms of the particular situation.

❖ Sample Size Determination When α *and* β are Fixed

Suppose we want to test $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1$. Then there are two sampling distribution from two normal population $N\left(\mu_0, \sigma_0^2\right)$ and $N\left(\mu_1, \sigma_1^2\right)$.

When H_0 is true the upper limit of the acceptance region (the point x) is determined by the relation

$$\bar{x} = \mu_0 + z_0 \frac{\sigma_0}{\sqrt{n}}$$

Under H_1, corresponding to an area equal to the specified value of β is the acceptance region under H_0. Considering H_1, we again determine the upper limit of the acceptance region by

$$\bar{x} = \mu_1 - z_1 \frac{\sigma_1}{\sqrt{n}}$$

Equating these two values we get,

$$\mu_0 + z_0 \frac{\sigma_0}{\sqrt{n}} = \mu_1 - z_1 \frac{\sigma_1}{\sqrt{n}}$$

$$\Rightarrow \qquad n = \frac{\sigma^2 \left(z_0 + z_1\right)^2}{\left(\mu_1 - \mu_0\right)^2} \qquad \text{when } \sigma_0 = \sigma_1 = \sigma$$

❖ Best Critical Region

Let C denote a subset of the sample space. Then C is called a best critical region of size α for testing the simple hypothesis $H_0 : \theta = \theta'$ against the alternative simple hypothesis $H_1 : \theta = \theta_n$ if for every subset A of the sample space for which

$$\mathrm{P}\{(x_1, x_2, \dots \dots \dots, x_n) \in A;\ H_0\} = \alpha$$

a) $\mathrm{P}\{(x_1, x_2, \dots \dots \dots, x_n) \in C;\ H_0\} = \alpha$

b) $\mathrm{P}\{(x_1, x_2, \dots \dots \dots, x_n) \in C;\ H_1\} \geq \mathrm{P}\{(x_1, x_2, \dots \dots \dots, x_n) \in A;\ H_1\}$

The best critical region is abbreviated as BCR.

❖ Most Powerful Test (MP test)

The test based on a BCR is called most powerful (MP) test. Let us consider the problem of testing a simple hypothesis $H_0: \theta = \theta_0$ against a simple alternative hypothesis $H_1: \theta = \theta_1$.

The critical region ω is the most powerful (MP) critical region of size α and the corresponding test is a most powerful test of level α for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$ if

$$P(x \in \omega | H_0) = \int_{\omega} L_0 dx = \alpha \dots \dots \dots (1)$$

$$P(x \in \omega | H_1) \geq P(x \in \omega_1 | H_1) \dots \dots \dots (2)$$

for every other critical region ω_1 satisfying (1).

❖ Uniformly Most Powerful Test (UMP test)

The region ω is called uniformly most powerful (UMP) critical region of size α and the corresponding test as uniformly most powerful test of level α for testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ $i.e. H_1: \theta = \theta_1 \neq \theta_0$ if

$$P(x \in \omega | H_0) = \int_{\omega} L_0 dx = \alpha \dots \dots \dots (1)$$

$$P(x \in \omega | H_1) \geq P(x \in \omega_1 | H_1) \dots \dots \dots (2)$$

for all $\theta \neq \theta_0$.

The region ω_1 satisfying (1).

❖ Unbiased Test and Unbiased Critical Region

Let us consider the testing of $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$. The critical region ω and consequently the test based on it is said to be unbiased if the power of the test exceeds the size of the critical region i.e. if

$$\text{Power of the test} \geq \text{Size of the critical region}$$

$$=> 1 - \beta \geq \alpha$$

$$P(x \in \omega | H_1) \geq P(x \in \omega | H_0)$$

- **State and Prove Nayman-Pearson Lemma. Or, How you determine BCR.**

Statement:

Let $x_1, x_2, \cdots\cdots, x_n$ be a random sample of size n from a distribution with pdf $f(x;\theta)$ where θ_0 and θ_1 are two possible values of θ. Denote the joint pdf of $x_1, x_2, \cdots\cdots, x_n$ by the likelihood function

$$L(\theta) = L(\theta;\ x_1, x_2, \cdots\cdots, x_n) = f(x_1, \theta).f(x_2, \theta) \ldots\ldots f(x_n, \theta)$$

If there exists a positive constant k and a subset w of the sample space such that

$(a) P[(x_1, x_2, \cdots\cdots, x_n) \in w|\theta_0] = \alpha$

$(b) \frac{L(\theta_0)}{L(\theta_1)} \le k\ for\ (x_1, x_2, \cdots\cdots, x_n) \in w$

$(c) \frac{L(\theta_0)}{L(\theta_1)} \ge k\ for\ (x_1, x_2, \cdots\cdots, x_n) \in \overline{w}$

Then w is a best critical region of size α for testing simple null hypothesis $H_0 : \theta = \theta_0$ against simple alternative hypothesis $H_1 : \theta = \theta_1$.

Proof:

Assume that there exists another critical region of size α, say w^* such that

$P(x \in w|H_0) = \int_w L(\theta_0)dx = \int_{w^*} L(\theta_0)dx = \alpha$

So, we have

$$\int_w L(\theta_0)dx - \int_{w^*} L(\theta_0)dx = 0$$

$$\Rightarrow \int_{w\cap\overline{w}^*} L(\theta_0)\,dx + \int_{w\cap w^*} L(\theta_0)dx - \int_{w\cap w^*} L(\theta_0)dx - \int_{\overline{w}\cap w^*} L(\theta_0)dx = 0$$

$$\Rightarrow \int_{w\cap\overline{w}^*} L(\theta_0)\,dx - \int_{\overline{w}\cap w^*} L(\theta_0)dx = 0 \cdots\cdots\cdots (i)$$

From (b) we have

$kL(\theta_1) \ge L(\theta_0)$ at each point in w and therefore for $w \cap \overline{w}^*$

$k\int_{w\cap\overline{w}^*} L(\theta_1)\,dx \ge \int_{w\cap\overline{w}^*} L(\theta_0)\,dx \cdots\cdots\cdots (ii)$

From (c) we have

$kL(\theta_1) \leq L(\theta_0)$ at each point in $\overline{w}$ and therefore for $\overline{w} \cap w^*$

$k \int_{\overline{w} \cap w^*} L(\theta_1)\, dx \leq \int_{\overline{w} \cap w^*} L(\theta_0)\, dx$

Multiplying both sides by -1 we get

$-k \int_{\overline{w} \cap w^*} L(\theta_1)\, dx \geq -\int_{\overline{w} \cap w^*} L(\theta_0)\, dx \cdots\cdots\cdots (iii)$

Adding equation (ii) and (iii) we get

$k \int_{w \cap \overline{w}^*} L(\theta_1)\, dx - k \int_{\overline{w} \cap w^*} L(\theta_1)\, dx \geq \int_{w \cap \overline{w}^*} L(\theta_0)\, dx - \int_{\overline{w} \cap w^*} L(\theta_0)\, dx$

$\Rightarrow k \int_{w \cap \overline{w}^*} L(\theta_1)\, dx - k \int_{\overline{w} \cap w^*} L(\theta_1)\, dx \geq 0$ (using equation (i))

$\Rightarrow k[\int_{w \cap \overline{w}^*} L(\theta_1)\, dx - \int_{\overline{w} \cap w^*} L(\theta_1)\, dx] \geq 0$

$\Rightarrow \int_{w \cap \overline{w}^*} L(\theta_1)\, dx - \int_{\overline{w} \cap w^*} L(\theta_1)\, dx \geq 0$

$\Rightarrow \int_{w \cap \overline{w}^*} L(\theta_1)\, dx \geq \int_{\overline{w} \cap w^*} L(\theta_1)\, dx$

$\Rightarrow \int_{w \cap \overline{w}^*} L(\theta_1)\, dx + \int_{w \cap w^*} L(\theta_1) dx \geq \int_{\overline{w} \cap w^*} L(\theta_1)\, dx + \int_{w \cap w^*} L(\theta_0) dx$ [Adding both sides by $\int_{w \cap w^*} L(\theta_1) dx$]

$\Rightarrow \int_{w} L(\theta_1)\, dx \geq \int_{w^*} L(\theta_1)\, dx$

$\therefore 1 - \beta_w \geq 1 - \beta_{w^*}$

Therefore w is the best critical region of size α for testing simple null hypothesis $H_0 : \theta = \theta_0$ against simple alternative hypothesis $H_1 : \theta = \theta_1$. (Proved)

- **Theorem:** Every most powerful (MP) or uniformly most powerful (UMP) critical region (CR) is necessarily unbiased.

(i) If W be an MPCR of size α for testing $H_0 \colon \theta = \theta_0$ against $H_1 \colon \theta = \theta_1$, then it is necessarily unbiased.

(ii) Similarly if W be an MPCR of size α for testing $H_0 \colon \theta = \theta_0$ against $H_1 \colon \theta \in \Theta_1$, then it is also necessarily unbiased.

Proof:

Since W be an MPCR of size α for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$, by Neyman-Pearson Lemma, we have for $\forall k > 0$

$$W = \{x: L(x, \theta_1) \geq kL(x, \theta_0)\} = \{x: L_1 \geq kL_0\}$$

$$\text{And } \overline{W} = \{x: L(x, \theta_1) < kL(x, \theta_0)\} = \{x: L_1 < kL_0\}$$

Where k is determined so that the size of the test is α. i.e.

$$P_{\theta_0}(W) = P(x \in W|H_0) = \int_W L_0 dx = \alpha \ldots \ldots \ldots (i)$$

To prove that W is unbiased, we have to show that

$$\text{Power of } W \geq \alpha \text{ i.e. } P_{\theta_1}(W) \geq \alpha \ldots \ldots \ldots (ii)$$

We have

$$P_{\theta_1}(W) = \int_W L_1 dx \geq k \int_W L_0 dx = k\alpha$$

$$\text{i.e. } P_{\theta_1}(W) \geq k\alpha; \ \forall k > 0 \ldots \ldots \ldots (iii)$$

Also

$$1 - P_{\theta_1}(W) = 1 - P(x \in W|H_1) = P(x \in \overline{W}|H_1)$$

$$= \int_{\overline{W}} L_1 dx < k \int_{\overline{W}} L_0 dx = kP(x \in \overline{W}|H_0) \qquad [Since\ on\ \overline{W},\ L_1 < kL_0]$$

$$= k[1 - P(x \in W|H_0)] = k(1 - \alpha)$$

$$\text{i.e. } 1 - P_{\theta_1}(W) < k(1 - \alpha); \ \forall k > 0 \ldots \ldots \ldots (iv)$$

Case (i) $k \geq 1$. If $k \geq 1$, then from (iii) we get

$$P_{\theta_1}(W) \geq k\alpha \geq \alpha$$

Therefore W is unbiased critical region.

Case (ii) $0 < k < 1$. If $0 < k < 1$, then from (iv) we get

$$1 - P_{\theta_1}(W) < (1 - \alpha)$$

$$\Rightarrow P_{\theta_1}(W) > \alpha$$

Therefore W is unbiased critical region.

Hence, MP critical region is unbiased.

(ii) If W is UMPCR of size α, then also the above proof holds if for θ_1 we write θ such that $\theta \in \Theta_1$. So we have

$$P_\theta(W) > \alpha \ ; \ \forall \theta \in \Theta_1$$

Therefore W is unbiased critical region.

❖ **Example-**1

Obtain best critical region of size α for testing simple hypothesis $H_0:\theta=\theta_0$ against $H_1:\theta=\theta_1$, where $x_1, x_2, \dots\dots\dots, x_n$ is a random sample of size n from $N(\theta,\sigma^2)$; σ^2 is known. Also find the power of the test.

Or, test the hypothesis $H_0:\theta=\theta_0$ against $H_1:\theta=\theta_1$ of size α in the distribution

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \qquad ; \qquad -\infty<x<\infty$$

where σ^2 is known. Hence find the power of the test.

Solution:

We have the likelihood functions are

$$L(x\,|\,H_0)=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta_0)^2}$$

$$\textit{and} \qquad L(x\,|\,H_1)=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta_1)^2}$$

$$\textit{Simply,} \quad L(x\,|\,H_i)=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{n}{2\sigma^2}\left\{s^2+(\bar{x}-\theta_i)^2\right\}} \qquad ; \qquad \textit{where,} \quad ns^2=\sum(x_i-\bar{x})^2$$

where s^2 *and* $\bar{x}$ are the sample variance and sample mean respectively.

According to Neyman-Pearson lemma, we have the BCR

$$\frac{L(x\,|\,H_0)}{L(x\,|\,H_1)}\le k$$

$$\Rightarrow \exp\left[-\frac{n}{2\sigma^2}\left\{(\bar{x}-\theta_0)^2+(\bar{x}-\theta_1)^2\right\}\right]\le k$$

$$\Rightarrow \exp\left[-\frac{n}{2\sigma^2}\left\{2\bar{x}(\theta_1-\theta_0)+\left(\theta_0^2-\theta_1^2\right)\right\}\right]\le k$$

$$\Rightarrow -\frac{n}{2\sigma^2}\left[2\bar{x}(\theta_1-\theta_0)+\left(\theta_0^2-\theta_1^2\right)\right]\le \ln k$$

$$\Rightarrow 2\bar{x}(\theta_1-\theta_0)+\left(\theta_0^2-\theta_1^2\right)\ge -\frac{2\sigma^2}{n}\ln k$$

$$\Rightarrow 2\bar{x}(\theta_1-\theta_0)\ge\left(\theta_1^2-\theta_0^2\right)-\frac{2\sigma^2}{n}\ln k$$

$$\Rightarrow \bar{x}(\theta_1-\theta_0)\ge\frac{\left(\theta_1^2-\theta_0^2\right)}{2}-\frac{\sigma^2}{n}\ln k \qquad \mathrm{L} \qquad \mathrm{L} \qquad \mathrm{L} \quad (1)$$

Case I:

If $\theta_1 > \theta_0$ then the BCR is determined by

$$\bar{x} \geq \frac{\theta_1 + \theta_0}{2} - \frac{\sigma^2}{n(\theta_1 - \theta_0)} lnk$$

$$\therefore\ \bar{x} \geq \lambda_1 (say) \cdots\cdots\cdots (2)$$

We know that

$$P\left[\bar{x} \geq \lambda_1 / H_0\right] = \alpha$$

$$\Rightarrow \int_{\lambda_1}^{\infty} f(\bar{x}) d\bar{x} = \alpha \qquad \left[under\ H_0 : \theta = \theta_0\right]$$

$$\Rightarrow \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \int_{\lambda_1}^{\infty} e^{-\frac{1}{2}\left(\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}}\right)^2} d\bar{x} = \alpha \qquad \left[\therefore \quad \bar{x} \sim N\left(\theta, \frac{\sigma^2}{n}\right)\right]$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{\frac{\lambda_1-\theta_0}{\sigma/\sqrt{n}}}^{\infty} e^{-\frac{1}{2}z^2} dz = \alpha \qquad \left[let \quad \frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} = z\right]$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{z_\alpha}^{\infty} e^{-\frac{1}{2}z^2} dz = \alpha \qquad \left[\sin ce \quad \frac{\lambda_1 - \theta_0}{\sigma/\sqrt{n}} = z_\alpha\right]$$

We have,

$$z_\alpha = \frac{(\lambda_1 - \theta_0)\sqrt{n}}{\sigma}$$

$$\Rightarrow \lambda_1 - \theta_0 = \frac{\sigma}{\sqrt{n}} z_\alpha$$

$$\Rightarrow \lambda_1 = \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$$

From equation (2), the BCR is $\bar{x} \geq \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$.

If $\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} \geq z_\alpha$ then we reject $H_0 : \theta = \theta_0$, otherwise we accept $H_0 : \theta = \theta_0$.

Case II:

If $\theta_1 < \theta_0$ then from equation (1) we have the BCR is

$$\bar{x}(\theta_0 - \theta_1) \ge \frac{\theta_1^2 - \theta_0^2}{2} - \frac{\sigma^2}{n} \ln k$$
$$\Rightarrow \bar{x} \ge -\frac{\theta_0 + \theta_1}{2} - \frac{\sigma^2}{n(\theta_0 - \theta_1)} \ln k$$
$$\Rightarrow \bar{x} \le \frac{\theta_0 + \theta_1}{2} + \frac{\sigma^2}{n(\theta_0 - \theta_1)} \ln k$$
$$\bar{x} \le \lambda_2 \ (say) \qquad \text{L} \qquad \text{L} \qquad \text{L} \qquad (3)$$

Again we have, $\quad P\left[\bar{x} \le \lambda_2 / H_0\right] = \alpha$

$$\Rightarrow \int_{-\infty}^{\lambda_2} f(\bar{x})\, d\bar{x} = \alpha \qquad [under\ H_0 : \theta = \theta_0]$$

$$\Rightarrow \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\lambda_2} e^{-\frac{1}{2}\left(\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}}\right)^2} d\bar{x} = \alpha \qquad \left[\because\ \bar{x} \sim N\left(\theta, \frac{\sigma^2}{n}\right)\right]$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\lambda_2-\theta_0}{\sigma/\sqrt{n}}} e^{-\frac{1}{2}z^2} dz = \alpha \qquad \left[let\ \ \frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} = z\right]$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{\frac{\lambda_2-\theta_0}{\sigma/\sqrt{n}}}^{\infty} e^{-\frac{1}{2}z^2} dz = 1-\alpha \qquad \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{z_{1-\alpha}}^{\infty} e^{-\frac{1}{2}z^2} dz = 1-\alpha \qquad \left[\sin ce\ \ \frac{\lambda_2-\theta_0}{\sigma/\sqrt{n}} = z_{1-\alpha}\right]$$

We have, $\quad z_{1-\alpha} = \frac{(\lambda_2 - \theta_0)\sqrt{n}}{\sigma} \qquad \Rightarrow \lambda_2 - \theta_0 = \frac{\sigma}{\sqrt{n}} z_{1-\alpha}$

$$\Rightarrow \lambda_2 = \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha}$$

By symmetry of normal distribution, we have,

$$z_{1-\alpha} = -z_\alpha \qquad \therefore \lambda_2 = \theta_0 - \frac{\sigma}{\sqrt{n}} z_\alpha$$

From equation (3), the BCR is $\bar{x} \le \theta_0 - \frac{\sigma}{\sqrt{n}} z_\alpha$. If $\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} \le -z_\alpha$ then we reject $H_0 : \theta = \theta_0$, otherwise we accept $H_0 : \theta = \theta_0$.

Power of the Test:

Case I:

If $\theta_1 > \theta_0$, by definition we have the power of the test is

$$power = P\left(\bar{x} \geq \lambda_1 / H_1\right) = \int_{\lambda_1}^{\infty} f(\bar{x})\, d\bar{x} \qquad [under\ H_1 : \theta = \theta_1]$$

$$= \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \int_{\lambda_1}^{\infty} e^{-\frac{1}{2}\left(\frac{\bar{x}-\theta_1}{\sigma/\sqrt{n}}\right)^2} d\bar{x} \qquad \left[\therefore \quad \bar{x} \sim N\left(\theta, \frac{\sigma^2}{n}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\lambda_1-\theta_1}{\sigma/\sqrt{n}}}^{\infty} e^{-\frac{1}{2}z^2} dz \qquad \left[let \quad \frac{\bar{x}-\theta_1}{\sigma/\sqrt{n}} = z\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\theta_0+\frac{\sigma}{\sqrt{n}}z_\alpha-\theta_1}{\sigma/\sqrt{n}}}^{\infty} e^{-\frac{1}{2}z^2} dz \qquad = \frac{1}{\sqrt{2\pi}} \int_{z_\alpha-\frac{\theta_1-\theta_0}{\sigma/\sqrt{n}}}^{\infty} e^{-\frac{1}{2}z^2} dz$$

Case II:

If $\theta_1 < \theta_0$, by definition we have the power of the test is

$$power = P\left(\bar{x} < \lambda_2 / H_1\right)$$

$$= \int_{-\infty}^{\lambda_2} f(\bar{x})\, d\bar{x} \qquad [under\ H_1 : \theta = \theta_1]$$

$$= \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\lambda_2} e^{-\frac{1}{2}\left(\frac{\bar{x}-\theta_1}{\sigma/\sqrt{n}}\right)^2} d\bar{x} \qquad \left[\therefore \quad \bar{x} \sim N\left(\theta, \frac{\sigma^2}{n}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\lambda_2-\theta_1}{\sigma/\sqrt{n}}} e^{-\frac{1}{2}z^2} dz \qquad \left[let \quad \frac{\bar{x}-\theta_1}{\sigma/\sqrt{n}} = z\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\theta_0-\frac{\sigma}{\sqrt{n}}z_\alpha-\theta_1}{\sigma/\sqrt{n}}} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z_\alpha-\frac{\theta_1-\theta_0}{\sigma/\sqrt{n}}} e^{-\frac{1}{2}z^2} dz$$

- **Example-**2

Test the hypothesis $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma = \sigma_1$ for size α in $N\left(0, \sigma^2\right)$. Hence find the power of the test.

Solution:

We have the likelihood function are

$$L\left(x \mid H_0\right) = \left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum\left(\frac{x_i}{\sigma_0}\right)^2}$$

$$and \quad L\left(x \mid H_1\right) = \left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum\left(\frac{x_i}{\sigma_1}\right)^2}$$

According to Neyman-Pearson lemma the BCR is given by

$$\frac{L\left(x \mid H_0\right)}{L\left(x \mid H_1\right)} \le k$$

$$\Rightarrow \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left[\frac{\sum x_i^2}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\right] \le k$$

$$\Rightarrow n \ln\frac{\sigma_1}{\sigma_0} + \frac{\sum x_i^2}{2}\left[\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right] \le \ln k$$

$$\Rightarrow \frac{\sum x_i^2}{2}\left[\frac{\sigma_0^2 - \sigma_1^2}{\sigma_0^2\sigma_1^2}\right] \le \ln k - n\ln\frac{\sigma_1}{\sigma_0}$$

$$\Rightarrow \sum x_i^2\left(\sigma_0^2 - \sigma_1^2\right) \le 2\sigma_0^2\sigma_1^2\left[\ln k - n\ln\frac{\sigma_1}{\sigma_0}\right] \qquad \text{L} \qquad \text{L} \qquad \text{L} \quad (1)$$

Case I:

If $\sigma_0 > \sigma_1$ then the BCR is

$$\sum x_i^2 \le \frac{2\sigma_0^2\sigma_1^2}{\left(\sigma_0^2 - \sigma_1^2\right)}\left[\ln k + n\ln\frac{\sigma_0}{\sigma_1}\right]$$

$$\Rightarrow \sum x_i^2 \le \lambda_1 \qquad \text{L} \qquad \text{L} \qquad \text{L} \quad (2)$$

We know that $\sum x_i^2$ is a $\chi^2\sigma_0^2$ with n degrees of freedom, we have,

$$P\left[\sum x_i^2 \le \lambda_1 / H_0\right] = \alpha$$

$$\Rightarrow \frac{1}{\left(2\sigma_0^2\right)^{n/2}\Gamma\frac{n}{2}}\int_0^{\lambda_1} e^{-\frac{\chi^2}{2\sigma_0^2}}\left(\chi^2\right)^{\frac{n}{2}-1} d\chi^2 = \alpha$$

$$\Rightarrow \frac{1}{2^{n/2}\Gamma\frac{n}{2}}\int_0^{\lambda_1/\sigma_0^2} e^{-\frac{y}{2}} y^{\frac{n}{2}-1} dy = \alpha \qquad \left[By\ letting\ \frac{\chi^2}{2\sigma_0^2} = y\right]$$

Using the table of values of incomplete gamma function the value of $\frac{\lambda_1}{\sigma_0^2} = k_1 \ (say)$ can be determined, from equation (2) we have the BCR is $\sum x_i^2 \le k_1\sigma_0^2$

If $\sum x_i^2 \le k_1\sigma_0^2$ then we reject $H_0 : \sigma = \sigma_0$, otherwise we accept $H_0 : \sigma = \sigma_0$.

Case II:

If $\sigma_0 < \sigma_1$ then the BCR is

$$\sum x_i^2 \ge \frac{2\sigma_0^2\sigma_1^2}{\left(\sigma_1^2 - \sigma_0^2\right)}\left[\ln k + n\ln\frac{\sigma_0}{\sigma_1}\right]$$

$$\Rightarrow \sum x_i^2 \ge \lambda_2 \ (say) \qquad \text{L} \qquad \text{L} \qquad \text{L} \qquad (3)$$

We know that $\sum x_i^2$ is a $\chi^2\sigma_0^2$ with n degrees of freedom, we have,

$$P\left[\sum x_i^2 \ge \lambda_2 / H_0\right] = \alpha$$

$$\Rightarrow \frac{1}{\left(2\sigma_0^2\right)^{n/2}\Gamma\frac{n}{2}}\int_{\lambda_2}^{\infty} e^{-\frac{\chi^2}{2\sigma_0^2}}\left(\chi^2\right)^{\frac{n}{2}-1} d\chi^2 = \alpha$$

$$\Rightarrow \frac{1}{2^{n/2}\Gamma\frac{n}{2}}\int_{\frac{\lambda_2}{\sigma_0^2}}^{\infty} e^{-\frac{y}{2}} y^{\frac{n}{2}-1} dy = \alpha \qquad \left[By\ letting\ \frac{\chi^2}{2\sigma_0^2} = y\right]$$

Using the table of values of incomplete gamma function the value of $\frac{\lambda_2}{\sigma_0^2} = k_2 \ (say)$ can be determined, from equation (2) we have the BCR is $\sum x_i^2 \ge k_2\sigma_0^2$

If $\sum x_i^2 \ge k_2\sigma_0^2$ then we reject $H_0 : \sigma = \sigma_0$, otherwise we accept $H_0 : \sigma = \sigma_0$.

Power of the test:

The power function of the test above test when $H_1 : \sigma = \sigma_1 < \sigma_0$

$$= P\left[\sum x_i^2 \le \lambda_1 / H_1\right]$$

$$= \frac{1}{\left(2\sigma_1^2\right)^{n/2}\Gamma\frac{n}{2}}\int_0^{\lambda_1} e^{-\frac{\chi^2}{2\sigma_1^2}}\left(\chi^2\right)^{\frac{n}{2}-1} d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}}\int_0^{\lambda_1/\sigma_1^2} e^{-\frac{y}{2}} y^{\frac{n}{2}-1} dy \qquad \left[By\ letting\ \frac{\chi^2}{2\sigma_1^2} = y\right]$$

$$\therefore Power = \frac{1}{2^{n/2}\Gamma\frac{n}{2}}\int_0^{\frac{k_1\sigma_0^2}{\sigma_1^2}} e^{-\frac{y}{2}} y^{\frac{n}{2}-1} dy$$

Similarly, the power function when $H_1 : \sigma = \sigma_1 > \sigma_0$

$$Power = \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \int_{\frac{k_2\sigma_0^2}{\sigma_1^2}}^{\infty} e^{-\frac{y}{2}} y^{\frac{n}{2}-1} dy$$

❖ **Example-**3

Let $x_1, x_2, \ldots \ldots, x_n$ denote a random sample from the distribution that has the $p.d.f.$

$$f(x\,;\,\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} \qquad ; \qquad -\infty < x < \infty$$

To test the simple hypothesis $H_0 : \theta = \theta' = 0$ against the alternative simple hypothesis $H_1 : \theta = \theta_n = 1$. Find the power of the test.

Solution:

According to Nayman-Pearson Lemma we have,

$$\frac{L(x\,|\,H_0)}{L(x\,|\,H_1)} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum x_i^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum (x_i-1)^2}} \le k$$

$$\Rightarrow \quad e^{\left\{-\frac{1}{2}\sum x_i^2 + \frac{1}{2}\left(\sum x_i^2 - 2\sum x_i + \sum 1\right)\right\}} \le k$$

$$\Rightarrow \quad e^{-\sum x_i + \frac{n}{2}} \le k \qquad\qquad \textit{is a best critical region}$$

$$\Rightarrow \quad -\sum x_i + \frac{n}{2} \le \ln k$$

$$\Rightarrow \quad \sum x_i \le \frac{n}{2} - \ln k$$

$$\Rightarrow \quad \sum x_i \le C \quad (say) \qquad\qquad where, \quad C = \frac{n}{2} - \ln k$$

$$\therefore \quad P\left\{\sum x_i \ge C\right\} = \alpha$$

$$\Rightarrow \quad P\left\{\bar{x} \ge \frac{C}{n}\right\} = \alpha$$

$$\Rightarrow \quad P\{\bar{x} \ge C_1 \,|\, H_1\} = \alpha \qquad\qquad \left[\frac{C}{n} = C_1 \quad (say)\right]$$

Since $X \sim N(\theta, 1) \qquad \therefore \qquad \overline{X} \sim N\left(\theta, \frac{1}{n}\right)$

For a given positive integer n, the size of the sample and a given significance level α, the number C_1 can be found from Table.

Hence, if the experimental values of $X_1, X_2, \dots\dots, X_n$ were respectively $x_1, x_2, \dots\dots, x_n$, we would compute $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$.

If $\bar{x} \geq C_1$, the simple hypothesis $H_0 : \theta = \theta' = 0$ would be rejected at the significance level α and if $\bar{x} < C_1$, the hypothesis H_0 would be accepted.

The power of the test is:

$$P\{\bar{x} \geq C_1 \mid H_1\} = \int_{C_1}^{\infty} \frac{1}{\sqrt{\frac{1}{n}}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\bar{x}-1}{1/\sqrt{n}}\right)^2} d\bar{x}$$

Let $\alpha = 0.05 \;\; and \;\; n = 25$

So, we have,

$$P\{\bar{x} \geq C_1 \mid H_0\} = \int_{C_1}^{\infty} \frac{1}{\sqrt{\frac{1}{n}}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\bar{x}}{1/\sqrt{n}}\right)^2} d\bar{x} = 0.05 \qquad \left[\begin{array}{ll} Let, \;\; \dfrac{\bar{x}}{1/\sqrt{n}} = Z & \Rightarrow \bar{x} = \dfrac{1}{\sqrt{n}} Z \\ & d\bar{x} = \dfrac{1}{\sqrt{n}} dZ \end{array}\right]$$

$$\Rightarrow \int_{\sqrt{n}C_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} dZ = 0.05 \qquad \left[\begin{array}{lll} if \;\; \bar{x} = C_1 & then & Z = \sqrt{n}C_1 \\ if \;\; \bar{x} = \infty & then & Z = \infty \end{array}\right]$$

From normal table, we have,

$$\sqrt{n}C_1 = 1.645$$

$$\Rightarrow \quad 5C_1 = 1.645 \qquad ; \quad Since \;\; n = 25$$

$$\Rightarrow \quad C_1 = 0.329$$

$$\therefore \quad P\{\bar{x} \geq 0.329\} = 0.05$$

Now the power of the test is:

$$P\{rejecting \;\; H_0 \;\; when \;\; H_0 \;\; is \;\; false\} = 1 - \beta$$

$$\Rightarrow \quad P\{\bar{x} \geq C_1 \mid H_1\} = \int_{C_1}^{\infty} \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\bar{x}-1}{1/\sqrt{n}}\right)^2} d\bar{x}$$

$$= \int_{0.329}^{\infty} \frac{5}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\bar{x}-1}{1/5}\right)^2} d\bar{x} \qquad \left[\begin{array}{ll} Let, \;\; \dfrac{\bar{x}-1}{1/5} = Z & \Rightarrow \bar{x} = Z/5 + 1 \\ & d\bar{x} = \dfrac{1}{5} dZ \end{array}\right]$$

$$= \int_{-3.355}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} dZ = 0.999 \qquad \left[\begin{array}{lll} if \;\; \bar{x} = 0.329 & then & Z = -3.355 \\ if \;\; \bar{x} = \infty & then & Z = \infty \end{array}\right]$$

- **Example-4**

The yield of a particular crop is assumed to the distributed as normal with mean μ and variance 4. A random sample of size 9 test plots gives an average yield of 25 units. Test the hypothesis that the true average yield $\mu = 20$ against the alternative hypothesis $\mu > 20$ at the 5% level of significance.

Solution:

Let X be the yield of a particular crop i.e.

$$X \sim N(\mu, 4)$$
$$\therefore \frac{\bar{x} - \mu}{2/\sqrt{n}} \sim N(0,1)$$

The hypothesis $H_0 : \mu = 20$ against $H_1 : \mu > 20$ at 5% level of significance, according to Neyman-Pearson lemma we have, if $\bar{x} \geq \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$ then we reject H_0, or if $\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \geq z_\alpha$, then we reject H_0.

Here $\bar{x} = 25$, $n = 9$, $\sigma = 2$, $\mu_0 = 20$ *and* $z_{0.05} = 1.64$

$$\frac{25 - 20}{2/3} = 7.5$$

Since $7.5 \geq 1.64$, so we reject $H_0 : \mu = 20$ in favor of $H_1 : \mu > 20$ on the basis of the sample.

- **Example-5**

The weights of the students in a particular grade are assumed to be a normal distribution with mean μ and variance 25. A random sample of 25 students and the total weight is equal to 1250 units. Test the hypothesis that $H_0 : \mu = 52$ against $H_1 : \mu < 52$ at the 1% level of significance.

Solution:

Let X be the weight of the students in a particular grade i.e.

$$X \sim N(\mu, 25)$$
$$\therefore \frac{\bar{x} - \mu}{5/\sqrt{n}} \sim N(0,1)$$

The hypothesis $H_0: \mu = 52$ against $H_1: \mu < 52$ at 1% level of significance, according to Neyman-Pearson lemma we have, if $\bar{x} \le \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha$ then we reject H_0, or if $\frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \le -z_\alpha$, then we reject H_0.

Here $\sum x = 1250$, $\bar{x} = 50$, $n = 25$, $\sigma = 25$, $\mu_0 = 52$ *and* $z_{0.01} = 2.58$

$$\frac{50 - 52}{5/5} = -2$$

Since $-2 \ge -2.58$, so we accept $H_0: \mu = 52$ in favor of $H_1: \mu < 52$ on the basis of the sample.

- **Example-6**

The proportion of adults living in a small town who are SSC passed is estimated to the $p = 0.30$. To test the hypothesis $H_0: p = 0.30$. A random sample of 15 students is selected. If the number of SSC passed in our sample is any where from 2-7, we shall accept $H_0: p = 0.30$, otherwise we shall conclude that $p \ne 0.30$. Now

a. Evaluate α assuming $p = 0.30$.
b. Evaluate β for the alternatives $p = 0.20$ and $p = 0.40$.

Solution:

Let X be the number of students who are SSC passed, so that $X \sim b(15, 0.30)$. Here the hypothesis $H_0: p = 0.30$ against $H_1: p \ne 0.30$, we have,

$$P(X = x) = \binom{15}{x}(0.30)^x (0.70)^{15-x}$$

The critical region is given by

$$\begin{aligned} & w = (X : x = 0,1 \ and \ 8 \ to \ 15) \\ and \quad & \bar{w} = (X : x = 2 \ to \ 7) \end{aligned}$$

Part a:

$$\begin{aligned} \alpha &= \text{probability of type I error} \\ &= \text{probability of rejecting } H_0 \text{ when } p = 0.30 \\ &= 1 - \sum_{x=2}^{7} \binom{15}{x} (0.30)^x (0.70)^{15-x} \\ &= 0.085 \end{aligned}$$

Part b:

$$\begin{aligned}\beta_1 &= \text{probability of type II error} \\ &= \text{probability of accepting } H_0 \text{ when } p = 0.20 \\ &= \sum_{x=2}^{7} \binom{15}{x} (0.20)^x (0.80)^{15-x} \\ &= 0.8286\end{aligned}$$

$$\begin{aligned}\beta_2 &= \text{probability of type II error} \\ &= \text{probability of accepting } H_0 \text{ when } p = 0.40 \\ &= \sum_{x=2}^{7} \binom{15}{x} (0.40)^x (0.60)^{15-x} \\ &= 0.782\end{aligned}$$

- **Example-7**

We wished to test the hypothesis that the mean weight of a population of people 140 pounds. Using $\sigma = 15$ lbs , α=0.05 and a sample of 36 people find

a. The values of $\bar{x}$ which would lead to rejection of the hypothesis or find critical region.

b. β, the type II error if $\mu = 150$ lbs use a two tailed test.

Solution:

We have the following information are given

$$\begin{aligned}H_0 : \mu &= 140 \text{ lbs} \\ \sigma &= 15 \text{ lbs} \\ \alpha &= 0.05 \\ n &= 36\end{aligned}$$

Part a:

According to Neyman-Pearson lemma we have the critical region are

$$\bar{x} \geq \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha \qquad and \qquad \bar{x} \leq \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha$$

$$\Rightarrow \bar{x} \geq 140 + \frac{15}{\sqrt{36}} \times 1.96 \qquad \Rightarrow \bar{x} \leq 140 - \frac{15}{\sqrt{36}} \times 1.96$$

$$\Rightarrow \bar{x} \geq 144.9 \qquad \Rightarrow \bar{x} \leq 135.1$$

Thus $H_0 : \mu = 140$ is rejected if $\bar{x} \geq 144.9$ and $\bar{x} \leq 135.1$.

Part b:

Now at $\bar{x} = 144.9$, we have $z = \dfrac{144.9 - 150}{15/\sqrt{36}} = -2.04$. And at $\bar{x} = 135.1$, we have $z = \dfrac{135.1 - 150}{15/\sqrt{36}} = -5.96$

Thus

$$\begin{aligned}\beta &= \text{probability of type II error} \\ &= \text{probability of accepting } H_0 \text{ when } \mu = 150 \\ &= \int_{135.1}^{144.9} f(x \mid H_1)\,dx \\ &= \int_{-5.96}^{-2.04} f(z)\,dz \\ &= (-\infty \le z \le -2.04) - (-\infty \le z \le -5.96) \quad = 0.0207\end{aligned}$$

❖ **Example-8**

A coin is thrown 10 times. Suppose that $H_0 : p = \frac{1}{2}$ is rejected in favor of $H_1 : p = \frac{2}{3}$ if 8 or more independent trials give heads where p denotes the probability of getting a head in any trial. Determine the sizes of type I and type II errors.

Solution:

Let X denotes the number of heads i.e. $X \sim b(10, p)$

The critical region is given by $w = (X : x \ge 8)$ such that

$$P(X = x) = \binom{10}{x}(p)^x (1-p)^{10-x}$$

We have

$$\begin{aligned}\alpha &= \text{probability of type I error} \\ &= \text{probability of rejecting } H_0 \text{ when it is acctually true} \\ &= \text{probability of rejecting } H_0 \text{ when } p = \frac{1}{2} \\ &= \sum_{x=8}^{10} \binom{10}{x}(0.50)^x (0.50)^{10-x} \quad = 0.054\end{aligned}$$

$\beta =$ probability of type II error

$=$ probability of accepting H_0 when H_1 is acctually true

$=$ probability of accepting H_0 when $p = \dfrac{2}{3}$

$$= \sum_{x=0}^{7} \binom{10}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{10-x} \qquad = 1 - \sum_{x=8}^{10} \binom{10}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{10-x} \qquad = 0.70$$

- **Example-9**

The consumption of electricity in a small town is assumed to be exponentially distributed with parameter θ. Determine the sizes of type I and type II errors if $H_0 : \theta = 1000 \text{ kw}$ is tested against $H_1 : \theta = 2000 \text{ kw}$ and if test criterion is as select any day at random if the consumption on that day is 4000 or more reject H_0 otherwise accept H_0.

Solution:

Let X be the consumption of electricity in a small town i.e. $X \sim \exp(\theta)$ so that we have,

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \qquad ; \ (x, \theta) > 0$$

Now, we have,

$\alpha =$ probability of type I error

$=$ probability of rejecting H_0 when it is acctually true

$=$ probability of rejecting H_0 when θ=1000

$$= \int_{4000}^{\infty} \frac{1}{1000} e^{-\frac{x}{1000}} dx$$

$= 0.018$

Again we have,

$\beta =$ probability of type II error

$=$ probability of accepting H_0 when H_1 is acctually true

$=$ probability of accepting H_0 when θ=2000

$$= \int_{0}^{4000} \frac{1}{2000} e^{-\frac{x}{2000}} dx$$

$= 0.865$

$\therefore power = 1 - \beta = 0.135$

❖ **Example-10**

For the previous example test

a. $\begin{array}{l} H_0 : \theta = 1000 \\ H_1 : \theta > 1000 \end{array}$ b. $\begin{array}{l} H_0 : \theta = 1000 \\ H_1 : \theta < 1000 \end{array}$ c. $\begin{array}{l} H_0 : \theta = 1000 \\ H_1 : \theta \neq 1000 \end{array}$

Draw the power curve.

Solution:

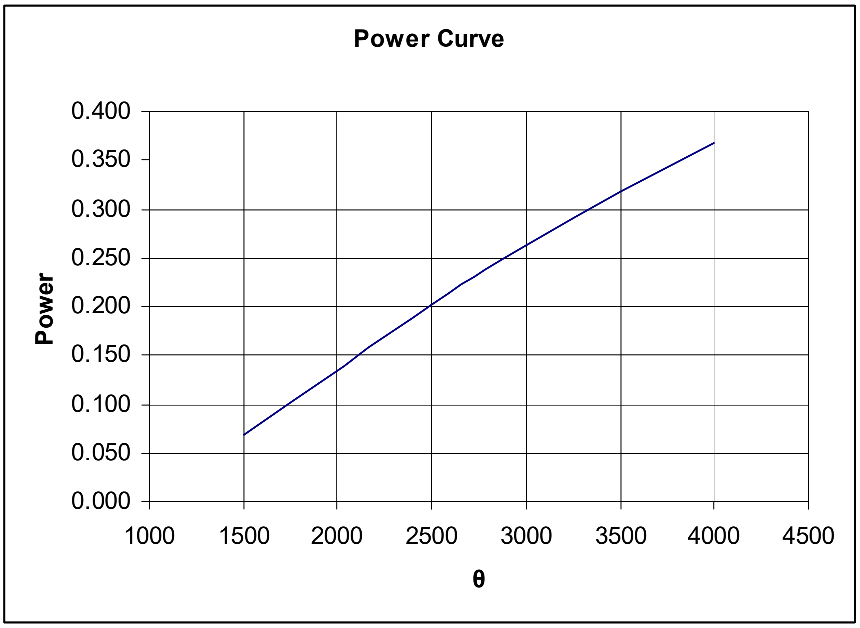
Part a:

We have $Power = \int_{4000}^{\infty} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$; where θ is any value less than 1000

Let us consider

θ	Power
1500	0.069
2500	0.202
3000	0.264
3500	0.319
4000	0.368

Now, we take the value of θ in x axis and the value of power in y axis we get the following curve

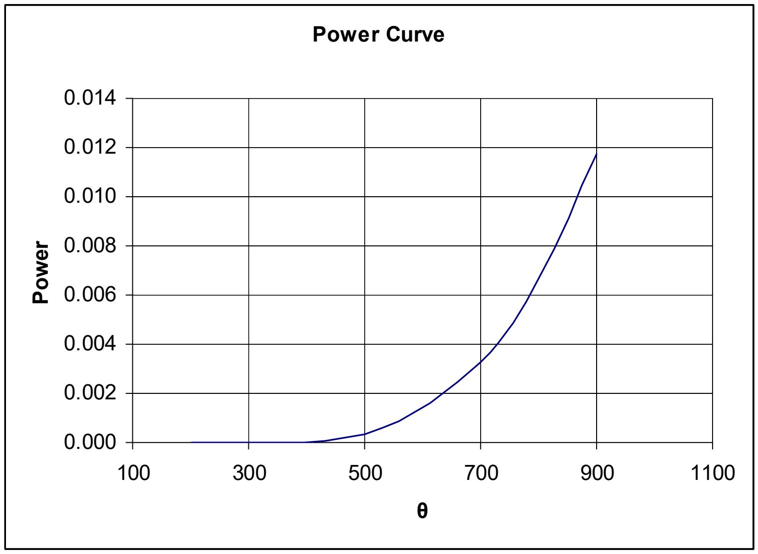


Part b:

We have $Power = \int_{4000}^{\infty} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$; where θ is any value greater than 1000

Let us consider

θ	Power
200	0.000
500	0.000
700	0.003
800	0.007
900	0.012



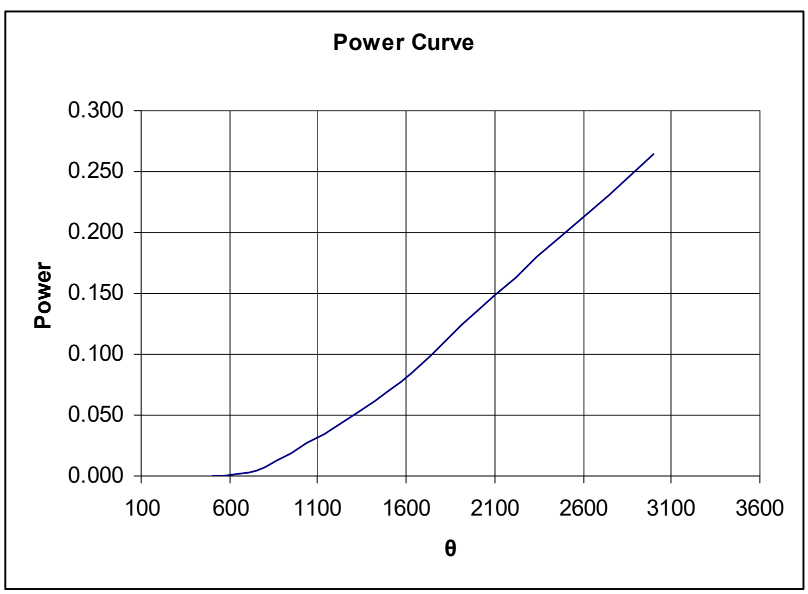
Now, we take the value of θ in x axis and the value of power in y axis we get the following curve.

Part c:

We have $Power = \int_{4000}^{\infty} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$; where θ is any value greater than 1000 or less than 1000

Let us consider

θ	Power
500	0.000
800	0.007
1500	0.069
2000	0.135
3000	0.264



Now, we take the value of θ in x axis and the value of power in y axis we get the following curve.