Random sample: Let $x_1, x_2, \ldots, x_n$ constitute a random sample on a random variable x if they are independent and each has the same distribution ~~of~~ as x. We will abbreviate this by saying that $x_1, x_2, \ldots, x_n$ are iid i.e independent and identically distributed.

Population: The collection of all units of a specific type in a given region at a particular time is termed as a population or universe.

Example, A population of Rajshahi university students, a population of books in a library, a population of tree of a certain region/country etc.

Sample: A sample is a representative part of the population that is taken and considered for study.

Example, some students of Rajshahi university, some books in a library. some trees in a certain contry etc.

Parameter: A parameter is an index associated with population. Any numerical value describing ~~the~~ a characteristic of a population is called a parameter.

Example, population mean μ, population variance σ^2, population proportion π etc.

Statistic: A statistic is a characteristic of a sample. Any function of sample observations or items is called statistic. Usually denoted by small english alphabet.

Example, sample mean $\bar{x}$, sample variance s^2, sample proportion p etc.

Random sample: Let $x_1, x_2, \ldots, x_n$ be n i.i.d. random variables each having pdf $f(x|\theta)$, then the random variables $x_1, x_2, \ldots, x_n$ are said to ~~be~~ constitute (গঠন করা) a random sample of size n from $f(x|\theta)$.

Here, $x_1, x_2, \ldots, x_n$ are the sample items.

Question: Explain the concept of estimation with the example.

Answer:

Estimation: Estimation is the process of finding an estimate or approximation, which is a value that is useable for some purpose even if input data may be incomplete, uncertain or unstable.

Another definition, Let x be a random variable which represent some characteristic of the elements in a population whose density function is assumed $f(x|\theta)$; θ is unknown parameter. Again, Let the values $x_1, x_2, \ldots, x_n$ of a random sample $x_1, x_2, \ldots, x_n$ from $f(x|\theta)$ can be observed sample values $x_1, x_2, \ldots, x_n$, it is desired to estimate the value of the unknown parameter θ or $\tau(\theta)$ (sample function of θ)

The estimation can be divided into two types

(i) Point estimation.

(ii) Interval estimation.

For example, If $f(x|\theta)$ is the normal density function that is

$$f(x|\theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] ; -\infty < x < \infty .$$

where the parameter θ is (μ, σ^2) and if it desired to estimate the mean that is $\tau(\theta) = \mu$ then the statistic $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a possible point estimator of $\tau(\theta) = \mu$.

Estimator: Any function of random sample $x_1, x_2, \ldots x_n$ that are being used/observed say $T_n(x_1, x_2, \ldots)$ is called a statistic. If it is used to estimate the unknown parameter θ of the distribution, it is called an estimator.

Example, sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, sample variance $s^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ are an estimator of the population mean μ and population variance σ^2 respectivily.

Estimate: A particular value of the estimator is called an estimate.

Example, the sample mean $\bar{x} = 5.6$ (say) is an estimate value of the estimator.

Point estimation: A point estimate is a single number that is used to estimate an unknown population parameter.

Another definition, suppose $(x_1, x_2, \ldots, x_n)$ is a samp from a density $f(x|\theta)$ where θ is unknown fixed value which can assume any value in one-dimensional real parameter space Ω. Let t be a functi of $x_1, x_2, \ldots, x_n$ so that t is a statistic and hen a random variable. If t is used to estimate θ then t is called a point estimator of θ. If the realized value of t from a sample is used for θ then t is called a point estimate of θ.

2023/6/11 23:03

For example, if $f(x|\theta)$ is the normal density function that is $f(x|\theta) = (1/\sigma\sqrt{2\pi})\exp[-\frac{1}{2}(x-\mu/\sigma)^2]$ where the parameter θ is (μ, σ^2) and it is desired to estimate the mean, that is $\tau(\theta) = \mu$ then the statistic $\bar{x} = n^{-1}.\sum_{i=1}^{n} x_i$ is a possible point estimator of $\tau(\theta) = \mu$.

Interval Estimation: The interval estimation is to define two statistic say $t_1(x_1, x_2, \cdots, x_n)$ and $t_2(x_1, x_2, \cdots, x_n)$ so that $\{t_1(x_1, x_2, \cdots, x_n), t_2(x_1, x_2, \cdots, x_n)\}$ constitutes (গঠন করা) an interval for which the probability can be determined that is contains the unknow $\tau(\theta)$.

For example if $f(x|\theta)$ is the normal density that is $f(x) = (\sigma\sqrt{2\pi})^{-1}\exp[-\frac{1}{2}(x-\mu/\sigma)^2]$ where the parameter θ is (μ, σ^2) and if it is desired to estimate the mean is $\tau(\theta) = \mu$. Then the statistic $\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$ is a possible point estimation of $\tau(\theta) = \mu$ and $(\bar{x} - 2\sqrt{s^2/n}, \bar{x} + 2\sqrt{s^2/n})$ is a possible interval estimator of $\tau(\theta) = \mu$, where $s^2 = (n-1)^{-1}.\sum_{i=1}^{n}(x_i - \bar{x})^2$.

⇒ **Propertise or criteria of a good estimator:**

(i) Unbaisedness
(ii) consistency
(iii) Efficiency
(iv) sufficiency.

(i) **Unbaisedness:** Any statistic whose mathematical expectation is equal to a parameter θ is called an unbaised estimator of the parameter θ. Otherwise the statistic is said to be baised.

Let t_n be a statistic calculated from a sample $(x_1, x_2, \ldots, x_n)$ of size n from density $f(x|\theta)$. If for al n and θ $E(t_n) = \theta$, then t_n is called an unbaised estimator of θ.

In case t_n be a biassed estimator the difference $E(t_n) - \theta$ is the amount of ~~bai~~ bias and $E(t_n - \theta)^2$ is called meansquare error. Meansquare-error of t_n = variance of t_n + bias².

For example, if a random sample $(x_1, x_2, \ldots, x_n)$ of size n is drawn from a normal distribution/population with mean θ and variance σ^2, then

$$E(\bar{x}) = \frac{1}{n}E(x_1 + x_2 + \cdots + x_n)$$
$$= \frac{1}{n}[E(x_1) + E(x_2) + \cdots + E(x_n)]$$
$$= \frac{1}{n}\cdot n\theta$$
$$= \theta$$
$$\therefore E(\bar{x}) = \theta.$$

And,
$$E(s^2) = \frac{1}{n-1}E\left[\sum(x_i - \bar{x})^2\right]$$
$$= \frac{\sigma^2}{(n-1)}E\left[\frac{\sum(x_i - \bar{x})^2}{\sigma^2}\right]$$
$$= \frac{\sigma^2}{(n-1)}E(\chi^2_{n-1})$$
$$= \sigma^2(n-1)^{-1}.(n-1)$$
$$= \sigma^2.$$

$\therefore E(\hat{\sigma}^2) = \sigma^2$

Thus, $\bar{x}$ and s^2 are an unbaised estimator of θ and σ^2 respectivily.

(ii) Consistency:

Let t_n be a statistic calculated from a sample $(x_1, x_2, \ldots, x_n)$ of size n from density $f(x|\theta)$.

If $\underset{n\to\infty}{P}[|t_n - \theta| < \varepsilon] = 1 - \delta$

where ε and δ are arbitrary small positive numbers then t_n is called a consistent estimator of θ.

consistency is a large sample property. It is not defined for a small sample. A statistic is said to be consistent estimator of the population parameter if it approaches the parameter as the sample size increases.

For example, if $x_1, x_2, \ldots, x_n$ is a random sample from a population with finite mean $E(x_i) = \mu < \infty$

Now, we have

$$\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$$

$$\therefore E(\bar{x}) = \mu$$

and $E(\bar{x}) = \mu$ as $n \to \infty$

Hence, sample mean $\bar{x}_n$ is always a consistent estimator of the population mean. μ.

(iii) Efficiency:

If $(x_1, x_2, \ldots, x_n)$ be a sample from density $f(x|\theta)$ and t be an unbaised consistent estimator of θ and furth no other estimations have variance less than that of t, then t is said to be the most efficient estimat of θ (also simply called efficient estimator of θ).

Let t^* be any other unbaised statistic. The efficier of t^* is the ratio of reciprocal of the variance of t^* to the amount of information in the data. Actuall the efficiency of t^* is measured by

$$e(t^*) = \frac{V(t)}{V(t^*)} \quad \ldots \ldots (1)$$

The efficiency of t^* represents the fraction of the relevant information available actually utilized by t^*.

Since $V(t) \leq V(t^*)$ the efficiency of any statistic vari between 0 to 1.

For example, Let $x \sim N(\mu, \sigma^2)$ and x_1, x_2, x_3 be a random sample, then

$$T_1 = \frac{x_1 + x_2 + x_3}{3} \sim N(\mu, \sigma^2/3)$$

$$T_2 = \frac{1}{2}(x_1 + x_2) \sim N(\mu, \sigma^2/2)$$

Here, both T_1 and T_2 are unbiased estimators of μ. But $\text{var}(T_1) < \text{var}(T_2)$ implies that T_1 is more efficient than T_2.

Sufficiency:

An estimator is said to be sufficient for a parameter if it contains all the information in the sample regarding the parameter.

If $T = t(x_1, x_2, \dots, x_n)$ is an estimator of a parameter θ, based on a sample $x_1, x_2, \dots, x_n$ of size n from the population with density $f(x|\theta)$ such that the conditional distribution of $x_1, x_2, \dots, x_n$ given T, is independent of θ, then T is sufficient estimator for θ.

For example, $x \sim B(n, \theta)$

sample: $x_1, x_2, \dots, x_n$

$\therefore f(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$

Now, $p(x) = \theta^{\Sigma x_i} (1-\theta)^{Nn - \Sigma x_i}$ $\boxed{\text{Bernoulli dist}^n}$

$\therefore p(\Sigma x_i) = \binom{n}{\Sigma x_i} \theta^{\Sigma x_i} (1-\theta)^{Nn - \Sigma x_i}$

Now,

$$\frac{p(x)}{p(\Sigma x_i)} = 1 / \binom{n}{\Sigma x_i}$$

which is independent of θ.

So, Σx_i is a sufficient estimator of θ.

Theorem: Cramer-Rao Lower Bound (CRLB):

Suppose

(i) $x_1, x_2, \dots, x_n$ are independent random variables each with density $f(x|\theta)$. $\theta \in \Omega$ an open interval on the real Line.

(ii) t is an estimator of parameter θ.

(iii) $E(t) = \theta + b(\theta)$ where $b(\theta)$ is the bias of t and is a differential function of θ.

(iv) The following regularity conditions hold

(a) for almost all x, $\frac{\partial L}{\partial \theta}$ exist $\forall \theta$.

(b) $\frac{\delta}{\delta\theta} \int \cdots \int L = \int \cdots \int \frac{\partial L}{\partial \theta}$ which is possible when limit of integration are independent of θ.

(c) $E\left[\frac{\partial \log L}{\partial \theta}\right]^2 > 0$ for $\theta \in \Omega$

(d) $\frac{\delta}{\delta\theta} \int \cdots \int tL$

$= \int \cdots \int t \frac{\partial L}{\partial \theta}$

Then for all $\theta \in \Omega$

Biase part [Unbiased হলে numerator 1 হয়ে যেতো

$$V(t) \geq \frac{[1 + b'(\theta)]^2}{nE\left[\frac{\partial \log f}{\partial \theta}\right]^2}$$

$$= \frac{[1 + b'(\theta)]^2}{E\left[\frac{\partial \log L}{\partial \theta}\right]^2}$$

$$= -\frac{''}{E\frac{\partial^2 \log L}{\partial \theta^2}}$$

where $b'(\theta)$ is the first differential of $b(\theta)$ with respect to θ.

P.T.O.

Question: Derive the Cramer Rao lower bound (CRLB) for the variance of an unbiased estimator t of the parameter θ.

or

State and prove the Cramer Rao Lower bound

Proof:

Statement: suppose

(i) $x_1, x_2, \ldots, x_n$ are independent random variables each with density $f(x|\theta)$. $\theta \in \Omega$ an open interval on the real line.

(ii) t is an estimator of θ.

(iii) $E(t) = \theta + b(\theta)$ where $b(\theta)$ is the biase of t and is a differentiable function of θ.

(iv) The following regularity conditions hold.

(a) for almost all x, $\frac{\partial L}{\partial \theta}$ (L is a likelyhood function) must exist for all $\theta \in \Omega$.

(b) $\frac{\partial}{\partial \theta}\int\cdots\int L = \int\cdots\int \frac{\partial L}{\partial \theta}$ which is possible when the limits of integration are independent of θ

(c) $E\left[\frac{\partial \log L}{\partial \theta}\right]^2 > 0$ for $\theta \in \Omega$

(d) $\frac{\partial}{\partial \theta}\int\cdots\int tL = \int\cdots\int t\frac{\partial L}{\partial \theta}$.

Then for all $\theta \in \Omega$

$$V(t) \geq \frac{[1+b'(\theta)]^2}{nE\left[\frac{\partial \log f}{\partial \theta}\right]^2}$$

$$= \frac{[1+b'(\theta)]^2}{E\left[\frac{\partial \log L}{\partial \theta}\right]^2}$$

$$= -\frac{[1+b'(\theta)]^2}{E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]}$$

where, $b'(\theta)$ is the first derivative of $b(\theta)$ w.r.to θ

Proof:

we know,

$$L = \prod_{i=1}^{n} f(x_i|\theta) \quad \cdots\cdots\cdots (i)$$

Since L is the joint density of the observation

$$\int\cdots\int L \, dx_1 \, dx_2 \ldots dx_n = 1 \quad \cdots\cdots (ii)$$

Now, suppose the first and second differentials of L exist. Then taking the first derivation of (ii) w.r.to θ on both sides.

$$\int\cdots\int \frac{\partial L}{\partial \theta} dx_1 \, dx_2 \ldots dx_n = 0$$

$$\text{or } \int\cdots\int \frac{\partial L}{\partial \theta}\cdot\frac{1}{L}\cdot L \, dx_1 \, dx_2 \ldots dx_n = 0$$

$$\text{or } \int\cdots\int \frac{\partial \log L}{\partial \theta}\cdot L \, dx_1 \, dx_2 \ldots dx_n = 0 \quad \text{———— (iii)}$$

$$\text{or } E\left[\frac{\partial \log L}{\partial \theta}\right] = 0 \quad \cdots\cdots\cdots (iii)$$

$$\text{or } E(\phi) = 0 \quad \text{where } \phi = \frac{\partial \log L}{\partial \theta}$$

Again ~~the~~ differentiating (iii) w.r.to θ.

$$\int\cdots\int\left[\frac{\partial \log L}{\partial\theta}\cdot\frac{\partial L}{\partial\theta}+L\,\frac{\partial^2\log L}{\partial\theta^2}\right]dx_1\,dx_2\cdots dx_n=0$$

$$\text{or, }\int\cdots\int\left[\frac{\partial \log L}{\partial\theta}\cdot\frac{\partial L}{\partial\theta}\cdot\frac{1}{L}\cdot L+L\cdot\frac{\partial^2\log L}{\partial\theta^2}\right]dx_1\,dx_2\cdots dx_n=0$$

$$\text{or, }\int\cdots\int\left[\left(\frac{\partial \log L}{\partial\theta}\right)^2\cdot L+L\cdot\frac{\partial^2\log L}{\partial\theta^2}\right]dx_1\,dx_2\cdots dx_n=0$$

$$\text{or, }\int\cdots\int\left(\frac{\partial \log L}{\partial\theta}\right)^2\cdot L\,dx_1\,dx_2\cdots dx_n+\int\cdots\int\frac{\partial^2\log L}{\partial\theta^2}\cdot L\,dx_1\,dx_2\cdots dx_n=0$$

$$\text{or, }E\left(\frac{\partial \log L}{\partial\theta}\right)^2+E\left(\frac{\partial^2\log L}{\partial\theta^2}\right)=0$$

$$\text{or }E\left(\frac{\partial \log L}{\partial\theta}\right)^2=-E\left(\frac{\partial^2\log L}{\partial\theta^2}\right).$$

Now,

$$E(t)=\theta+b(\theta)$$
$$=\int\cdots\int tL\cdot dx_1\,dx_2\cdots dx_n.$$

$$\therefore\ \frac{\partial E(t)}{\partial\theta}=[1+b'(\theta)]=\int\cdots\int t\,\frac{\partial L}{\partial\theta}\cdot dx_1\,dx_2\cdots dx_n.$$
$$=\int\cdots\int t\cdot\frac{\partial\log L}{\partial\theta}\cdot L\,d\underset{\sim}{x}$$
$$=E\left(t,\frac{\partial\log L}{\partial\theta}\right)$$
$$=E(t,\phi)\quad\text{since }\phi=\frac{\partial\log L}{\partial\theta}.$$

$$=\mathrm{Cov}(t,\phi)$$
since $E(\phi)=0$.

$$\text{or, }[1+b'(\theta)]^2=[\mathrm{Cov}(t,\phi)]^2$$
$$\leq V(t).V(\phi)$$
by schwartz's inequality.

Therefore,

$$V(t)\geq\frac{[1+b'(\theta)]^2}{V(\phi)}$$
$$=\frac{[1+b'(\theta)]^2}{E\left[\frac{\partial\log L}{\partial\theta}\right]^2}$$
$$=-\frac{[1+b'(\theta)]^2}{E\left[\frac{\partial^2\log L}{\partial\theta^2}\right]}$$
$$=-\frac{[1+b'(\theta)]^2}{nE\left[\frac{\partial\log f}{\partial\theta}\right]^2}$$

In case, t is an unbaised estimator of θ. i.e $E(t)=$.

Then,

$$V(t)\geq\frac{1}{V(\phi)}$$

(proved)

Proof: we know

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$
$$= L(\theta|x_i)$$

Since L is the ~~density~~ joint density of the observations

$$\int\cdots\int L \, dx_1 \, dx_2 \cdots dx_n = 1 \quad \cdots\cdots\cdots (i)$$

Now suppose the first and second differentials of L exist. Then taking first derivative w.r.to θ on both sides.

$$\int\cdots\int \frac{\partial L}{\partial \theta} dx_1 \, dx_2 \cdots dx_n = 0$$

or $$\int\cdots\int \left(\frac{\partial L}{\partial \theta} \cdot \frac{1}{L}\right) L \, dx_1, dx_2 \cdots dx_n = 0$$

or $$\int\cdots\int \frac{\partial \log L}{\partial \theta} \cdot L \, dx_1 \cdot dx_2, \cdots, dx_n = 0$$

or $$E\left(\frac{\partial \log L}{\partial \theta}\right) = 0 \quad \cdots\cdots\cdots (ii)$$

Differentiating (ii) again

$$\int\cdots\int \left[\frac{\partial \log L}{\partial \theta} \cdot \frac{\partial L}{\partial \theta} + L \cdot \frac{\partial^2 \log L}{\partial \theta^2}\right] dx_1 \cdots dx_n = 0$$

or $$\int\cdots\int \left[\frac{\partial \log L}{\partial \theta} \cdot \frac{\partial L}{\partial \theta} \cdot \frac{1}{L} \cdot L + L \cdot \frac{\partial^2 \log L}{\partial \theta^2}\right] dx_1 \cdots dx_n = 0$$

or $$\int\cdots\int \left[\left(\frac{\partial \log L}{\partial \theta}\right)^2 \cdot L + L \cdot \frac{\partial^2 \log L}{\partial \theta^2}\right] dx_1 \cdots dx_n = 0$$

or $$\int\cdots\int \left(\frac{\partial \log L}{\partial \theta}\right)^2 \cdot L \, dx_1 \cdots dx_n + \int\cdots\int \frac{\partial^2 \log L}{\partial \theta^2} \cdot L \, dx_1 \cdots dx_n = 0$$

or $$E\left[\frac{\partial \log L}{\partial \theta}\right]^2 + E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right] = 0$$

or $$E\left[\frac{\partial \log L}{\partial \theta}\right]^2 = -E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right].$$

Now,

$$E(t) = \theta + b(\theta)$$
$$= \int\int t \, L \, dx_1 \, dx_2 \cdots dx_n$$

or $$\frac{\partial E(t)}{\partial \theta} = 1 + b'(\theta)$$
$$= \int\cdots\int t \frac{\partial L}{\partial \theta} dx_1 \, dx_2 \cdots dx_n.$$
$$= \int\cdots\int t \frac{\partial L}{\partial \theta} \cdot \frac{1}{L} \cdot L \, dx_1 \, dx_2 \cdots dx_n.$$
$$= \int\cdots\int t \cdot \frac{\partial \log L}{\partial \theta} \cdot L \cdot dx_1 \, dx_2 \cdots dx_n.$$
$$= E\left(t \cdot \frac{\partial \log L}{\partial \theta}\right)$$
$$= E(t\phi) \quad \text{where } \phi = \frac{\partial \log L}{\partial \theta}.$$
$$= cov(t \cdot \phi) \quad \text{since } E(\phi) = 0.$$

or $$[1 + b'(\theta)]^2 = [cov(t \cdot \phi)]^2$$
$$\leq v(t) \cdot v(\phi)$$

by schwartz's inequality.

Therefore,

$$v(t) \geq \frac{[1 + b'(\theta)]^2}{v(\phi)}$$
$$= \frac{[1 + b'(\theta)]^2}{E\left[\frac{\partial \log L}{\partial \theta}\right]^2}$$

$$= -\frac{[1+b'(\theta)]^2}{E\frac{\partial^2 \log L}{\partial \theta^2}}$$

$$= \frac{[1+b'(\theta)]^2}{nE\left(\frac{\partial \log f}{\partial \theta}\right)^2}.$$

(proved)

In case t is an unbiased estimator $E(t)=\theta$

and $V(t) \geq \frac{1}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$

$$= \frac{1}{V(\phi)} \quad \cdots\cdots\cdots \text{①}$$

The condition for equality

we have

$$[1+b'(\theta)]^2 = [cov(t,\phi)]^2$$

$$\text{Also } [1+b'(\theta)]^2 \leq V(t).V(\phi)$$

Hence for equality the following condition should satisfy

$$[Cov(t,\phi)]^2 = V(t).V(\phi)$$

when $\rho^2_{\phi t} = 1$, where $\rho_{t\phi}$ is the correlation coefficient between t and ϕ.

P.T.O.

If $\rho^2_{t\phi} = 1$ t and ϕ are Linearly related and the relationship is of the form

$$\phi = At + B. \cdots\cdots\cdots\cdots (*)$$

where A and B are constants and can be function of θ.

Taking expectation on both side in equation (*), then

$$E(\phi) = AE(t) + B$$

$$\text{or, } 0 = AE(t) + B \cdots\cdots\cdots (**)$$

Now (*) − (**), then

$$A[t - E(t)] = \phi$$

$$\text{or } E(\phi^2) = A^2 E[t - E(t)]^2 \quad \text{[squaring and taking expectation on both]}$$

$$\text{or } E(\phi^2) = A^2 V(t)$$

$$\text{or } V(\phi) = A^2 V(t)$$

$$\text{or, } V(t) = \frac{V(\phi)}{A^2}$$

$$\text{or } V(t) = \frac{1}{V(t).A^2} \quad \left[\because V(\phi) = \frac{1}{V(t)}\right]$$

$$\text{or } [V(t)]^2 = \frac{1}{A^2}$$

$$\text{or } V(t) = \frac{1}{A}$$

Thus A is the reciprocal of the variance of MVB unbiased estimator t.

Question: If the statistic t be such that $\phi = \frac{\partial \log L}{\partial \theta}$ (where, $L = \prod_{i=1}^{n} f(x|\theta)$ is the likelyhood function of θ) can be expressed in the form $\frac{\partial \log L}{\partial \theta} = A(t - E(t))$ or $A(t-\theta)$, then t is an MVB unbiased estimator of θ, with variance $\frac{1}{A}$.

or

Establish the condition under which minimum variance unbiased estimator (MVBUE) attains?

or

Find the minimum variance bound unbiased estimator (MVBUE).

Proof:

From Cramer Rao inequality, we know.

$$v(t) \geq \frac{[1+b'(\theta)]^2}{E\left[\frac{\partial \log L}{\partial \theta}\right]^2}$$

$$\text{or, } v(t) \geq \frac{[1+b'(\theta)]^2}{v\left(\frac{\partial \log L}{\partial \theta}\right)}$$

$$\text{or, } v(t) \geq \frac{[1+b'(\theta)]^2}{v(\phi)}$$

where, $\phi = \frac{\partial \log L}{\partial \theta}$.

The condition for equality, we have

$$[1+b'(\theta)]^2 = [Cov(t,\phi)]^2$$

Also, $[1+b'(\theta)]^2 \leq v(t) \cdot v(\phi)$

Now,

$$[Cov(t,\phi)]^2 = v(t) \cdot v(\phi)$$

This is satisfied when $\rho_{t\phi}^2 = 1$, where t and ϕ $\rho_{t\phi}$ is the correlation coefficient between t and ϕ. If $\rho_{t\phi} = 1$, t and ϕ are linearly related and relationship is of the form.

$$\phi = At + B \quad \cdots\cdots (1.1)$$

Taking expectation on both sides in (1.1), we have

$$E(\phi) = A\,E(t) + B.$$

$$0 = A.E(t) + B \quad \cdots\cdots (1.2).$$

Subtracting (1.2) from (1.1) we have.

$$\phi = A[t - E(t)]$$

$$\text{or, } \phi^2 = A^2[t - E(t)]^2$$

$$\text{or, } E(\phi^2) = A^2 E[t - E(t)]^2$$

$$= A^2 v(t)$$

$$\text{or } v(t) = \frac{E(\phi^2)}{A^2} \qquad \because E(\phi) = 0$$

$$= \frac{v(\phi)}{A^2}$$

$$\therefore v(t) = \frac{v(\phi)}{A^2} \quad \cdots\cdots (1.3).$$

Also, we know

$$v(t) = \frac{1}{v(\phi)} \quad \text{since } t \text{ is MVB estimator}$$

$$\therefore v(\phi) = \frac{1}{v(t)}$$

Substituting this in (1.3), we have.

$$v(t) = \frac{1}{A^2 \cdot v(t)}$$

$$\text{or, } [v(t)]^2 = \frac{1}{A^2}$$

$$\text{or } \quad v(t) = \frac{1}{A}$$

$$\therefore v(t) = \frac{1}{A}$$

Thus, A is the reciprocal of the variance of MVBUE of t.

(showed)

Question: Let $x \sim N(\mu, \sigma^2)$. Find the MVBUE of μ.

Answer:

Given, $x \sim N(\mu, \sigma^2)$, then the density function

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\{(x-\mu)/\sigma\}^2\right] \; ; -\infty < x < \infty$$

we know

$$L = \prod_{i=1}^{n} f(x_i|\mu, \sigma^2)$$

$$= (2\pi\sigma^2)^{-n/2} \, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Taking log on bothsides, we have

$$\log L = \log c - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \qquad \text{where } c = (2\pi\sigma^2)^{-n/2}$$

$$= \log c - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Now, taking derivative w.r.to μ, we have.

$$\frac{\partial \log L}{\partial \mu} = 0 - \frac{2}{2\sigma^2}\sum(x_i-\mu)^{2-1}(-1)$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$= \frac{1}{\sigma^2}(\sum x_i - n\mu)$$

$$= \frac{1}{\sigma^2}(n\bar{x} - n\mu)$$

$$= \frac{n}{\sigma^2}(\bar{x} - \mu)$$

which can be expressed as $\frac{\partial \log L}{\partial \theta} = A(t-\theta)$ where $A = n/\sigma^2$ and variance $v(t) = A^{-1} = \sigma^2/n$.

Therefore, we can say that $\bar{x}$ is the MVBUE of μ with variance σ^2/n.

Question: Let $x \sim E(1/\theta)$. Find the MVBUE of θ.

Answer:

Given, $x \sim E(1/\theta)$, then the density function.

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta} \; ; \quad x = 0, 1, 2 \ldots$$

we know the likelyhood function

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \prod_{i=1}^{n} \left[\frac{1}{\theta} e^{-x_i/\theta}\right]$$

$$= (1/\theta)^n e^{-\frac{1}{\theta}\Sigma x_i}$$

$$\log L = n \log \frac{1}{\theta} - \frac{1}{\theta}\Sigma x_i = -n \log \theta - \frac{1}{\theta}\Sigma x_i$$

$$\therefore \frac{\partial \log L}{\partial \theta} = -n \frac{1}{\theta} + \frac{\Sigma x_i}{\theta^2}$$

$$= -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2}$$

$$= \frac{-n\theta + n\bar{x}}{\theta^2}$$

$$= \frac{1}{\theta^2} n (\bar{x} - \theta)$$

$$= \frac{n}{\theta^2} (\bar{x} - \theta)$$

$$\therefore \frac{\partial \log L}{\partial \theta} = \frac{n}{\theta^2} (\bar{x} - \theta)$$

which can be expressed as $\frac{\partial \log L}{\partial \theta} = A(t - \theta)$ where $A = n/\theta^2$ and variance $V(\theta) = A^{-1} = \frac{\theta^2}{n}$

Therfore, we can say that $\bar{x}$ is the MVBUE of θ with variance θ^2/n.

Question: If $x \sim B(n, \theta)$. Find the MVBUE of θ.

Answer:

we are given $x \sim B(n, \theta)$, then the pmf is

$$f(x|n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \; ; \quad x = 0, 1, 2, \ldots, n$$

we know the likelyhood function

$$L = \prod_{i=1}^{n} f(x_i|n, \theta)$$

$$= \prod_{i=1}^{n} \left[\binom{n}{x_i} \theta^{x_i} (1-\theta)^{n-x_i}\right]$$

$$= \prod_{i=1}^{n} \binom{n}{x_i} \theta^{\Sigma x_i} (1-\theta)^{\Sigma(n-x_i)}$$

Taking log on both sides.

$$\log L = \log\left[\prod_{i=1}^{n} \binom{n}{x_i}\right] + \Sigma x_i \log \theta + \Sigma(n-x_i) \log(1-\theta)$$

$$\therefore \frac{\partial \log L}{\partial \theta} = 0 + \Sigma x_i \cdot \frac{1}{\theta} + \Sigma(n-x_i) \cdot \frac{1}{(1-\theta)} \cdot (-1)$$

$$= \frac{\Sigma x_i}{\theta} - \frac{\Sigma(n-x_i)}{(1-\theta)}$$

$$= \frac{\Sigma x_i(1-\theta) - \Sigma(n-x_i) \cdot \theta}{\theta(1-\theta)}$$

$$= \frac{\Sigma x_i - \theta\Sigma x_i - \theta n^2 + \theta\Sigma x_i}{\theta(1-\theta)}$$

$$= \frac{\Sigma x_i - \theta n^2}{\theta(1-\theta)}$$

$$= \frac{n\bar{x} - \theta n^2}{\theta(1-\theta)}$$

$$= \frac{n(\bar{x} - n\theta)}{\theta(1-\theta)}$$

$$= \frac{n}{\theta(1-\theta)}(\bar{x} - n\theta) = \frac{n}{\theta(1-\theta)} \cdot n\left(\frac{\bar{x}}{n} - \theta\right)$$

which can be expressed in the for $A(t-\theta)$ where $A = n/\theta(1-\theta)$ and variance $v(t) = \theta(1-\theta)/n$.

Therefore, we can say that $\bar{x}$ is the MVBUE of $n\theta$ with variance $(\theta(1-\theta)/n)$.

Question: For Bernoulli distribution. Find MVBUE of θ.

Answer:

The pmf of bernoulli distribution is,

$$f(x|\theta) = \theta^x (1-\theta)^{1-x} \quad ; x = 0, 1$$

We know,

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{\sum_{i=1}^{n}(1-x_i)}$$

$$\log L = \Sigma x_i \log\theta + \Sigma(1-x_i) \log(1-\theta)$$

$$\therefore \frac{\partial \log L}{\partial \theta} = \Sigma x_i \frac{1}{\theta} + \Sigma(1-x_i)\frac{1}{(1-\theta)} \cdot (-1)$$

$$= \frac{\Sigma x_i}{\theta} - \frac{\Sigma(1-x_i)}{(1-\theta)}$$

$$= \frac{\Sigma x_i(1-\theta) - \theta\Sigma(1-x_i)}{\theta(1-\theta)}$$

$$= \frac{\Sigma x_i - \theta\Sigma x_i - \theta . n + \theta\Sigma x_i}{\theta(1-\theta)}$$

$$= \frac{\Sigma x_i - \theta n}{\theta(1-\theta)}$$

$$= \frac{n\bar{x} - n\theta}{\theta(1-\theta)}$$

$$= \frac{n(\bar{x} - \theta)}{\theta(1-\theta)}$$

$$= \frac{n}{\theta(1-\theta)}(\bar{x} - \theta)$$

which can be expressed as $A(t-\theta)$ where $A = \frac{n}{\theta(1-\theta)}$ and variance $v(t) = \frac{\theta(1-\theta)}{n}$.

Therefore, we can say that $\bar{x}$ is the MVBUE of θ with variance $\theta(1-\theta)/n$.

Ans

Question: Find MVBUE of θ for poission distribution.

Answer:

We know the pmf of poission distribution.

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!} \quad ; x = 0, 1, 2, \ldots$$

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we know

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \frac{e^{-n\theta} \cdot \theta^{\Sigma x_i}}{\prod_{i=1}^{n} (x_i)!}$$

$$\log L = -n\theta + \Sigma x_i \log \theta - \log\left[\prod_{i=1}^{n} (x_i)!\right]$$

$$\log L = -n\theta + \Sigma x_i \log \theta - \log c$$

$$\therefore \frac{\partial \log L}{\partial \theta} = -n + \Sigma x_i \frac{1}{\theta}$$

$$= -n + \frac{n\bar{x}}{\theta}$$

$$= \frac{-n\theta + n\bar{x}}{\theta}$$

$$= \frac{n}{\theta}(\bar{x} - \theta)$$

$$\therefore \frac{\partial \log L}{\partial \theta} = \frac{n}{\theta}(\bar{x} - \theta)$$

Therefore, we can say that $\bar{x}$ is the MVBUE of θ with variance θ/n.

Problem: If $x \sim N(0, \sigma^2)$. Then, find the MVBUE of σ^2.

or

A random sample $x_1, x_2, \dots, x_n$ is taken from a normal population with mean 0 and variance σ^2. Examine if $\Sigma x_i^2/n$ is a MVBUE of σ^2.

Solution:

Since $x \sim N(0, \sigma^2)$

Then, the density function:

$$f(x|\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2} \quad ; -\infty < x < \infty$$

We know,

$$L = \prod_{i=1}^{n} f(x|\sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\Sigma x_i^2}$$

Taking log on both sides, we have.

$$\log L = \frac{n}{2}\log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\Sigma x_i^2 \cdot \log e$$

$$= \frac{n}{2}\log\frac{1}{2\pi} - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma x_i^2$$

Now, differentiating, we get

$$\frac{\partial \log L}{\partial \sigma^2} = 0 - \frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{\Sigma x_i^2}{2\sigma^4}$$

$$= \frac{\Sigma x_i^2}{2\sigma^4} - \frac{n}{2}\frac{1}{\sigma^2}$$

$$= \frac{n}{2\sigma^4}\left[\frac{\Sigma x_i^2}{n} - \sigma^2\right]$$

We can write the form in the following term

$$\frac{\partial \log L}{\partial \theta} = A[t - \theta]$$

where, $A = \frac{n}{2\sigma^4}$, and variance $v(t) = \frac{2\sigma^4}{n}$.

Therefore, we can say that $\Sigma x_i^2/n$ is an MVBUE of σ^2 with variance. $\frac{2\sigma^4}{n}$.

Hence,

The MVB of t' where t' is an unbiased estimator of σ is given by.

$$(\text{MVB of } \sigma^2) \cdot \left(\frac{\partial g(\theta)}{\partial \theta}\right)^2$$

$$= \frac{2\sigma^4}{n} \cdot \frac{1}{4\sigma^2} \qquad \because \frac{\partial g(\theta)}{\delta\sigma} = \frac{1}{2\sigma}.$$

$$= \frac{\sigma^2}{2n}.$$

Thus, an MVB of σ is $\frac{\sigma^2}{2n}$ which is not attain.

⇒ x is an $N(\mu, \sigma^2)$ variate. Find the MVB of unbaised estimator of σ^2 when μ is known.

Sol^n. Given that

x is an $N(\mu, \sigma^2)$ variate when μ is known (here, $\mu = 0$) then, $N(0, \sigma^2)$.

Then the pdf of x is

$$f(x|\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \quad ; -\infty < x < \infty$$

we know.

$$L = \prod_{i=1}^{n} f(x|\sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} x^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\Sigma x_i^2}$$

Taking log on both we have.

$$\log L = \frac{n}{2}\log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\Sigma x_i^2 \log e.$$

$$= \frac{n}{2}\log\frac{1}{2\pi} - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma x_i^2$$

$$\therefore \frac{\partial \log L}{\partial \sigma^2} = 0 - \frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{\Sigma x_i^2}{2\sigma^4}$$

$$= \frac{\Sigma x_i^2}{2\sigma^4} - \frac{n}{2}\cdot(1/\sigma^2)$$

$$= \frac{n}{2\sigma^4}\left[\frac{\Sigma x_i^2}{n} - \sigma^2\right]$$

Therefore, we can say that $\Sigma x_i^2/n$ is the MVBUE of σ^2 with variance $2\sigma^4/n$.

Problem: x is an $N(\mu, \sigma^2)$ variate. Find an MVB of unbiased estimator of σ^2 when μ is unknown.

Answer:

Given that

x is an $N(\mu, \sigma^2)$, when μ is unknown.

Then the density function of x is

$$f(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; \; -\infty < x < \infty.$$

Now, the likehood function is

$$L = \prod_{i=1}^{n} f(x|\mu, \sigma^2).$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{\frac{-1}{2}\frac{\sum_{i=1}^{n}(x-\mu)^2}{\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[\frac{-1}{2\sigma^2}\sum(x-\mu)^2\right]$$

$$\log L = \frac{n}{2}\log\frac{1}{(2\pi\sigma^2)} - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2 \log e$$

$$= \frac{n}{2}\log\frac{1}{(2\pi\sigma^2)} - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$\frac{\partial \log L}{\partial \sigma^2} = 0 - \frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{\sum(x_i-\mu)^2}{2\sigma^4}$$

$$= \frac{\sum(x_i-\mu)^2}{2\sigma^4} - \frac{n}{2}\cdot\frac{1}{\sigma^2}$$

$$= \frac{n}{2\sigma^4}\left[\frac{\sum(x_i-\mu)^2}{n} - \sigma^2\right]$$

$$\therefore \frac{\partial \log L}{\partial \sigma^2} = \frac{n}{2\sigma^4}\left[\frac{\sum(x_i-\mu)^2}{n} - \sigma^2\right]$$

which can be expressed in the form as $\frac{\partial \log L}{\partial \theta} = A[t-\theta]$. where $A = n/2\sigma^4$.

Therefore, we can say that $\frac{\sum(x_i-\mu)}{n}$ is a the MVBUE of σ^2 with variance $2\sigma^4/n$.

Question: Estiblish the method of finding MVB for a unbiased estimator intended to estimate function of a parameter.

Answer:

Suppose we have found an MVB unbiased estimator θ. This case, we use MVB of unbiased estimator of g a function of θ.

Let, $E(t') = g(\theta)$

Now,

$$\text{MVB of } t = \frac{\left[\frac{\delta E(t)}{\partial\theta}\right]^2}{nE\left[\frac{\partial \log f}{\partial\theta}\right]^2}$$

$$= \frac{1}{nE\left[\frac{\partial \log f}{\partial\theta}\right]^2} \qquad \because E(t)=\theta,\ \frac{\partial E(t)}{\partial\theta}=1$$

and

$$\text{MVB of } t' = \frac{\left[\frac{\partial E(t')}{\delta g(\theta)}\right]^2}{nE\left[\frac{\partial \log f}{\partial g(\theta)}\right]^2}$$

$$= \frac{1}{nE\left[\frac{\partial \log f}{\partial g(\theta)}\right]^2} \qquad \because E(t') = g(\theta),\ \frac{\partial E(t')}{\partial g(\theta)} = 1$$

$$= \frac{1}{E\left[\frac{\partial \log L}{\partial g(\theta)}\right]^2}$$

$$= \frac{1}{E\left[\frac{\partial \log L}{\partial\theta}\cdot\frac{\partial\theta}{\partial g(\theta)}\right]^2}$$

$$= \frac{1}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2 \left(\frac{\partial \theta}{\partial g(\theta)}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2 E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2 E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2} \cdot \frac{1}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2} \cdot MVB(t)$$

$$= MVB(t) \cdot \left(\frac{\partial g(\theta)}{\partial \theta}\right)^2$$

$$\therefore MVB \text{ of } t' = MVB(t) \cdot \left(\frac{\partial g(\theta)}{\partial \theta}\right)^2$$

(showed)

Example: x is an $N(0, \sigma^2)$ variate. Find an MVB unbiased estimator of σ.

Answer:

Given that:

x is an $N(0, \sigma^2)$ variate.

Let, $\theta = \sigma^2$ and $g(\theta) = \sigma$

Now, we know the likelyhood function

$$L = \prod_{i=1}^{n} f(x|\sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(x-\mu/\sigma\right)^2};$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2}$$

Taking log on both sides. we have.

$$\log L = \frac{n}{2}\log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2$$

$$= \frac{n}{2}\log\frac{1}{2\pi} - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2$$

Now, differentiating ~~lol~~ $\log L$ with respect to σ^2,

$$\frac{\partial \log L}{\partial \sigma^2} = 0 - \frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\Sigma(x_i-\mu)^2$$

$$= \frac{1}{2\sigma^4}\Sigma(x_i-\mu)^2 - \frac{n}{2\sigma^2}$$

$$= \frac{1}{2\sigma^4}\Sigma x_i^2 - \frac{n}{2\sigma^2} \quad \text{Since } \mu = 0$$

$$= \frac{n}{2\sigma^4}\left[\frac{\Sigma x_i^2}{n} - \sigma^2\right]$$

$$\frac{\partial \log L}{\partial \sigma^2} = \frac{n}{2\sigma^4}\left[\frac{\Sigma x_i^2}{n} - \sigma^2\right]$$

which can be expressed in the form $\frac{\partial \log L}{\partial \theta} = A(t - E(t))$

where $A = n/2\sigma^4$ and $V(t) = 2\sigma^4/n$.

Therefore, we can say. $\Sigma x_i^2/n$ is an MVB estimator of σ^2.

Again, the MVB of t' where t' is an unbaised estimate of σ is given by

$$(\text{MVB of } \sigma^2) \cdot \left[\frac{\partial g(\theta)}{\partial \theta}\right]^2$$

$$= \frac{2\sigma^4}{n} \cdot \frac{1}{4\sigma^2}$$

$$= \frac{\sigma^2}{2n}.$$

Thus, an MVB of σ is $\sigma^2/2n$ which, which doesn't attained the $(CRLB)_{t'}$.

Example: To find the MVB unbiased estimator of θ when x is a $G(\frac{1}{\theta}, r)$ variate and r is known.

Answer:

We are given

x is a $G(\frac{1}{\theta}, r)$ variate and r is known

Then, the density function of x is

$$f(x|1/\theta, r) = \frac{e^{-x/\theta} \cdot x^{r-1}}{\theta^r \sqrt{r}}$$

P.T.O

We know, the likelyhood function is

$$L = \prod_{i=1}^{n} f(x|1/\theta, r)$$

$$= \prod_{i=1}^{n} \frac{e^{-x/\theta} \cdot x^{r-1}}{\theta^r \sqrt{r}}$$

$$= \frac{e^{-\frac{1}{\theta}\Sigma x_i} \prod_{i=1}^{n} x^{r-1}}{\theta^{nr} (\sqrt{r})^n}$$

$$\log L = -\frac{1}{\theta}\Sigma x_i + \log\left[\prod_{i=1}^{n} x^{r-1}\right] - nr \log\theta - n\log(\sqrt{r})$$

Taking derivative with respect to θ, we have

$$\frac{\partial \log L}{\partial \theta} = \frac{\Sigma x_i}{\theta^2} + 0 - \frac{nr}{\theta}$$

$$= \frac{n}{\theta^2}\left[\frac{\Sigma x_i}{n} - \theta r\right]$$

$$= \frac{n}{\theta^2}\left[\bar{x} - \theta r\right]$$

$$= \frac{nr}{\theta^2}\left[\frac{\bar{x}}{r} - \theta\right]$$

Hence, we say that $\frac{\bar{x}}{r}$ is an MVB unbiased estimat of θ with variance θ^2/nr.

Example: x is a $P(\mu)$ variate. To find the MVB of unbiased estimator of μ^2.

Answer:

We are given

x is a $P(\mu)$ variate, then the density function.

$$f(x|\mu) = \frac{e^{-\mu} \cdot (\mu)^x}{x!}.$$

we know the likelyhood function.

$$L = \prod_{i=1}^{n} f(x|\mu)$$

$$= \prod_{i=1}^{n} \frac{e^{-\mu}.(\mu)^x}{x!}$$

$$= \frac{e^{-n\mu}.(\mu)^{\Sigma x_i}}{\prod_{i=1}^{n}(x_i!)}$$

Taking log on both sides, we have

$$\log L = -n\mu + \Sigma x_i \log \mu - \log\left[\prod_{i=1}^{n}(x_i!)\right]$$

$$\frac{\partial \log L}{\partial \mu} = -n + \frac{\Sigma x_i}{\mu} - 0$$

$$= \frac{\Sigma x_i}{\mu} - n.$$

$$= \frac{\Sigma x_i - n\mu}{\mu}.$$

$$= \frac{n\bar{x} - n\mu}{\mu}.$$

$$= \frac{n(\bar{x} - \mu)}{\mu}$$

$$= \frac{n}{\mu}[\bar{x} - \mu]$$

which can be expressed in the form $\frac{\partial \log L}{\partial \theta} = A[t - \theta]$
where $A = n/\mu$.

Therefore, $\bar{x}$ is an MVB unbiased estimator with variance μ/n.

Again,

Let $\theta = \mu$ and $g(\theta) = \mu^2$

Now, $\frac{\partial g(\theta)}{\partial \theta} = 2\mu.$

Now, the MVB of μ^2 is given as.

$$(\text{MVB of } \hat{\mu}) \left[\frac{\partial g(\theta)}{\partial \theta}\right]^2$$

$$= \frac{\mu}{n}.(2\mu)^2$$

$$= \frac{4\mu^3}{n}.$$

which is the MVB unbiased estimator of μ^2.

Note:

For 2μ, we let $\theta = \mu$ and $g(\theta) = 2\mu$.

Then $2t$ is the ~~minim~~ MVBUE of 2μ with variance $4.A$.

Similarly, $3\theta.(3t+4).(3\theta+4)$

Asymptotically Most efficient Estimator:
If $\lim_{n\to\infty} e(t^*) = \frac{MVB}{V(t^*)} = 1$, then t^* is called an asymptotically most efficient estimator.

Question: Relationship between MVBE and MLE.

Solution:
Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from $f(x|\theta)$, where θ is a parameter.

By definition of likelyhood function.

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

Let T be the MVBE of θ, then

$$\frac{\partial \log L}{\partial \theta} = A[t - E(t)] = A[t - \theta]; \quad A = \frac{1}{V(t)} \quad \text{--- (I)}$$

But we know MLE for solving θ is

$$\frac{\partial \log L}{\partial \theta} = 0 \quad \text{--- (II)}$$

From (I) and (II) we have

$$A[t-\theta] = 0$$
$$\Rightarrow t - \theta = 0$$
$$\therefore \hat{\theta} = t$$

which implies that MLE of θ is t.
Therefore, we coincide that if an MVBE exists then ~~B~~ MLE is the ~~MB~~ MVBE.

Sufficiency

Question: Define sufficient statistic.

Answer:
Sufficient Statistic: Let $f(x|\theta)$ be the density of a random variable x where θ is known fixed param and $\theta \in \Omega$. Let $x_1, x_2, \ldots, x_n$ be a random sample from this density. Let t and t' are two statistic such that t' is not a function of t. If the conditional distribution of t' for given t be independent of θ, it is called a sufficient statistic for θ.

Question: Prove that, sufficient statistic is unique.

Proof:
Let $x_1, x_2, \ldots, x_n$ be a sample from density $f(x|\theta)$. t be a sufficient statistic for θ.

Let there be another distinct statistic t_1 which is a sufficient statistic for θ. Then

$$h(t, t_1|\theta) = g(t|\theta)\, h(t_1|t)$$
$$= g(t_1|\theta)\, h(t|t_1)$$

t and t_1 are thus functionally related as

$$t = K(t_1, \theta).$$

Since t and t_2 are function of sample values. On t is functionally related to t_1.

Thus, a sufficient statistic is unique.

Theorem: If an MV unbiased estimator exists it is always unique. irrespective of whether any bound is attain.

Proof:
Let t_1 and t_2 be both MV unbiased estimators of θ each with variance V. consider a new estimator
$\bar{t} = \frac{t_1+t_2}{2}$.

Now,

$$
\begin{aligned}
V(\bar{t}) &= \frac{1}{4}\{V(t_1)+V(t_2)+2\,cov(t_1,t_2)\} \\
&\leq \frac{1}{4}[V(t_1)+V(t_2)+2\{V(t_1).V(t_2)\}^{1/2}] \\
&\leq \frac{1}{4}[V+V+2V] \\
&\leq \frac{1}{4}[4V] \\
&\leq V
\end{aligned}
$$

which contradicts the assumption that both t_1 and t_2 have minimum variance.

Now, $V(\bar{t}) = V$ provided.

$$
\begin{aligned}
\therefore Cov(t_1,t_2) &= \sqrt{V(t_1)\,V(t_2)}. \\
&= \sqrt{V(t_1).V(t_1)} \\
&= V(t_1).
\end{aligned}
$$

which is true when t_1 is identically equal to t_2.
Hence, MV unbiased estimator is unique.

(Showed)