ব্র্যাইট ফটোষ্ট্যাট
দোকান নং-০৪ সাইকেল গ্যারেজ মার্কেট
রাজশাহী বিশ্ববিদ্যালয়
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Department of Statistics
B.S.C. Honours Part-II, Examination - 2014
Subject: statistics
Course: B. Stat. 201: Statistical method

1. a) What are the basic difference between i) Sample and population ii) parameter and statistic and iii) sampling and parent distributions.

b) Let x_1 and x_2 be independent χ^2 variates with n_1 and n_2 degrees of freedoms respectively then show that i) $U = \frac{x_1}{x_1+x_2}$ and $V = x_1+x_2$ are independently distributed. ii) U follows $\beta_1(\frac{n_1}{2}, \frac{n_2}{2})$ distribution and ii) V follows χ^2 distribution with (n_1+n_2) d.f.

Answer: a) i) The basic differences between sample and population are given bellow:-

Sample	Population
1. A sample consists one or more observations drawn from the population	1. A population includes all of the elements from a set of data.
2. Measurable characteristic of a sample is called statistic	2. A measurable characteristic of a population such as mean or standard deviation are called parameter.
3. The mean of a sample is denoted by the symbol $\bar{x}$.	3. The mean of a population is denoted by M.
4. The sample consists of n objects.	4. The population consists of N objects.

There is a lot of differences between parameter and statistic. Some of them are shown below:-

parameter	statistic
1. parameter is the characteristic of an entire population	1. statistic is the characteristic of a sample.
2. The parameter is a fixed measure which describes the target population	2. The statistic is a fixed measure which describes the sample with population
3. Parameter is a fixed and unknown numerical value.	3. statistic is a variable and known variable.
4. In population parameter, μ represents mean	4. In population statistic $\bar{x}$ represents mean
5. In parameter standard deviation is labeled as σ, variance is represented by σ^2 and population size is indicated by N	5. In statistic standard deviation is labeled as s, variance is represented by s^2 and population size is denoted by n.

iii) The differences between sampling distⁿ and parent distribution is given below:-

Sampling distⁿ	parent distⁿ
i) The probability distribution of sample is called sampling distribution	i) The probability distribution of parameter is called parent distribution.
ii) It is derived from parent distribution	ii) It is not derived from sampling distribution.
iii) It is the special case of parent distribution	iii) It is not the special case of sampling distⁿ
iv) χ^2, F and t distribution are the sampling distⁿ	iv) Normal distⁿ is parent distⁿ
v) It is the distribution of statistic	v) It is the distribution of variable

b) Since x_1 and x_2 be independent χ^2 variate with n_1 and n_2 degrees of freedoms respectively then the pdf of x_1 and x_2 are given below:-

$$f(x_1) = \frac{e^{-x_1/2}\,(x_1)^{n_1/2-1}}{2^{n_1/2}\,\Gamma\frac{n_1}{2}} \quad ; \; 0 \le x_1 < \infty$$

$$f(x_2) = \frac{e^{-x_2/2}\,(x_2)^{n_2/2-1}}{2^{n_2/2}\,\Gamma\frac{n_2}{2}} \quad ; \; 0 \le x_2 < \infty$$

Now the joint pdf of x_1 and x_2 is

$$f(x_1 x_2) = f(x_1) \cdot f(x_2)$$

$$= \frac{e^{-x_1/2} (x_1)^{n_1/2 - 1}}{2^{n_1/2} \sqrt{\frac{n_1}{2}}} \; \frac{e^{-x_2/2} (x_2)^{n_2/2 - 1}}{2^{n_2/2} \sqrt{\frac{n_2}{2}}}$$

$$= \frac{e^{-\frac{1}{2}(x_1+x_2)} (x_1)^{n_1/2 - 1} (x_2)^{n_2/2 - 1}}{2^{\frac{n_1+n_2}{2}} \sqrt{\frac{n_1}{2}} \cdot \sqrt{\frac{n_2}{2}}}$$

let, $u = \frac{x_1}{x_1 + x_2}$

$u = \frac{x_1}{v}$

$\Rightarrow x_1 = uv$

and, $v = x_1 + x_2$

$\Rightarrow v = uv + x_2$

$\Rightarrow x_2 = v - uv$

$\therefore x_2 = v(1-u)$

when, $x_1 = 0$ then $u = 0$

$x_1 = \alpha$ " $u = \alpha$

when $x_2 = 0$ then $v = 0$

$x_2 = \alpha$ " $v = \alpha$

Now. Jacobian transformation is,

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= |v(1-u) + uv|$$

$$= |v - uv + uv|$$

$$= |v| = v$$

Now the pdf of u and v is

$$f(u,v) = \frac{e^{-v/2} (uv)^{n_1/2 - 1} \{v(1-u)\}^{n_2/2 - 1}}{2^{\frac{n_1+n_2}{2}} \sqrt{\frac{n_1}{2}} \cdot \sqrt{\frac{n_2}{2}}} \cdot |J|$$

$$= \frac{e^{-v/2} (uv)^{\frac{n_1}{2} - 1} \{v(1-u)\}^{n_2/2 - 1}}{2^{\frac{n_1+n_2}{2}} \sqrt{\frac{n_1}{2}} \cdot \sqrt{\frac{n_2}{2}}} \; v$$

$$= \frac{e^{-v/2} u^{n_1/2 - 1} (1-u)^{n_2/2 - 1} v^{\frac{n_1}{2} + \frac{n_2}{2} - 1}}{2^{\frac{n_1+n_2}{2}} \sqrt{\frac{n_1}{2}} \sqrt{\frac{n_2}{2}}}$$

$$= \frac{u^{\frac{n_1}{2} - 1} (1-u)^{n_2/2 - 1} e^{-v/2} v^{n_1/2 + \frac{n_2}{2} - 1}}{2^{\frac{n_1+n_2}{2}} \sqrt{\frac{n_1}{2}} \cdot \sqrt{\frac{n_2}{2}}}$$

$$= f(u) \, f(v) \quad (\because f(u) \sim \beta_1(n_1/2, n_2/2), \; f(v) \sim \chi^2_{(n_1+n_2)})$$

Now, the marginal pdf of u is,

$$f(u) = \frac{u^{\frac{n_1}{2} - 1} (1-u)^{\frac{n_2}{2} - 1}}{2^{\frac{n_1+n_2}{2}} \sqrt{\frac{n_1}{2}} \cdot \sqrt{\frac{n_2}{2}}} \int_0^{\alpha} e^{-v/2} v^{\frac{n_1+n_2}{2} - 1} dv$$

$$= \frac{u^{\frac{n_1}{2}-1}(1-u)^{\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\cdot\Gamma\frac{n_2}{2}} \frac{\Gamma\frac{n_1+n_2}{2}}{(1/2)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{u^{\frac{n_1}{2}-1}(1-u)^{\frac{n_2}{2}-1}}{\frac{\Gamma\frac{n_1}{2}\cdot\Gamma\frac{n_2}{2}}{\Gamma\frac{n_1+n_2}{2}}\{u+(1-u)\}^{\frac{n_1+n_2}{2}}} \quad \left[\because \{u+1-u\}^{\frac{n_1+n_2}{2}} = 1\right.$$

$$= \frac{u^{\frac{n_1}{2}-1}(1-u)^{\frac{n_2}{2}-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\{u+(1-u)\}^{\frac{n_1+n_2}{2}}}$$

$$= \frac{u^{\frac{n_1}{2}-1}(1-u)^{\frac{n_2}{2}-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}$$

Hence u follows $\beta_1\left(\frac{n_1}{2},\frac{n_2}{2}\right)$ dist^n [Ans of (ii)]

Now the marginal pdf of v is

$$f(v) = \int_u f(u,v)\,du$$

$$= \int_0^\infty \frac{u^{\frac{n_1}{2}-1}(1-u)^{n_2/2-1}e^{-v/2}v^{\frac{n_1}{2}+n_2/2-1}}{2^{\frac{n_1+n_2}{2}}\Gamma n_1/2\,\Gamma n_2/2}\,du$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma n_2/2}\int_0^\infty \frac{u^{\frac{n_1}{2}-1}(1-u)^{n_2/2-1}e^{-v/2}v^{\frac{n_1}{2}+\frac{n_2}{2}-1}}{1}\,du$$

$$= \frac{e^{-v/2}v^{\frac{n_1}{2}+\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma n_2/2}\int_0^\infty u^{\frac{n_1}{2}-1}(1-u)^{n_2/2-1}\,du$$

optional

*[If the limit of u = 0 to 1 then it follow beta dist^n of first kind. the rang of u should be 0 to 1.] */

$$= \frac{e^{-v/2}v^{\frac{n_1}{2}+\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma n_2/2}\int_0^1 u^{\frac{n_1}{2}-1}(1-u)^{\frac{n_2}{2}-1}\,du$$

$$= \frac{e^{-v/2}v^{n_1/2+n_2/2-1}}{2^{\frac{n_1+n_2}{2}}\Gamma n_1/2\,\Gamma n_2/2}\beta\left(\frac{n_1}{2},n_2/2\right)$$

Here the dist^n function $\frac{e^{-v/2}v^{n_1/2+n_2/2-1}}{2^{\frac{n_1+n_2}{2}}\Gamma n_1/2\,\Gamma n_2/2}$

follows χ^2- dist^n with (n_1+n_2) d.f. . Ans/

2. a) Define cumulative distribution function (CDF). Mention some properties of univariate and bivariate CDF's.

b) Let X be a random variable with CDF given by $F_X(x) = (1-pe^{-\lambda x}), \; x>0$. Show that $E(X) = p/\lambda$.

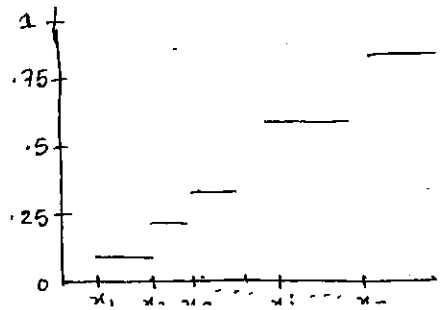
c) Discuss the moment generating function (m.g.f) technique for finding the distn of function of random variable.

2 **Answer:** a) Cumulative distribution function CDF:

The cumulative distribution function of the random variable X, denoted by $F(x)$, is defined to be that function with domain the real line and counterdomain the interval $[0,1]$ which satisfies

$$F(x) = P[X \le x]$$

Graphical representation of CDF:



properties of univariate CDF:

i) $F(x)$ is monotonic increasing function, i.e $F(a) \le F(b)$, when $a \le b$.

ii) The limit of $F(x)$ to the left is 0 and to the right is 1: That is,

a) $\lim_{x \to -\infty} F(x) = F(-\infty) = 0$ and b) $\underset{x \to \infty}{Lt} F(x) = F(\infty) = 1$

iii) $F(x)$ is continuous from the right, that is $\underset{0<h<\infty}{Lt} F(x+h) = F(x)$.

properties of bivariate CDF:

2. b) Given that,

$$CDF = F_X(x) = (1-pe^{-\lambda x}) \quad ; x>0$$

we know, by definition

$$f(x) = \frac{d}{dx} F_X(x)$$

$$= \frac{d}{dx}(1-pe^{-\lambda x})$$

$$= -pe^{-\lambda x}.(-\lambda)$$

$$= \lambda p e^{-\lambda x}$$

Now by the definition of mathematical expectation

$$E(x) = \int_x x f(x)\,dx \quad ; x>0$$

$$= \int_0^\infty x.\lambda p e^{-\lambda x}\,dx$$

$$= p\lambda \int_0^\infty e^{-\lambda x}.x^{2-1}\,dx$$

$$= p\lambda \frac{\sqrt{2}}{\lambda^2} \quad \left[\because \int_0^\infty e^{-ax}x^{n-1}dx = \frac{\sqrt{n}}{a^n}\right]$$

$$= p/\lambda$$

$\therefore E(X) = p/\lambda$ [showed]

2. c) Moment generating technique:

Moment generating function (m.g.f): Let X denote a random variable with probability density function $f(x)$, if continuous; probability mass function $p(x)$, if discrete then,

$$M_X(t) = \text{The moment generating function of } x.$$

$$= E(e^{tx})$$

$$= \begin{cases} \int_{-\alpha}^{\alpha} e^{tx} f(x)\,dx & ; \text{if } x \text{ is continuous} \\ \sum e^{tx} p(x) & ; \text{if } x \text{ is discrete} \end{cases}$$

The distribution of a random variable X is described by either,

i) The density function $f(x)$ if x is continuous probability mass function $p(x)$ if x is discrete

ii) The camulative distribution function $F(x)$.

iii) The moment generating function $M_X(t)$.

Properties: a) Let X be a random variable with moment generating function $M_x(t)$

let $Y = bx + a$, then

$$M_Y(t) = E[e^{(bx+a)t}]$$

$$= e^{at}E(e^{xbt})$$

$= e^{at} m_x(bt)$

b) let X and Y be two independent random variables with moment generating function $m_x(t)$ and $m_y(t)$, then

$$m_{x+y}(t) = E\left[e^{(x+y)t}\right]$$
$$= E\left[e^{tx} \cdot e^{yt}\right]$$
$$= E\left[e^{tx}\right] \cdot E\left[e^{yt}\right]$$
$$= m_x(t) \cdot m_y(t)$$

c) let X and Y be two random variables with moment generating function $m_x(t)$ and $m_y(t)$ and two distribution functions $F_x(X)$ and $F_y(Y)$ respectively.

let, $m_x(t) = m_y(t)$ then $F_x(X) = F_y(Y)$

This ensures that the distribution of a random variable can be identified by its moment generating function.

3. a) suppose X and Y independent random variables each uniformly distributed over the interval $(0,1)$. Find the pdf of XY and X/Y respectively. Also, compute $E(XY)$ and $E(X/Y)$

b) Let X_1 and X_2 have a joint pdf

$$f(x_1, x_2) = \begin{cases} x_1 x_2/36 & ; x_1, x_2 = 1, 2, 3 \\ 0 & ; \text{otherwise} \end{cases}$$

Find the joint pdf of $Y_1 = X_1 X_2$ and $Y_2 = X_2$. Also, find the marginal pdf of Y_1.

c) Let $f(x) = \begin{cases} 2x e^{-x^2} & ; 0 < x < \alpha \\ 0 & ; \text{otherwise} \end{cases}$

Find the pdf of $Y = X^2$.

<u>Answer:</u> a) Given that,

$$f(x) = 1$$
$$f(y) = 1$$
$$\therefore f(x,y) = 1$$

Now, Let us make a transformation

$U = XY$ and $V = X/Y$

$= X \cdot \frac{X}{V}$

$= \frac{x^2}{v}$

$\Rightarrow x^2 = uv$

$\therefore x = \sqrt{uv}$

$\therefore Y = \frac{x}{v}$

$= \frac{\sqrt{uv}}{v}$

$= \frac{\sqrt{u}}{\sqrt{v}}$

$\therefore Y = \sqrt{\frac{u}{v}}$

when, $x=0$ then $u=0$

$x=1$ then $u=1$

and, when $Y = 0$, then $v=\infty$

$Y=1$; $v=$

$$|J| = \left|\frac{\partial(x,Y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{1}{2v\sqrt{u/v}} \\ \frac{\sqrt{u}}{2\sqrt{v}} & -\frac{u}{2v^2(\sqrt{u/v})} \end{vmatrix}$$

$$= \frac{-u\sqrt{v}}{2.2\sqrt{u}\,v^2(\sqrt{u/v})} - \frac{\sqrt{u}}{2\sqrt{v}} \cdot \frac{1}{2v\sqrt{u/v}}$$

$$\therefore |J| = \left|\frac{-1}{4v} - \frac{1}{2v}\right| = \left|-\frac{3}{4}v\right| = \frac{3}{4}v$$

$$\therefore f(u,v) = f(x,y) \times |J| = \frac{3}{4}v$$

?

(b) Given that

$$f(x_1,x_2) = \begin{cases} \frac{x_1 x_2}{36} & \text{for } x_1, x_2 = 1,2,3 \\ 0 & ; \text{ otherwise} \end{cases}$$

The joint pdf of x_1 and x_2 is

(x_1,x_2)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
$f(x_1,x_2)$	1/36	1/18	1/12	1/18	1/9	1/6	1/12	1/6	1/4

Now the joint pdf of $y_1 = x_1 x_2$ and $y_2 = x_2$ is

(y_1,y_2)	(1,1)	(2,2)	(3,3)	(2,1)	(4,2)	(6,3)	(3,1)	(6,2)	(9,3)
$f(y_1,y_2)$	1/36	1/18	1/12	1/18	1/9	1/6	1/12	1/6	1/4

Hence the marginal pdf of y_1 is

y_1	1	2	3	4	6	9
$f(y_1)$	1/36	1/9	1/6	1/9	1/3	1/4

optional

/* $E(y_1) = \sum y_1 P(y_1)$

$= 1\times 1/36 + 2\times 1/18 + 3\times 1/6 + 4\times 1/9 + 6\times 1/3 + 9\times 1/4$

$= 5.44$ */

$V(y_1) = ?$

(e) Given that pdf of x is

$$f(x) = 2xe^{-x^2} \quad ; 0<x<\alpha$$
$$= 0 \quad ; \text{otherwise}$$

Now,
$$y = x^2$$
$$\Rightarrow x = \sqrt{y}$$
$$= y^{1/2}$$
$$\Rightarrow dx = \frac{1}{2} y^{-1/2} dy \quad ; 0<y<\alpha$$
$$\therefore |J| = \left|\frac{\partial x}{\partial y}\right| = |1/2\, y^{-1/2}| = 1/2\, y^{-1/2}$$

the pdf of y is
$$g(y) = 2y^{1/2} e^{-y} \cdot 1/2\, y^{-1/2}$$
$$= e^{-y} \quad ; 0<y<\alpha$$
$$= 0 \quad ; \text{otherwise}$$

which is the pdf of $y = x^2$ Ans

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4.(a) Define chi-square statistic. Mention its important properties. Show that χ^2 dist^n tends to normal dist^n for large degree of freedom.
If $f(x,y) = 4xy\, e^{-(x^2+y^2)}$; $x>0$, $y>0$. Find the dist^n of $Z = \sqrt{X^2+Y^2}$.

Answer: (a) Chi-square variate: A square of standard normal variate is known as χ^2-variate with 1 d.f. These of $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and $Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2$

$Z^2 = \frac{(X-\mu)^2}{\sigma^2}$ is a χ^2-variate with 1 d.f.

If X_i $(i=1,2,\dots n)$ are independent normal variate with mean μ_i and variance σ_i^2 then $\chi^2 = \Sigma\left(\frac{X_i-\mu_i}{\sigma_i}\right)^2$ is known as χ^2-variate with n d.f.

Properties of χ^2-dist^n:

i) χ^2 dist^n is a continuous type of dist^n and its range is 0 to α.

ii) The mean and variance of χ^2-distⁿ for n d.f are n and 2n respectively.

iii) χ^2- distⁿ is positively skewed and lepto-kurtic.

iv) The mode of χ^2- distⁿ for n d.f is $(n-2)$

v) χ^2- distⁿ is the limiting case of normal distribution.

vi) Gamma distribution is the special case of χ^2- distribution.

vii) Incomplete gamma distⁿ is the special case of χ^2 distⁿ.

viii) The sum of two χ^2 variate is χ^2 variate

ix) $\sqrt{2\chi^2}$ follows approximately normal distⁿ with mean $\sqrt{2n-1}$ and variance unity.

x) $\chi^2/n^{1/3}$ follows approximately normal distⁿ with mean $1-2/9n$ and variance $2/9n$ is known as wilson ferty's result (1931)

xi) Two independent variate χ_1^2 and χ_2^2 follow χ^2- distⁿ with n_1 and n_2 d.f. Then χ_1^2/χ_2^2 is β_2 variate with some parameter then $\beta_2(n_1/2, n_2/2)$

<u>χ^2- distⁿ tends to normal distⁿ for large d.f :-</u>

Let us consider χ^2 distⁿ with n d.f.

the mean of χ^2 distⁿ, $E(\chi^2) = n$

the variance of χ^2 " $v(\chi^2) = 2n$

Now, let, $z = \dfrac{\chi^2 - E(\chi^2)}{\sqrt{v(\chi^2)}}$

$$= \frac{\chi^2 - n}{\sqrt{2n}} \sim N(0,1)$$

Then the m.g.f of z is,

$$M_z(t) = E[e^{zt}]$$

$$= E\left[e^{\frac{(\chi^2-n)t}{\sqrt{2n}}}\right]$$

$$= e^{-\frac{tn}{\sqrt{2n}}} E\left[e^{t\frac{\chi^2 t}{\sqrt{2n}}}\right]$$

$$= e^{-\sqrt{n}t/2} E\left[e^{\frac{t\,\chi^2 t}{\sqrt{2n}}}\right]$$

Now,

$$E\left[e^{-\frac{\chi^2 t}{\sqrt{2n}}}\right] = \left[1 - \frac{2t}{\sqrt{2n}}\right]^{-n/2}$$

$$\therefore M_z(t) = e^{-\frac{\sqrt{n}t}{\sqrt{2}}}\left(1-\frac{2t}{\sqrt{2n}}\right)^{-n/2}$$

Taking Log both sides we've,

$$K_z(t) = \log M_z(t)$$

$$= -\frac{\sqrt{n}t}{\sqrt{2}} - \frac{n}{2}\log\left(1-\frac{2t}{\sqrt{2n}}\right)$$

$$= \frac{-\sqrt{n}t}{\sqrt{2}} - \frac{n}{2}\left(\frac{-\sqrt{2}t}{\sqrt{n}} - \frac{(\sqrt{2/n}\,t)^2}{2!} - \frac{(\sqrt{2/n}\,t)}{3!}\right.$$

$$= -\frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{2}t + \frac{t^2}{2} + 0 - \text{terms contain } \sqrt{n} \text{ is the denominator}$$

$$\therefore \lim_{n\to\infty} K_z(t) = \frac{t^2}{2} = \lim_{n\to\infty}\log M_z(t)$$

$$\Rightarrow \lim_{n\to\infty} M_z(t) = e^{t^2/2}$$

which is the moment generating f^n of the standard normal variate for large sample and χ^2-dist^n tends to normal dist^n.

Solution: Given that,

$$f(x,y) = 4xy\, e^{-(x^2+y^2)} \quad ; x>0,\ y>0$$

Also,

$$z = \sqrt{x^2+y^2}$$

Let, $v = x$

$\therefore x = v$

$$\Rightarrow z^2 = x^2+y^2$$

$$\Rightarrow y = \sqrt{z^2-x^2}$$

$$\therefore y = \sqrt{z^2-v^2}$$

Now Jacobian of transformation is given by,

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{z}{\sqrt{z^2-v^2}} & \frac{-v}{\sqrt{z^2-v^2}} \end{vmatrix}$$

$$= \left|\frac{-z}{\sqrt{z^2-v^2}}\right|$$

$$= \frac{z}{\sqrt{z^2-v^2}}$$

The joint density of z and v is

$$g(z,v) = 4v\sqrt{z^2-v^2}\, e^{-z^2}|J|$$

$$= 4v\,v\,\sqrt{z^2-v^2}\; e^{-z}\; \frac{z}{\sqrt{z^2-v^2}}$$

$$= 4vz\,e^{-z^2} \quad ; z>0, v>0, 0\le v\le z\le \alpha$$

$$\therefore g(z) = \int_v g(v,z)\,dv$$

$$= \int_0^z 4zve^{-z^2}dv$$

$$= 4ze^{-z^2}\int_0^z v\,dv$$

$$= 4ze^{-z^2}\left[\frac{v^2}{2}\right]_0^z$$

$$= 4ze^{-z^2}\left[\frac{z^2}{2}-0\right]$$

$$= 2z^3e^{-z^2} \quad ; 0\le v< z \quad A$$

which is the distⁿ of $z=\sqrt{x^2+y^2}$

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5. (a) Define F-variate and its important uses.
Let x be a beta variate of 1st kind with parameters n_1 and n_2. Find the distⁿ of

$$F=\frac{n_2x}{n_1(1-x)}$$

suppose x_1 and x_2 be two independent random variables from $f(x)=e^{-x}$; $0<x<\alpha$. obtain the distⁿ of $F=\frac{x_1}{x_2}$.

Solution: (a) F-variate: F-variate is the ratio of two independent χ^2 variate with their respective degrees of freedom
If x and y are two independent chi-square variate with n_1 and n_2 degrees of freedom respectively then the F-variate is

$$F=\frac{X/n_1}{Y/n_2} \quad \text{where, } X\sim\chi^2_{n_1},\; Y\sim\chi^2_{n_2}$$

which is the F-distⁿ with n_1 and n_2 d.f.
Then the density function of F-distⁿ is given below-

$$f(F)=\frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}(F)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \quad ; 0\le F<\alpha$$

$$n_1>0;\ n_2>0$$

Important uses of F-dist^n: The important applications or uses of F-dist^n are given below

i) F-dist^n is used to test the equality of several mean.

ii) It is used to test the equality of population variance.

iii) It is used for testing the significance of observed multiple correlation coefficient

iv) It is used for testing the significance of observed sample correlation ratio.

v) F-dist^n is used to test the linearity of regression.

Solution: Since x be a beta variate of first kind with parameter n_1 and n_2.

Then the pdf of beta variate of first kind is given below-

$$f(x) = \frac{x^{n_1/2-1}(1-x)^{n_2/2-1}}{\beta(\frac{n_1}{2}, n_2/2)} \quad ; 0 \le x \le 1$$

let us make the transformation

$$x = \frac{n_1 F}{n_2 + n_1 F}$$

$$\Rightarrow n_2 x + n_1 F x = n_1 F$$

$$\Rightarrow n_1 F x - n_1 F = -n_2 x$$

$$\Rightarrow F(n_1 - n_1 x) = n_2 x$$

$$F = \frac{n_2}{n_1}\left(\frac{x}{1-x}\right) \quad \text{---(1)}$$

Now, $$\frac{dx}{dF} = \frac{(n_2 + n_1 F)n_1 - n_1^2 F}{(n_2 + n_1 F)^2}$$

$$= \frac{n_1 n_2 + n_1^2 F - n_1^2 F}{(n_2 + n_1 F)^2}$$

$$= \frac{n_1 n_2}{(n_2 + n_1 F)^2}$$

Now the jacobian

$$|J| = \left|\frac{dx}{dF}\right| = \frac{n_1 n_2}{(n_2 + n_1 F)^2}$$

Now the density of F is

$$f(F) = f(x) \cdot |J|$$

$$= \frac{\left(\frac{n_1 F}{n_2 + n_1 F}\right)^{\frac{n_1}{2}-1}\left(1 - \frac{n_1 F}{n_2 + n_1 F}\right)^{n_2/2-1}}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} \cdot \frac{n_1 n_2}{(n_2 + n_1 F)^2}$$

$$= \frac{\left(\frac{n_1 F}{n_2 + n_1 F}\right)^{\frac{n_1}{2}-1}\left(\frac{n_2}{n_2 + n_1 F}\right)^{n_2/2-1}(n_1 n_2)}{\beta(\frac{n_1}{2}, \frac{n_2}{2})(n_2 + n_1 F)^2}$$

$$= \frac{(n_1)^{\frac{n_1}{2}-1+1} F^{\frac{n_1}{2}-1} n_2^{-n_1/2} \left(1+\frac{n_1}{n_2}F\right)^{-\left(\frac{n_1+n_2}{2}\right)}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \; ; 0 \le F < \alpha$$

which is the pdf of F- $dist^n$ with n_1 and n_2

Solution: Given that,

$$f(x) = e^{-x} \; ; 0 < x < \alpha$$

$$u = x_1/x_2$$

Now,
The joint pdf of x_1 and x_2 is

$$f(x_1, x_2) = e^{-(x_1+x_2)}$$

let us make the transformation

$$v = x_1 + x_2 \quad \text{and} \quad u = x_1/x_2$$

$$\therefore x_1 = u x_2 = \frac{uv}{1+u}$$

$$v = x_2(1+u), \quad x_2 = \frac{v}{1+u}$$

The Jacobian of transformation

$$J = \frac{\partial(x_1, x_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v\left[\frac{1}{1+u} - \frac{u}{(1+u)^2}\right] & \frac{u}{1+u} \\ -\frac{v}{(1+u)^2} & \frac{1}{1+u} \end{vmatrix}$$

$$= \frac{v}{(1+u)^2} \; ; 0 < v < \alpha, \; 0 < u < \alpha$$

$$\therefore g(u,v) = e^{-v} \frac{v}{(1+u)^2}$$

$$h(u) = \frac{1}{(1+u)^2} \int_0^{\alpha} e^{-v} v \, dv = \frac{\sqrt{2}}{(1+u)^2} \; ; 0 < u < \alpha$$

$$= \frac{\left(\frac{2}{2}\right)^{2/2-1} u^{2/2-1}}{\beta(2/2, 2/2)\left(1+2/2\, u\right)^{\frac{2+2}{2}}} \; ; 0 < u < \alpha$$

which is the density function of $F_{2,2}$ i.e F $dist^n$ with $n_1 = 2$ and $n_2 = 2$ degrees of freedom.

6.(a) suppose $x_1, x_2, \cdots x_n$ be a random sample from $N(\mu, \sigma^2)$

Let, $\bar{x} = \sum_{i=1}^{n} x_i/n$ and $s^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)$, then show that,

i) $\bar{x} \sim N(\mu, \sigma^2/n)$

ii) $(n-1)s^2/\sigma^2 \sim \chi^2_{(n-1)}$

iii) $\bar{x}$ and $(n-1)s^2/\sigma^2$ are stochastically indepe

(b) compute $E(s^2)$ and $v(s^2)$

(c) If $P = P_r(\chi_2^2 > \chi_0^2)$, then $\chi_0^2 = 2\log(1/p)$.

Answer: (a) solution: i) Given that,

$$\bar{x} = \sum_{i=1}^{n} x_i/n$$

$$\therefore \bar{x} = \frac{x_1+x_2+\cdots+x_n}{n} \quad \text{------(i)}$$

Taking expectation both sides we have,

$$E(\bar{x}) = E\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

$$= \frac{1}{n}\left[E(x_1)+E(x_2)+\cdots+E(x_n)\right]$$

$$= \frac{1}{n}\left[\mu+\mu+\cdots+\mu\right]$$

$$= \frac{n\mu}{n}$$

$$\therefore E(\bar{x}) = \mu$$

Taking variance in (i) in both sides we've,

$$v(\bar{x}) = v\left[\frac{x_1+x_2+\cdots+x_n}{n}\right]$$

$$= \frac{1}{n^2}\left[v(x_1)+v(x_2)+\cdots+v(x_n)\right]$$

$$= \frac{1}{n^2}\left(\sigma^2+\sigma^2+\cdots+\sigma^2\right)$$

$$= \frac{n\sigma^2}{n^2}.$$

$$\therefore v(\bar{x}) = \sigma^2/n$$

$$\therefore \bar{x} \sim N(\mu, \sigma^2/n) \quad \text{(showed)}$$

ii) solution: Given that,

$$s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$$

when, $n=2$, then

$$s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2-1}$$

$$= (x_1-\bar{x})^2 + (x_2-\bar{x})^2$$

$$= \left(x_1 - \frac{x_1+x_2}{2}\right)^2 + \left(x_2 - \frac{x_1+x_2}{2}\right)^2 \quad \left[\bar{x} = \frac{x_1+x_2}{2}\right]$$

$$= \left(\frac{2x_1-x_1-x_2}{2}\right)^2 + \left(\frac{2x_2-x_1-x_2}{2}\right)^2$$

$$= \left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{x_2-x_1}{2}\right)^2$$

$$= 2\left(\frac{x_1-x_2}{2}\right)^2$$

$$= (x_1-x_2)^2/2$$

$$\frac{(n-1)s^2}{\sigma^2} = \frac{(x_1-x_2)^2}{2\sigma^2}$$

$$= \left[\frac{(x_1-x_2)}{\sqrt{2\sigma^2}}\right]^2$$

Now, $x_1 \sim N(\mu, \sigma^2)$

$x_2 \sim N(\mu, \sigma^2)$

$$E(x_1-x_2) = E(x_1) - E(x_2)$$
$$= \mu - \mu = 0$$

$$V(x_1-x_2) = V(x_1) + V(x_2)$$
$$= \sigma^2 + \sigma^2$$
$$= 2\sigma^2$$

$\therefore x_1 - x_2 = N(0, 2\sigma^2)$

$$\Rightarrow \frac{(x_1-x_2) - E(x_1-x_2)}{\sqrt{V(x_1-x_2)}} \sim N(0, 2\sigma^2)$$

$$\Rightarrow \frac{x_1-x_2}{\sqrt{2\sigma^2}} \sim N(0, 2\sigma^2)$$

$\therefore \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ where, $n=2$

$\therefore \frac{(n-1)s^2}{\sigma^2}$ has a chi-square distⁿ with $(n-1)$ d.f.

$\boxed{\text{see hand note of sir}}$ → best way - 551 pg

ii) Solution: we have to prove $\bar{x}$ and $(n-1)s^2/\sigma^2$ are stochastically independent.

To prove this theorem, let $n=2$, then

$$\bar{x} = \frac{x_1+x_2}{2}$$

and,
$$\sum_{i=1}^{2}(x_i-\bar{x})^2 = (x_1-\bar{x})^2 + (x_2-\bar{x})^2$$

$$= \left(x_1 - \frac{x_1+x_2}{2}\right)^2 + \left(x_2 - \frac{x_1+x_2}{2}\right)^2$$

$$= \left(\frac{2x_1-x_2-x_2}{2}\right)^2 + \left(\frac{2x_2-x_1-x_2}{2}\right)^2$$

$$= \left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{x_2-x_1}{2}\right)^2$$

$$= 2\left(\frac{x_1-x_2}{2}\right)^2$$

$$= \frac{(x_1-x_2)^2}{2}$$

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Here, $\bar{x}$ is a function of (x_1+x_2)

Then by definition of m.g.f we've

$$M_{x_1+x_2}(t_1) = E[e^{(x_1+x_2)t_1}]$$
$$= E[e^{t_1x_1} \cdot e^{t_2x_2}]$$
$$= E[e^{t_1x_1}] \cdot E(e^{t_1x_2}]$$
$$= M_{x_1}(t_1) \cdot M_{x_2}(t_1)$$

we know that,

$$M_{x_1}(t_1) = e^{t_1^2/2}$$
$$M_{x_2}(t_1) = e^{t_1^2/2}$$

$$M_{x_1+x_2}(t_1) = M_{x_1}(t_1) \cdot M_{x_2}(t_1)$$
$$= e^{t_1^2/2} \cdot e^{t_1^2/2}$$
$$= e^{2t_1^2/2}$$
$$= e^{t_1^2}$$

Again, $\sum_{i=1}^{n}(x_i-\bar{x})^2$ is a function of (x_1-x_2) then by the form of m.g.f we'vee,

$$M_{x_1-x_2}(t_2) = E[e^{(x_1-x_2)t_2}]$$
$$= e^{t^2/2}$$

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Again,

$$M_{(x_1+x_2,\, x_1-x_2)}(t_1,t_2) = E[e^{(x_1+x_2)t_1+(x_1-x_2)t_2}]$$
$$= E[e^{x_1t_1+x_2t_1+x_1t_2-x_2t_2}]$$
$$= E[e^{(t_1+t_2)x_1+(t_1-t_2)x_2}]$$
$$= e^{t_1^2} \cdot e^{t_2^2}$$
$$= M_{x_1+x_2}(t_1) \cdot M_{x_1-x_2}(t_2)$$

Since, the joint m.g.f into the product of the marginal m.g.f (x_1+x_2) and (x_1-x_2) are independent.

and, so $\bar{x}$ and $\sum_{i=1}^{n}(x_i-\bar{x})^2$ are independent

Hence $\bar{x}$ and $\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{6}$ or $\frac{(n-1)s^2}{6^2}$ are also independent.

Given that,

$$s^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)$$

Here, $$f(s^2) = \frac{(n/2\sigma^2)^{\frac{n-1}{2}}}{\Gamma\frac{n-1}{2}} e^{-\frac{s^2}{2\sigma/n}} (s^2)^{\frac{n-1}{2}-1};$$

$f(s^2) \rightarrow$ from hand note $\rightarrow$ chi-squar dist^n – pg–552

$$E(s^2) = \int_{s^2} s^2 f(s^2)\, ds^2$$

$$= \int_0^\alpha s^2 \frac{\left(\frac{n}{2\sigma^2}\right)^{\frac{n-1}{2}}}{\Gamma\frac{n-1}{2}} e^{-s^2/2\sigma^2/n} (s^2)^{\frac{n-1}{2}-1} ds^2 \quad ; 0 < s^2 < \alpha$$

$$= \frac{\left(\frac{n}{2\sigma^2}\right)^{\frac{n-1}{2}}}{\Gamma\frac{n-1}{2}} \int_0^\alpha e^{-\frac{s^2 n}{2\sigma^2}} (s^2)^{\frac{n+1}{2}-1} ds^2$$

$$= \frac{(n/2\sigma^2)^{\frac{n-1}{2}}}{\Gamma\frac{n-1}{2}} \frac{\Gamma\frac{n+1}{2}}{\left(\frac{n}{2\sigma^2}\right)^{\frac{n+1}{2}}}$$

$$= \frac{n^{n/2} \cdot n^{-1/2}\, \Gamma\frac{n+1}{2}\, (2)^{n/2} (2)^{1/2} (\sigma^2)^{n/2} (\sigma^2)^{1/2}}{(2)^{n/2} \cdot (2)^{-1/2} \cdot (\sigma^2)^{n/2}\, \Gamma\frac{n-1}{2}\, n^{n/2} n^{1/2} (\sigma^2)^{-1/2}}$$

$$= \frac{\sqrt{2\sigma^2}\, \Gamma\frac{n+1}{2} \cdot \sqrt{2\sigma^2}}{\sqrt{n}\sqrt{n}\, (\Gamma\frac{n-1}{2}} = \frac{2\sigma^2\, \Gamma\frac{n+1}{2}}{n\, \Gamma\frac{n-1}{2}}$$

and $V(s^2) = E(s^2)^2 - \{E(s^2)\}^2$ ———— (*)

Now, $E(s^2)^2 = \int_{s^2} (s^2)^2 f(s^2)\, ds^2 \quad ; 0 < s^2 < \alpha$

$$= \int_0^\alpha (s^2)^2 \frac{(n/2\sigma^2)^{\frac{n-1}{2}}}{\Gamma\frac{n-1}{2}} e^{-s^2/2\sigma^2/n} (s^2)^{\frac{n-1}{2}-1} ds^2$$

$$= \int_0^\alpha \frac{(n/2\sigma^2)^{\frac{n-1}{2}}}{\Gamma\frac{n-1}{2}} e^{-s^2/2\sigma^2 n} (s^2)^{\frac{n+3}{2}-1} ds^2$$

$$= \frac{(n/2\sigma^2)^{\frac{n-1}{2}}}{\Gamma\frac{n-1}{2}} \int_0^\alpha e^{-s^2\frac{n}{2\sigma^2}} (s^2)^{\frac{n+3}{2}-1} ds^2$$

$$= \frac{(n/2\sigma^2)^{\frac{n-1}{2}}}{\Gamma\frac{n-1}{2}} \frac{\Gamma\frac{n+3}{2}}{(n/2\sigma^2)^{\frac{n+3}{2}}}$$

$$= \left(\frac{2\sigma^2}{n}\right)^2 \quad \text{[on simplication]}$$

from (*) we have

$$V(s^2) = \left(\frac{2\sigma^2}{n}\right)^2 - \left\{\frac{2\sigma^2\, \Gamma\frac{n+1}{2}}{n\, \Gamma\frac{n-1}{2}}\right\}^2 \quad \text{(Ans)}$$

$$= 2\sigma^2 - 4(\sigma^2)^2 n(n-1)^2 \quad \left(\because \Gamma\tfrac{n+1}{2} = \tfrac{\Gamma n+1}{\Gamma n}\right)$$

[on simplification]

(c) Given that,

$$p = Pr(\chi_2^2 > \chi_0^2)$$

Now, the pdf of χ^2- dist^n with 2 d.f is

$$f(\chi_2^2) = \left[\frac{1}{2^{n/2}\sqrt{n/2}} \exp(-\chi^2/2).(\chi^2)^{(n/2-1)}\right]_{n=2}$$

$$= \frac{1}{2}\exp(-\chi^2/2) \cdot \quad 0 \le \chi^2 < \infty$$

$$\therefore P = P(\chi_2^2 > \chi_0^2) = \int_{\chi_0^2}^{\infty} \frac{1}{2}\exp(-\chi^2/2)d\chi^2$$

$$= \frac{1}{2}\frac{\exp(-\chi^2/2)}{-1/2}\Bigg|_{\chi_0^2}^{\infty}$$

$$= +\exp(-\chi_0^2/2)$$

$$\Rightarrow \log_e P = -\chi_0^2/2$$

$$\Rightarrow \chi_0^2 = -2\log_e P$$

$$\therefore \chi_0^2 = 2\log_e\left(\frac{1}{P}\right) \quad A$$

7. a) Define student t-statistic and its density function.

b) Obtain first four raw moments and central moments of t-dist^n and hence find its skewness and kurtosis.

c) Find the relationship between t-statistic and F- statistic.

Answer:- Student-t statistic: If $x_1, x_2, \dots x_n$ be a random sample of size n from a normal population with mean μ and variance σ^2. Then student's t- is defined by the statistic

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n}}$$

where, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$ is an unbiased estimator of the population variance σ^2 and it follows student's t- dist^n with $v=(n-1)$ d.f with probability density function

$$f(t) = \frac{1}{\sqrt{v}\,\beta(1/2, v/2)\,(1+t^2/v)^{\frac{v+1}{2}}} \quad ; -\infty < t < \infty$$

(b) For t-$dist^n$ there exist two types of m.g.f

i) odd order moment

ii) Even order moment.

i) Odd order moment: By the definition of raw moment we have $(2r+1)$th raw moment about origin is given by,

$$\mu'_{2r+1} = E[t^{2r+1}]$$

$$= \int_t t^{2r+1} f(t)\,dt$$

$$= \int_t t^{2r+1} \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})(1+\frac{t^2}{n})^{\frac{n+1}{2}}}\,dt$$

$$= \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \int_{-\alpha}^{\alpha} \frac{t^{2r+1}}{(1+\frac{t^2}{n})^{\frac{n+1}{2}}}\,dt$$

since, $\int_{-\alpha}^{\alpha} \frac{t^{2r+1}}{(1+\frac{t^2}{n})^{\frac{n+1}{2}}}$ is odd order function therefore the integral part equal to zero.

$$= \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \cdot 0$$

$$= 0$$

$$\therefore \mu'_{2r+1} = 0$$

Now, putting $r = 0, 1$ we have,

$$\mu'_{2\times 0+1} = \mu'_1 = 0$$

$$\mu'_{2\times 1+1} = \mu'_3 = 0$$

Even order moments:

By the definition of raw moment we have $2r$-th raw moment about origin is given by.

$$\mu'_{2r} = E[t^{2r}]$$

$$= \int_t t^{2r} f(t)\,dt$$

$$= 2\int_0^{\alpha} t^{2r} \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})(1+\frac{t^2}{n})^{\frac{n+1}{2}}}\,dt$$

$$= \frac{2}{\sqrt{n}\beta(\frac{1}{2},\frac{n}{2})} \int_0^\alpha \frac{t^{2r}}{(1+t^2/n)^{\frac{n+1}{2}}} dt$$

This integral is absolutely convergent i[f] $2r<n$

Let, $\frac{t^2}{n} = m$

$\Rightarrow t^2 = mn$

$\Rightarrow t = \sqrt{mn}$

$$\frac{dt}{dm} = \frac{1}{2\sqrt{mn}} \cdot n$$

$$= \frac{\sqrt{n}.\sqrt{n}}{2\sqrt{m}.\sqrt{n}}$$

$$= \frac{\sqrt{n}}{2\sqrt{m}}$$

$t=0$ then $m=0$

$t=\alpha$ then $m=\alpha$

Now, $t^{2r} = (\sqrt{mn})^{2r} = \{(\sqrt{mn})^2\}^r$

$= m^r . n^r$

[H]ence,

$$\mu'_{2r} = \frac{2}{\sqrt{n}\beta(\frac{1}{2},\frac{n}{2})} \int_0^\alpha \frac{m^r n^r}{(1+m)^{\frac{n+1}{2}}} dm \frac{\sqrt{n}}{2\sqrt{m}}$$

$$= \frac{n^r}{\beta(\frac{1}{2},\frac{n}{2})} \int_0^\alpha \frac{m^r}{(1+m)^{\frac{n+1}{2}} m^{1/2}} dm$$

$$= \frac{n^r}{\beta(\frac{1}{2},\frac{n}{2})} \int_0^\alpha \frac{m^{r-1/2}}{(1+m)^{\frac{n+1}{2}}} dm$$

$$= \frac{n^r}{\beta(\frac{1}{2},\frac{n}{2})} \int_0^\alpha \frac{m^{(r+1/2)-1}}{(1+m)^{(n/2-r)+(r+1/2)}} dm$$

$$= \frac{n^r}{\beta(\frac{1}{2},\frac{n}{2})} \beta\{(r+\tfrac{1}{2}),(\tfrac{n}{2}-r)\}$$

$$= \frac{n^r}{\beta(\frac{1}{2},\frac{n}{2})} \frac{\Gamma(r+\frac{1}{2})\Gamma(\frac{n}{2}-r)}{\Gamma(r+\frac{1}{2}+\frac{n}{2}-r)}$$

$$= \frac{n^r}{\beta(\frac{1}{2},\frac{n}{2})} \frac{\Gamma(r+\frac{1}{2})\Gamma(\frac{n}{2}-r)}{\Gamma\frac{n+1}{2}}$$

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$$= \frac{n^r \Big/ \Gamma\frac{n+1}{2}}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}} \cdot \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\frac{n+1}{2}}$$

$$= \frac{n^r\,\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}$$

which is the required moment

let, $r=1$

$$\mu'_{2.1} = \frac{n\,\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}$$

$$= \frac{n\,\Gamma\left(\frac{n}{2}-1\right)\left(\frac{1}{2}\right)\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)}$$

$$= \frac{n\cdot\frac{1}{2}}{\frac{n}{2}-1}$$

$$= \frac{n}{n-2}$$

$$\therefore \mu'_2 = \frac{n}{n-2} \quad \text{for } n>2$$

and when, $r=2$ we have,

$$\mu'_4 = \frac{n^2\,\Gamma\left(\frac{n}{2}-2\right)\Gamma\left(\frac{1}{2}+2\right)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}\right)\Gamma\frac{1}{2}}$$

$$= \frac{n^2\,\Gamma\left(\frac{n}{2}-2\right)}{\Gamma\frac{1}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\Gamma\left(\frac{n}{2}-2\right)}$$

$$= \frac{n^2\,\Gamma\left(\frac{n}{2}-2\right)\left(\frac{1}{2}+1\right)\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\Gamma\left(\frac{n}{2}-2\right)}$$

$$= \frac{n^2\;3/4}{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)}$$

$$= \frac{n^2\;3/4}{\left(\frac{n-1}{2}\right)\left(\frac{n-4}{2}\right)}$$

$$= \frac{3n^2}{(n-1)(n-4)}$$

$$\therefore \mu'_4 = \frac{3n^2}{(n-1)(n-4)}$$

Now, variance, $\mu_2 = \mu'_2 - (\mu'_1)^2$

$$= \frac{n}{n-2} - 0$$

$$= n/n-2$$

Now,

$$\text{skewness, } \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{\left(\frac{n}{n-2}\right)^3} = 0$$

$$\text{Kurtosis, } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\frac{3n^2}{(n-2)(n-4)}}{\frac{n^2}{(n-2)^2}}$$

$$= \frac{3n^2}{(n-2)(n-4)} \cdot \frac{(n-2)^2}{n^2}$$

$$= \frac{3(n-2)}{(n-4)}$$

$$= \frac{3n-6}{n-4}$$

$$= \frac{3(n-4)+6}{n-4}$$

$$= 3 + \frac{6}{n-4}$$

$$\therefore \beta_2 = 3 + \frac{6}{n-4}$$

Comment: since, $\beta_1 = 0$, then the dist^n is symmetric and $\beta_2 > 3$ then the dist^n is leptokurtic.

(c) Relation between t-statistic and F-statistic:

Solution: we know, the pdf of F-dist^n with n_1 and n_2 degrees of freedom is,

$$f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} F^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \quad ; 0 \le F < \alpha$$

Let, $n_1 = 1$ and $n_2 = n$ then

$$f(F) = \frac{\left(\frac{1}{n}\right)^{1/2} F^{1/2-1}}{\beta\left(1/2, n/2\right)\left(1+F/n\right)^{\frac{n+1}{2}}}$$

$$= \frac{n^{-1/2} F^{-1/2}}{\beta(1/2, n/2)\left(1+F/n\right)^{\frac{1+n}{2}}}$$

Let, $F = t^2$

$dF = 2t\,dt$

$$\frac{dF}{dt} = 2t$$

$$\therefore |J| = \left|\frac{\partial F}{\partial t}\right| = 2t \qquad \left[\begin{array}{l}\text{when, } F=0;\ t=0 \\ F=\alpha;\ t=\alpha\end{array}\right]$$

$$f(t) = \frac{(1/n)^{1/2}(t^2)^{-1/2}}{\beta(1/2, n/2)\left(1+t^2/n\right)^{\frac{(1+n)}{2}}} \cdot 2t \quad ; 0 < t < \alpha$$

$$= \frac{2}{n^{1/2}\beta(1/2, n/2)(1+t^2/n)^{\frac{1+n}{2}}}$$

$$= \frac{2}{\sqrt{n}\,\beta(1/2, n/2)(1+t^2/n)^{\frac{1+n}{2}}} \quad ; 0<t<\alpha$$

This function is not one to one then the f^n is even function, so the pdf of t-dist^n is $f(t) = \frac{1}{\sqrt{n}\,\beta(1/2, n/2)(1+t^2/n)^{\frac{1+n}{2}}}$; $-\alpha \leq t < \alpha$

which is the pdf of student-t dist^n with n d.f. Hence $t^2 \sim F(1,n)$

8. (a) what are order statistics? Explain why order statistics are not independent. obtain the joint probability density function (pdf) of $X_{i:n}$ and $X_{j:n}$ $(1 \leq i < j \leq n)$ from the joint pdf of all n order statistics. Hence or otherwise, find the joint pdf of smallest and largest order statistics $f_{1,n:n}(x_1, x_n)$. Show that, $\iint f_{1,n:n}(x_1, x_n)\, dx_n\, dx_1 = 1$.

where, $-\alpha < x_1 < x_n < \alpha$

Answer to the question No-08

Answer:- Order statistics:

Let $x_1, x_2, \ldots x_p, \ldots x_n$ is a random sample of size n from an absolutely continuous population with probability density function $f(x)$ and cumulative distribution function $F(x)$. If the sample values be arranged in order of magnitude as follows-

$$x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{p:n} \leq \ldots \leq x_{n:n} \quad \text{——(1)}$$

Then, the set of new random variable given by (1) are called the order statistics drawn from the population.

where,
$x_{1:n}$ = First order or smallest order statistics

$x_{2:n}$ = 2nd order statistics

$\vdots$

$x_{n:n}$ = nth order statistics or Largest order statistics.

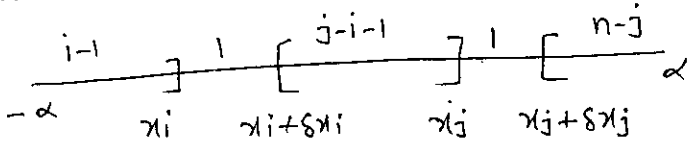
order statistics are dependent:

The order statistics are not independent, because the random sample are arranged in the ascending order by their magnitude in the random sample of order statistics so the random sample of order statistics are dependent.

The joint probability density function of $X_{i:n}$ and $X_{j:n}$ $(1\le i<j\le n)$:

In order to derive the joint density f^n of two order statistics $X_{i:n}$ and $X_{j:n}$ $(1\le i<j\le n)$

let us first visualize the event. $(x_i < x_{i:n} \le x_i+\delta x_i,\ x_j < x_{j:n} \le x_j+\delta x_j)$ as follows:



$x_p \le x_i$ for $i-1$ for the x_p's, $x_i < x_p < x_i+\delta x_i$ for exactly one of the x_p's, $x_i+\delta x_i < x_p \le x_j$ for $j-i-1$ of the x_p's, $x_j < x_p \le x_j+\delta x_j$ for exactly on of the x_p's and $x_p > x_j+\delta x_j$ for the remaining $n-j$ of the x_p's,

By considering δx_i and δx_j to be both small, we may write.

$$P(x_i < x_{i:n} < x_i+\delta x_i,\ x_j < x_{j:n} < x_j+\delta x_j)$$

$$= \frac{n!}{(i-1)!\,1!\,(j-i-1)!\,1!\,(n-j)!}\left[F(x_i)\right]^{i-1}\left[F(x_i+\delta x_i)-F(x_i)\right]$$

$$\left[F(x_j)-F(x_i+\delta x_i)\right]^{j-i-1}\left[F(x_j+\delta x_j)-F(x_j)\right]$$

$$\left[1-F(x_j+\delta x_j)\right]^{n-j} + O\{(\delta x_i)^2\delta x_j\} + O\{(\delta x_i)(\delta x_j)^2\}$$

$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\left[F(x)\right]^{i-1}\left[F(x_j)-F(x+\delta x)\right]^{j-i-1}$$

$$\left[1-F(x_j+\delta x_j)\right]^{n-j}\{F(x_i+\delta x_i)-F(x_i)\}\{F(x_j+\delta x_j)-F(x_j)\}$$

$$+O\{(\delta x_i)^2\delta x_j\}+O\{(\delta x_i)(\delta x_j)^2\} \qquad (*)$$

here $O((\delta x_i)^2\delta x_j)$ and $O((\delta x_i)(\delta x_j)^2)$ are higher order terms which corresponding to the probabilities of the event of having more than one x_p ~~is~~ in the interval $(x_i, x_i+\delta x_i)$

and at least one x_p in the interval $(x_j, x_j+\delta x_j)$ and of the event having one x_p in the interval $(x_i, x_i+\delta x_i)$ and more than one x_p in the interval $(x_j, x_j+\delta x_j)$ respectively. From equation (*) we may derive the joint density f^n of $x_{i:n}$ and $x_{j:n}$ $(1 \le i < j \le n)$ to be

$$f_{i,j:n}(x_i\, x_j) = \underset{\partial x_i \to 0\, \partial x_j \to 0}{\alpha t} \left\{ \frac{P(x_i < x_{i:n} < x_i+\delta x_i,\ x_j < x_{j:n} < x_j+\delta x_j)}{\partial x_i\, \partial x_j} \right.$$

$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \left[F(x_i)\right]^{i-1} \left\{F(x_j) - F(x_i+\delta x_i)\right\}^{j-i-1}$$

$$\left[1 - F(x_j+\delta x_j)\right]^{n-j} \underset{\partial x_i \to 0}{\alpha t} \left[\frac{F(x_i+\delta x_i) - F(x_i)}{\delta x_I}\right]$$

$$\underset{\partial x_j \to 0}{\alpha t} \left[\frac{F(x_j+\delta x_j) - F(x_j)}{\delta x_j}\right] + 0 + 0$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \left[F(x)\right]^{i-1} \left[F(x_j) - F(x_i+\delta x_i)\right]^{j-i-1} \left[1 - F(x_j+\delta x_j)\right]^{n-j} f(x_i)\, f(x_j)$$

$$-\alpha < x_i < x_j < \alpha$$

which is the joint pdf of $x_{i:n}$ and $x_{j:n}$

Proof:- We know, that the joint density f of the smallest and largest order statistics

$$f_{1,n:n}(x_1, x_n) = \frac{n(n-1)}{1} \left[F(x_n) - F(x_1)\right]^{n-2} f(x_1)\, f(x_n)$$

$$-\alpha < x_1 < x_n < \alpha$$

$$= n(n-1)\left[F(x_n) - F(x_1)\right]^{n-2} f(x_1)\, f(x_n)$$

$$; -\alpha < x < x_n < \alpha$$

Now,

$$\int_{x_n}\int_{x_1} f_{1,n:n}(x_1, x_n)\, dx_1\, dx_n$$

$$= \int_{-\alpha}^{\alpha}\int_{-\alpha}^{x_n} n(n-1)\left[F(x_n) - F(x_1)\right]^{n-2} f(x_1)\, f(x_n)\, dx_1\, dx_n$$

$$= n(n-1) \int_{-\alpha}^{\alpha}\int_{-\alpha}^{x_n} \left[F(x_n)\right]^{n-2} \left[1 - \frac{F(x_1)}{F(x_n)}\right]^{n-2} f(x_1)\, dx_1\, f(x_n)\, dx_n$$

$$= n(n-1) \int_{-\alpha}^{\alpha} \left[F(x_n)\right]^{n-2} \int_{-\alpha}^{\alpha} \left[1 - \frac{F(x_1)}{F(x_n)}\right]^{n-2} f(x_1)\, dx_1\, f(x_n)\, dx_n$$

Let, $\dfrac{F(x_1)}{F(x_n)} = z_1$ $\qquad x_1 = -\alpha \Rightarrow z_1 = 0$

$\qquad x_1 = x_n \Rightarrow z_1 = 1$

$$\frac{1}{F(x_n)} f(x_1)\, dx_1 = dz_1$$

$$= n(n-1)\int_{-\alpha}^{\alpha}[F(x_n)]^{n-2}\int_0^1 [1-z_1]^{n-2} F(x_n)\, dz_1\, f(x_n)\, dx_n$$

$$= n(n-1)\int_{-\alpha}^{\alpha}[F(x_n)]^{n-1}\int_0^1 (1-z_1)^{n-2} dz_1\, f(x_n)\, dx_n$$

$$= n(n-1)\int_{-\alpha}^{\alpha}[F(x_n)]^{n-1}\left[\frac{(1-z_1)^{n-1}}{n-1}(-1)\right]_0^1 f(x_n)\, dx_n$$

$$= n\int_{-\alpha}^{\alpha}[F(x_n)]^{n-1}\left[-(1-1)^{n-1}+(1-0)^{n-1}\right] f(x_n)\, dx_n$$

$$= n\int_{-\alpha}^{\alpha}[F(x_n)]^{n-1} f(x_n)\, dx_n$$

let, $F(x_n) = z_2$ | $x_n = -\alpha \to z_2 = 0$

$f(x_n)dx_n = dz_2$ | $x_n = \alpha \to z_2 = 1$

$$= n\int_0^1 z_2^{n-1}\, dz_2$$

$$= n\left[\frac{z_2^n}{n}\right]_0^1$$

$$= [1^n - 0]$$

$$= \underline{1}$$

Hence,

$$\iint f_{1,n:n}(x_1, x_n)\, dx_n\, dx_1 = \underline{1} \quad [\text{proved}]$$

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